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Combination-Difference Synchronization of Fractional Order Chaotic Duffing Oscillator and Financial Systems With Parameter Mismatch

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Abstract: *This research work is born out of the desire to design an effective synchronization scheme that could give a better understanding of the coordination of multiple processes and effective communication among various components of a complex system or between different groups of complex systems As a result, this research work presents combination-difference synchronization of fractional order chaotic (FOC) systems with parameter mismatches evolving from different initial conditions. Using the FOC Duffing oscillators and FOC financial systems as a paradigm, the backstepping technique is applied to design control laws for the achievement of combination-difference synchronization. These control laws enable the differences between the sums of the variables of the drive systems and differences of the variables of the response systems to converge to zero which confirms the achievement of combination difference synchronization. Numerical simulations provided confirm the effectiveness of the combination-difference synchronization technique. This result could be used to explain different interactions among particles and neurons of the same or different dynamical behaviour*

Keywords: *Parameter mismatch; Combination-difference synchronization; fractional order systems; backstepping technique; Duffing oscillators; financial systems.*

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1.0 Introduction

Fractional order systems are a generalized form of integer order systems. Due to memory and other hereditary properties, fractional order systems describe the dynamic behaviour of real-life systems more accurately than the integer order model (Bagley and Torvik, 1984; Bagley, 1998; Podlubny, 1999; Hilifer, 2001). For instance, the mathematical theory of fractional order differential equation has been proven as an effective theory for the analysis of complex dynamic behaviour in physical, biological, and chemical systems (Gerhards and Schlerf, 2018; Huang *et al*., 2018; Pandey *et al*., 2022). It is worthy of note that coupled fractional order systems give better insight into understanding complex dynamical behaviour exhibited by natural and artificial systems. One of the most investigated coupled dynamical behaviour of FOC systems is synchronization, this is a result of its potential applications to several life situations (Lei and Zheng, 2015; Bettayeb *et al*., 2017; Rajagopal *et al*., 2020; Batiha *et al*., 2021; Rahman *et al*., 2021).

The quest for synchronization that could give a better explanation of interaction among different groups of interconnected complex dynamical systems with parameter mismatch motivated this research work. As a result, a new synchronization scheme called combination-difference synchronization has been developed to synchronize to groups of chaotic systems with parameters mismatched. This new synchronization scheme is meant to give a better understanding of synchronization phenomena in real-life systems. This new synchronization developed for fraction order systems is reported here for the first time to the best of our knowledge. So, this paper aims to investigate of combination-difference synchronization behaviour of coupled fractional order with parameter mismatch evolving from different initial conditions systems via a nonlinear control technique.

Meanwhile, the synchronization of nonlinear dynamical systems has been fully developed due to its applications in various fields of study. Several methods of synchronization of nonlinear dynamical systems have been investigated such as linear feedback, nonlinear feedback, active control methods, backstepping technique, optimal control, finite-time, and others (Razminia and Baleanu, 2013; Li and Chen, 2014; Wu and Baleanu, 2014; Golmankhaneh *et al*., 2015; Boulkroune *et al.,* 2016; Maheri and Arifin, 2016; Nourian and Balochian, 2016; Li and Zhang, 2016; Shao *et al*., 2016; Soukkou *et al*., 2016; Jajarmi *et al*., 2017; Huang *et al*., 2018; Akif *et al*., 2021). Backstepping has been proven to be very effective in the synchronization of identical systems, non-identical systems, and systems of different dimensions (Ojo *et al.,* 2013; Ogunjo *et al*., 2017; Shen *et al*, 2019: Dongmo *et al*., 2022; Majdoul *et al.,*2022).

The backstepping control technique is chosen as the synchronization method as a result of its excellent performance in synchronization of nonlinear dynamical systems. Many synchronization schemes such as combination synchronization, combination-combination synchronization, difference synchronization and difference-difference have been investigated (Korsch *et al* ., 2007; Runzi *et al.,* 2011; Runzi and Yinglan, 2012; Runzi and Yinglan, 2012; Sun *et al*., 2013; Sun *et al*., 2013; Ojo *et al.,* 2015; Ojo *et al.,* 2016; Ojo *et al.,* 2022).

Most of these synchronization schemes have been reported on integer order systems but only a few reports are available on fractional order systems particularly, FOC systems with parameter mismatch. It should be noted that parameter mismatch is an important factor to be considered in synchronization since no two or more systems can be the same. Therefore, parameter mismatch synchronization is challenging and realistic as a result of differences in system parameters. Hence, the synchronization result of parameter mismatch systems gives a better understanding of the practical implementation of synchronization. This research work reports combination-difference synchronization of fraction order chaotic systems with parameter mismatch for the first time to the best of our knowledge.

The paper is organized as follows: Section 2 provides a brief system description. Sections 3 and 4 deal with combination-difference synchronization of duffing oscillators and financial systems with parameter mismatch respectively. Section 5 concludes the research paper.

2.0 Combination-Difference Synchronization Scheme

Combination-difference synchronization for arbitrary two drive and response systems are described using the nonlinear dynamical systems in equations (1)- (4) below. The drive systems are

$$D^q x = Ax + f(x) \quad (1)$$
$$D^q y = Ay + f(y) \quad (2)$$

The response nonlinear systems are

$$D^q z = A'z + f(z) + u(x, y, z, w) \quad (3)$$
$$D^q w = B'w + f(w) + v(x, y, z, w) \quad (4)$$

where $x = (x_1, x_2, \dots, x_n)^T \in \Re^m, y = (y_1, y, \dots, y_n)^T \in \Re^n, z = (z_1, z, \dots, z_n)^T \in \Re^p$ and $w = (w_1, w_2, \dots, w_n)^T \in \Re^q$ are the state vectors of system (1)-(4) respectively. $A \in \Re^m \times \Re^m, B \in \Re^n \times \Re^n, A' \in \Re^p \times \Re^p, B' \in \Re^q \times \Re^q$. $f(x), f(y), f(z), f(w)$ are nonlinear functions of the systems which are continuous and differentiable. The control functions to be designed that would enable combination-synchronization are $u(x, y, z, w)$ and $v(x, y, z, w)$.

Definition 1: *The order of the drive and the response systems are the same as assumed in this section. There exists error e such that* $e = \lim_{t\to\infty} \|(w + z) - (x - y)\| = 0$, *Then, the drive systems (1), (2) and the response systems (3), and (4) achieve combination-difference synchronization.*

3.0 Brief system description

3.1 Description of Duffing Oscillator

The fractional order Duffing model gives a good description of mechanical systems. The model of fractional order Duffing considered in this research work depends on the nature of nonlinearity, damping term, external excitation and linear term. This fractional order duffing exhibits different responses such as sustained single-period oscillations, multiple-period oscillations, and chaos. Applications of the Duffing equations include modelling of harmonically excited spring equations with nonlinear restoring force; suspended mass on a parallel combination of dashpot; and equation of rolling ships (Runzi and Yinglan, 2012; Runzi and Yinglan, 2012; Sun *et al*., 2013). The model used in this paper is described mathematically as follows

$$D^q x_1 = x_2$$
$$D^q x_2 = -bx_2 - \alpha x_1 - \beta x_1^3 + f \quad (5)$$

where q is the order of derivative which is between $0 < q < 1$. f is the amplitude of the forcing variable, ω is the frequency of the forcing term, β is the coefficient of the nonlinear term, b is the amplitude of the damping term and α is the coefficient of the linear term. The phase portrait of the fractional is shown in Fig. 1.

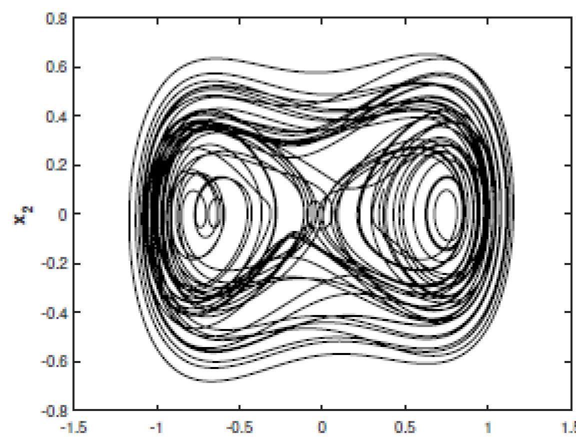

Fig. 1: The phase portrait of the fractional order Duffing for $b = 0.01, \alpha = -0.5, \beta = 1.0, \omega = 0.79, q_1 = 0.98, q_2 = 0.98$.

3.2 Description of financial system

The nonlinear dynamical system theory has been applied to solve some complex problems in finance, stock market, and economics which arise from complicated interaction between several market and socio-economic variables. A detailed description of the FOC financial system can be found in (Sun *et al*., 2013). The fractional order model of the financial system used in this paper is represented by the equations below.

$$D^q x = z + (y - a)x$$
$$D^q y = 1 - by - x^2$$
$$D^q z = -x - cz \quad (6)$$

The system describes the time variation of the interest x, investment y, and price index z. The parameters a, b, and c are the saving amount, the cost per unit investment, and the elasticity of demand of the market. The phase portrait of the fractional order financial system is presented in Fig. 2.

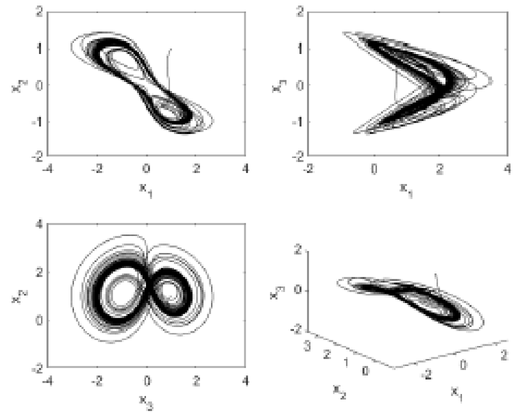

Fig. 2: The phase portrait of the fractional order financial system $\boldsymbol{a = 0.3}$, $\boldsymbol{b = 0.1}$, $\boldsymbol{c = 1.0}$, $\boldsymbol{q_1 = 0.9}$, $\boldsymbol{q_2 = 0.9}$, $\boldsymbol{q_3 = 0.9}$.

4.0 Combination-Difference Synchronization of Fractional Order Duffing Oscillators with Parameter Mismatch

The drive nonlinear duffing systems are as follows

$$D^q x_1 = x_2$$
$$D^q x_2 = -bx_2 - \alpha x_1 + f_1$$
$$D^q x_3 = x_4$$
$$D^q x_4 = -bx_4 - \alpha x_3 + f_2 \quad (7)$$

where; $f_1 = -\beta x_1^3 + f\cos\omega t$ and $f_2 = -\beta x_3^3 + f\cos\omega t$.

The response systems with parameter mismatch are as represented in the following equations

$$D^q x_5 = x_6 + u_1$$
$$D^q x_6 = -(b + \Delta b)x_6 - (\alpha + \Delta\alpha)x_5 + f_3 + u_2$$
$$D^q x_7 = x_8 + u_3$$
$$D^q x_8 = -(b + \Delta b)x_8 - (\alpha + \Delta\alpha)x_7 + f_4 + u_4 \quad (8)$$

where; $f_3 = -(\beta + \Delta\beta)x_5^3 + f\cos\omega t$ and $f_4 = -(\beta + \Delta\beta)x_7^3 + f\cos\omega t$.

In order to achieve the goal of combination synchronization, the following error systems are derived from the drive and the response systems above

$$e_1 = (x_5 + x_7) - (x_1 - x_3)$$
$$e_2 = (x_6 + x_8) - (x_2 - x_4) \quad (9)$$

Fractional order differential of (9) concerning time yields

$$D^q e_1 = e_2 + u_1 + u_3$$
$$D^q e_2 = -be_2 - \alpha e_1 - \Delta b(x_6 + x_8) - \Delta\alpha(x_5 + x_7) - f_1 + f_2 + f_3 + f_4 + u_2 + u_4 \quad (10)$$

Step 1: Consider the stability of the first error dynamics of 10 by letting $e_1 = z_1$ and regarding $e_2 = \alpha(z_1)$ as virtual work, a Lyapunov function in this direction is chosen as the Lyapunov function $v_1 = \frac{1}{2}z_1^2$ and its fractional-order time derivative is given as

$$D^q v_1 = D^q z_1^2 = z_1 D^q z_1 = z_1(\alpha(z_1)) + u_1 + u_3 \quad (11)$$

(11) must be negative definite in order to establish the stability of the subsystem. As a result, let $\alpha(z_1) = -z_1$ and $u_1 = u_3 = 0$. Making right substitution (11) becomes

$$D^q v_1 = -z_1^2 \le 0 \quad (12)$$

The estimated error between in assumed virtual work is given as:

$$z_2 = e_2 - \alpha(z_1) = e_2 + z_1 \qquad (13)$$

The following subsystems are obtained as:

$$D^q z_1 = z_2 - z_1$$
$$D^q z_2 = -b(z_2 - z_1) - \alpha z_1 - \Delta b(x_6 + x_8) - \Delta\alpha(x_5 + x_7) - f_1 + f_2 + f_3 + f_4 + u_2 + u_4 + z_2 - z_1 \qquad (14)$$

Step 2: In line with above subsystems 14, a Lyapunov function is chosen as $v_2 = v_1 + \frac{1}{2}z_2^2$and its fractional-order time derivative is

$$D^q v_2 = D^q v_1 + z_2 D^q z_2$$
$$D^q z_2 = -z_1^2 + z_2[-b(z_2 - z_1) - \alpha z_1 - \Delta b(x_6 + x_8) - \Delta\alpha(x_5 + x_7) - f_1 + f_2 + f_3 + f_4 + u_2 + u_4 + z_2 - z_1] \qquad (15)$$

In order to achieve a stable stability condition $D^q v_2 = -z_1^2 - kz_2^2 < 0$ (negative definite) in 15 then, we choose

$$u_2 + u_4 = b(z_2 - z_1) + \alpha z_1 - \Delta b(x_6 + x_8) - \Delta\alpha(x_5 + x_7) + f_1 - f_2 - f_3 - f_4 - kz_2 \qquad (16)$$

For simplicity, let $u_2 = u_4$ then,

$$u_2 = u_4 = 0.5[b(z_2 - z_1) + \alpha z_1 + \Delta b(x_6 + x_8) + \Delta\alpha(x_5 + x_7) + f_1 - f_2 - f_3 - f_4 - kz_2] \qquad (17)$$

With the control functions chosen as (17) then, stable combination-difference synchronization of the drive and response systems is achieved. The coupled Parameter mismatch of the FOC Duffing oscillator with the derived control functions is numerically computed via the Runge-Kutta algorithm implemented on MATLAB software. The system parameters $b = 0.01$, $\alpha = -0.5$, $\beta = 1.0$, $f = 0.095$, $\omega = 0.79$, $\Delta b = 0.03$, $\Delta\alpha = -0.2$, $\Delta\beta = 0.8$, $\Delta f = 0.5$, $q_1 = 0.98$, $q_2 = 0.98$ with initial conditions 0.01, 0.01, 1, 1, 0.5, 2, 0.1, 0.2 are employed. The error dynamics of the error variables as shown in Fig. 3 show that the error variable moves chaotically when the control functions are deactivated at $0 < t < 50$ and stabilized at zero when the control function is activated at $t < 50$. The reduction of error functions to zero is an indication of global synchronization between the systems. Another evidence of global synchronization is shown in Fig. 4. Identical dynamic behaviour was achieved between the drive and the response systems when control functions were applied $t < 50$.

5.0 Combination-Difference Synchronization of Financial Systems With Parameter Mismatch

The drive systems for the combination-difference synchronization is represented by

$$D^q x_1 = x_3(x_2 - a)\, x_1$$
$$D^q x_2 = 1 - bx_2 - x_1^2$$
$$D^q x_3 = -x_1 - cx_3$$
$$D^q x_4 = x_6 + (x_5 - \alpha)x_4$$
$$D^q x_5 = 1 - bx_5 - x_4^2$$
$$D^q x_6 = -x_4 - cx_6 \qquad (18)$$

The response systems with parameter mismatch are as follows

$$D^q x_7 = x_9 + (x_8 - (a + \Delta a)\,)x_7 + u_1$$
$$D^q x_2 = 1 - (b + \Delta b)x_8 - x_7^2 + u_2$$
$$D^q x_9 = -x_7 - \ (c + \Delta c)x_9 + u_3$$

$$D^q x_{10} = x_{12} + \big(x_{11} - (a + \Delta a)\big)x_{10} + u_4$$
$$D^q x_{11} = 1 - (b + \Delta b)x_{11} - x_{11}^2 + u_5$$
$$D^q x_{12} = -x_{10} - (c + \Delta c)x_{12} + u_6 \quad (19)$$

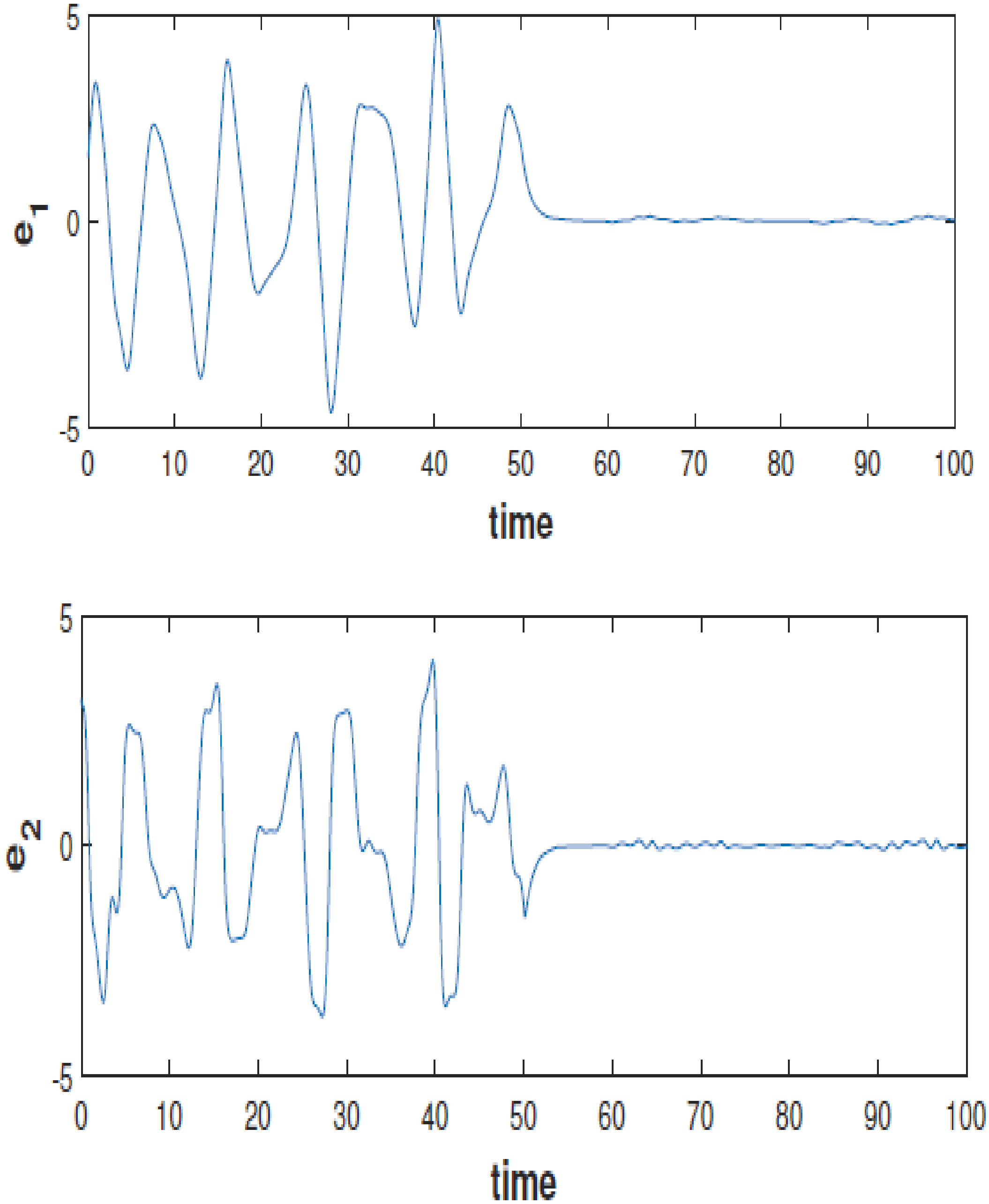

Fig. 3: The duffing error dynamics for combination-difference synchronization of the FOC duffing oscillators with parameter mismatch evolving from different initial conditions with the control function applied for $t < 50$.

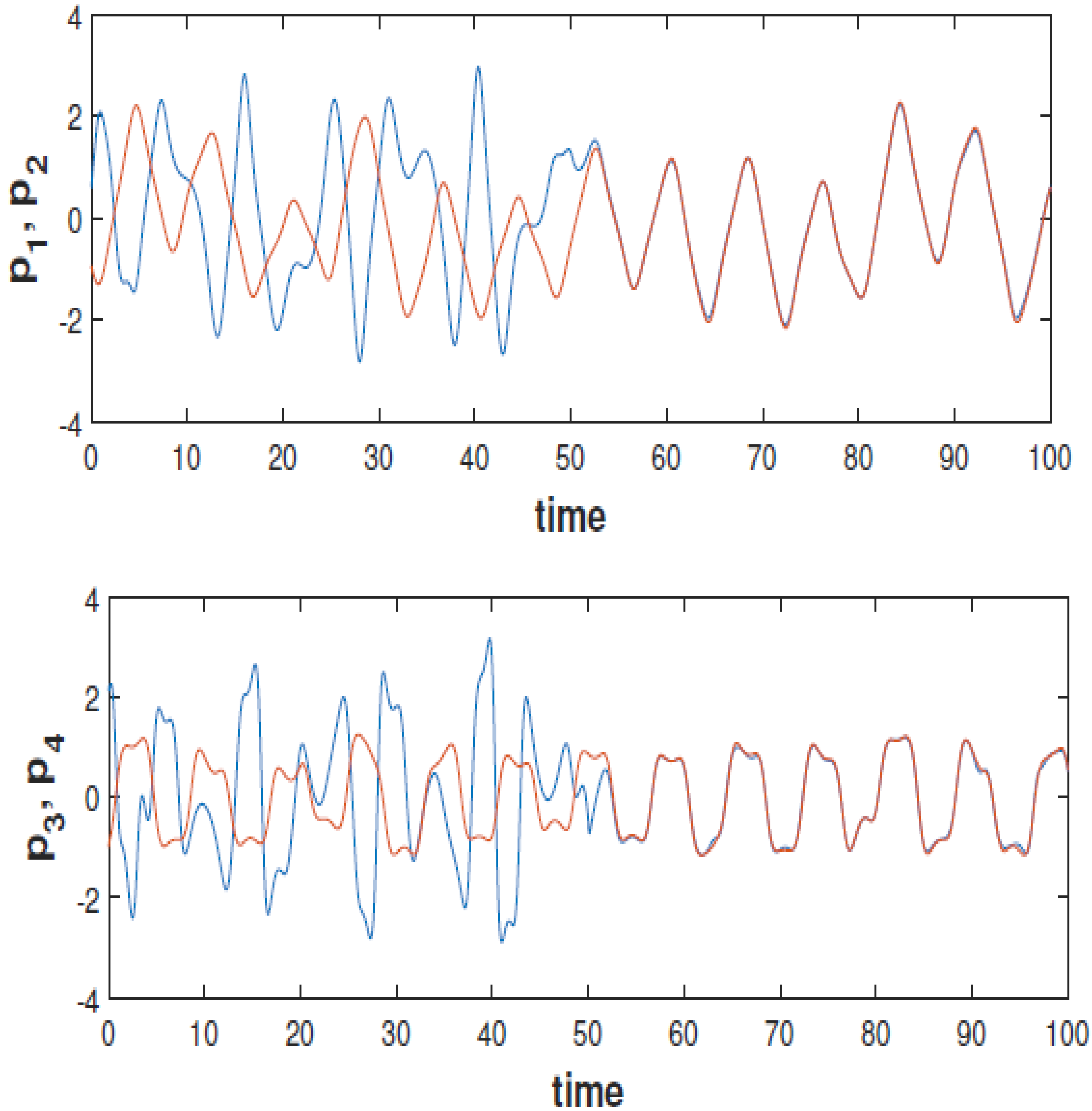

Fig. 4: Dynamics of the Duffing oscillators state variables for combination-difference synchronization of the FOC duffing oscillators with parameter mismatch evolving from different initial conditions with the control function applied for $t < 50$.

The following error systems are derived concerning the drive and response systems

$$e_1 = (x_7 + x_{10}) - (x_1 - x_4)$$
$$e_2 = (x_8 + x_{11}) - (x_2 - x_5)$$
$$e_3 = (x_9 + x_{12}) - (x_3 - x_6) \qquad (20)$$

The fractional order derivative of (20) is given as

$$D^q e_1 = e_3 - \alpha e_1 - \Delta a(x_7 + x_{10}) + f_1 + u_1 + u_4$$
$$D^q e_2 = 2 - b e_2 - \Delta b(x_8 + x_{11}) + f_2 + u_2 + u_5$$
$$D^q e_3 = -e_1 - c e_3 - \Delta c(x_9 + x_{12}) + u_3 + u_6 \qquad (21)$$

where $f_1 = x_7 x_8 + x_{10} x_{11} - x_1 x_2 + x_4 x_5$

$f_2 = -x_7^2 - x_{10}^2 + x_1^2 - x_4^2$

Step 1: Consider the stability of the first error dynamics of (21) by letting $e_1 = p_1$ and Regarding $e_3 = \alpha_3(p_1)$ as virtual work, a Lyapunov function in this direction is chosen as a Lyapunov function $v_1 = \frac{1}{2}p_1^2$ and its fractional-order time derivative is given as

$$D^q v_1 = D^q p_1^2 = p_1(\alpha_3(p_1) - \alpha p_1 - \Delta a(x_7 + x_{10}) + f_1 + u_1 + u_4) \quad (22)$$

(11) must be negative definite to establish the stability of the subsystem. As a result, let $\alpha_3(p_1) = 0$ and

$$u_1 + u_4 = \alpha p_1 - \Delta a(x_7 + x_{10}) - f_1 - p_1 \quad (23)$$

Making right substitution (22) becomes

$$D^q v_1 = -p_1^2 \leq 0 \quad (24)$$

Step 2: The estimative error between in assumed virtual work is given as $p_3 = e_3 - \alpha_3(p_1) = e_3$ since $\alpha_3(p_1) = 0$ The following sub-systems are obtained having made necessary substitutions

$$D^q p_1 = p_3$$
$$D^q p_3 = -p_1 - cp_3 - \Delta c(x_9 + x_{12}) + u_3 + u_6 \quad (25)$$

In line with above subsystems 25, a Lyapunov function is chosen as $v_3 = v_1 + \frac{1}{2}p_3^2$ and its fractional-order time derivative is

$$D^q v_3 = D^q v_1 + p_3\, D^q p_3$$
$$= -p_1^2 + p_3(-p_1 - cp_3 - \Delta c(x_9 + x_{12}) + u_3 + u_6) \quad (26)$$

In order to achieve a stable stability condition $D^q v_3 = -p_1^2 - cp_3^2 < 0$ (negative definite) in (26) then, we choose

$$u_3 + u_6 = p_1 + \Delta c(x_9 + x_{12}) \quad (27)$$

Step 3: Let $e_2 = \alpha_2(p_{1,}, p_2)$where $\alpha_2(p_{1,}, p_2)$ is virtual work then, $p_{2,} = e_{2,} - \alpha_2(p_{1,}, p_2)$. In order to establish the goal of combination-difference synchronization, a Lyapunov function is choosing as $v_2 = v_3 + \frac{1}{2}p_2^2$ with its fractional order time derivative given as:

$$D^q v_2 = D^q v_3 + p_2\, D^q p_2$$
$$= -p_1^2 + cp_3^2 + p_2(-bp_2 - \Delta b(x_8 + x_{11}) + f_2 + u_2 + u_5) \quad (28)$$

If $u_2 + u_5 = -f_2 + bp_2 + \Delta b(x_8 + x_{11}) - p_2$ and $\alpha_2(p_{1,}, p_2) = 0$ then, Lyapunov function Becomes $D^q v_2 = -p_1^2 + cp_3^2 + p_2^2 \leq 0$ is negative definite which shows that the coupled systems of the drive and the response systems are stable and stable synchronization is achieved. For simplicity of the control function we have

$$u_2 = u_5 = 0.5(-f_2 + bp_2 + \Delta b(x_8 + x_{11}) - p_2) \quad (29)$$

The coupled parameter mismatch of the FOC financial systems with the derived control functions are numerically computed via the Runge-Kutta algorithm implemented on MATLAB software. The system parameters $a = 0.3$, b =0.1, $c = 1.0$, $\Delta a = 0.2$, $\Delta b = 0.15$, $\Delta c = 1.5$, $q_1 = 0.9$, $q_2 = 0.9$, $q_3 = 0.9$ with initial conditions 1, 1, 1, 1.5, 1.5, 1.5, 2, 2, 2, 2.5, 2.5, 2.5, are employed. The error dynamics of the error variables as shown in Fig. 5 show that the error variable moves chaotically when the control functions are deactivated at $0 < t < 5$ and stabilized at zero when the control functions are activated at $t > 10$. The reduction of error functions to zero indicates global synchronization between the systems. Another evidence of global synchronization is shown in Fig. 6.

Identical dynamic behaviour was achieved between the drive and the response systems when control functions were applied $t > 10$.

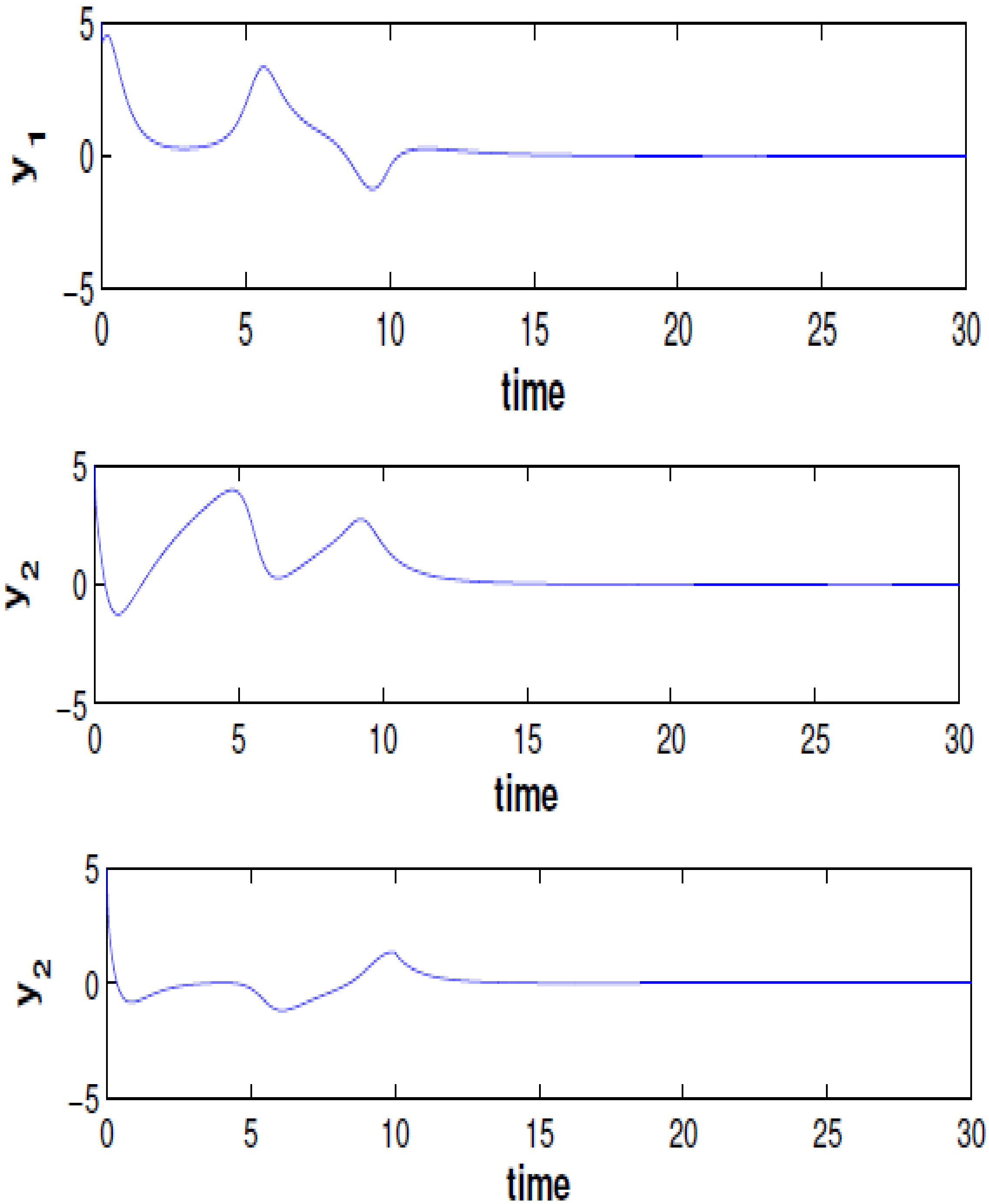

Fig. 5: The financial systems error dynamics for combination-difference synchronization of the FOC financial systems with parameter mismatch evolving from different initial condition with the control function applied for $t > 10$.

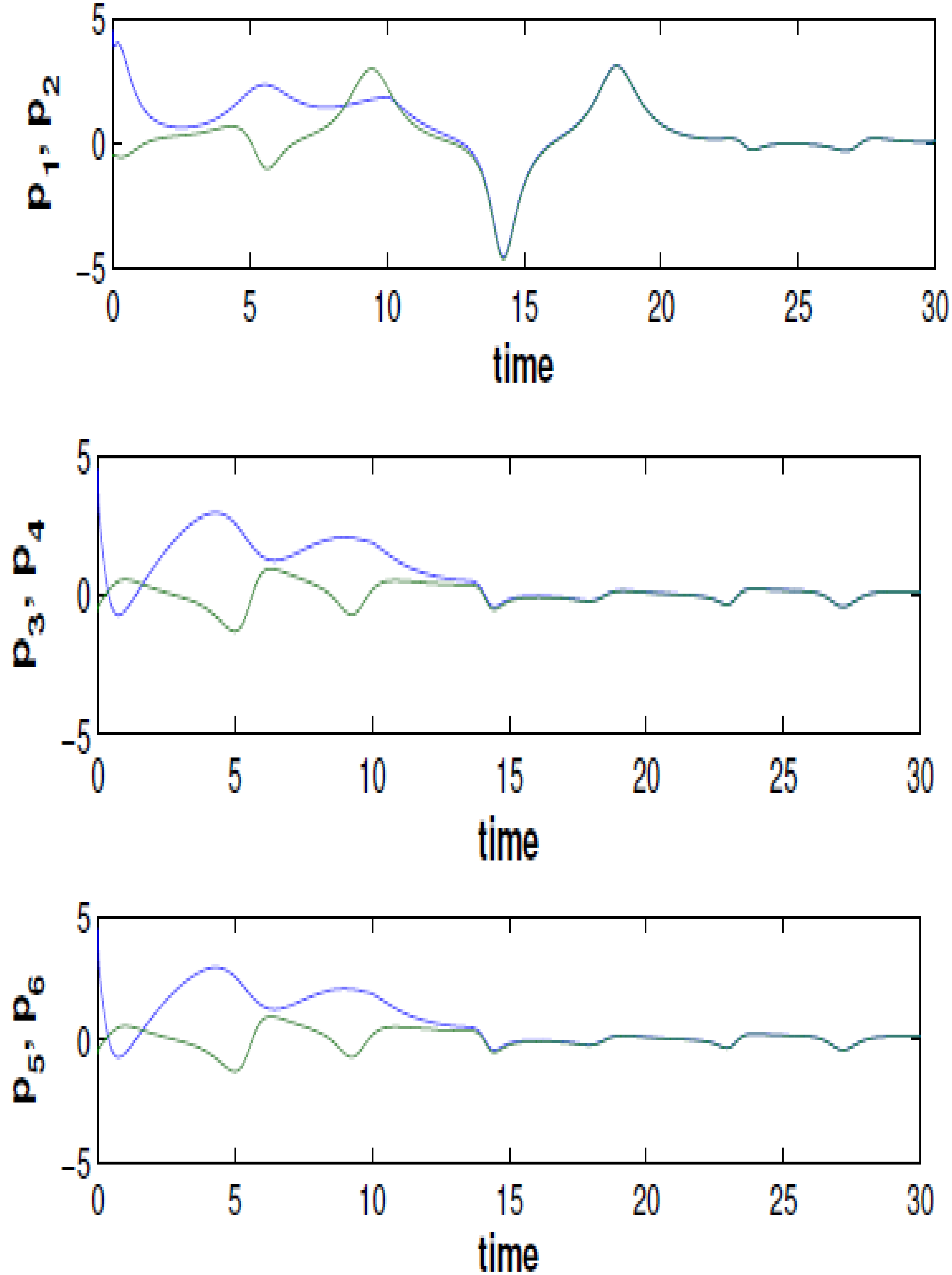

Fig. 6: Dynamics of the financial systems state variables for combination-difference synchronization of the FOC financial systems with parameter mismatch evolving from different initial condition with the control function applied for $t > 10$.

6.0 Conclusion

A new synchronization scheme called combination-difference has been established via application backstepping technique to identical FOC duffing oscillator and identical FOC financial systems with parameter mismatch evolving from different initial conditions. The results obtained show that the scheme is effective and can be used to achieve synchronization among several groups of complex dynamical systems either integer order or fractional order systems. Also, the successful implementation of this synchronization scheme has increased the number of existing synchronization schemes. This synchronization scheme is developed for fractional order systems for the first time to the best of our knowledge. hence, adding to the body of knowledge. The new synchronization scheme would have potential applications in biological systems where many organs need to synchronize or anti-synchronize to function effectively.

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Compliance with Ethical Standards Declarations

The authors declare that they have no conflict of interest.

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All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data public.

Competing interests

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Authors' Contribution

KSO conceptualised the work and contributed to the first draft. MBA, EA and ABA partook in investigative study while all authors were involved in manuscript develop and review.