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A Note On The Proofs Of Cramer's Formula

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Abstract: *The work of Gabriel Cramer (1704-1752) that yielded the formula for solving an arbitrary number of unknown in a square linear system of equations has witnessed in the recent past, several methods of proofs regardless of the supposed high computational cost. It is our purpose in this research to proffer an alternative method of proof to Cramer's formula for solving square linear system of equations.*

Keywords: Cramer's method, Determinant, system of linear equations, modify Cramer's method.

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1.0 Introduction

The Swiss mathematician Gabriel Cramer (1704-1752) published the rule for solving n by n system of linear equations which has come to bear his name (Debnath, 2013) though literature has it that the rule has been in use before his publication (Mc-Laurin, 1748; Boyer, 1966) and (Hedman , 1999) still the credit goes to him considering his contribution in giving the concept a finishing touch. The subject of the solution to the problem of solving a system of linear equations is particularly of interest in all fields of science and engineering (Ufuoma, 2013, 2019). Before now, several scholars have given various versions of proof of the Cramer's formula with the knowledge that the finding fathers; Colin Maclaurin (1698-1746) and Gabriel Cramer (1704-1752) to whom this formula is attributed did not give a formal proof of the formula rather, explained how to build the formula for more general cases. It is important to note that both Cramer and Maclaurin wrote down the solution of a system of 3 linear equations with 3 unknowns, as ratios of two quantities, which does not necessarily translate to the notion of the determinant as a closed-form function, as introduced by Alexandre-Theophile Vandermonde (1735-1796) in 1771 (Vandermonde, 1771). However, the rule given by Maclaurin to choose the appropriate sign for each summand is wrong; on the contrary, Cramer's idea to count the number of transpositions (derangements) in the permutation attached to a given term flawlessly reproduces the right one. Hence one may conclude that Cramer's Rule is genuine due to Cramer (Kosinsky, 2001). Thus, the gap we proposed to fill in this work is to provide an alternative proof to Cramer's formula for solving square linear system of equations which is of independent interest when compared with the existing proofs in the literature. In particular, amongst the authors that proved this formula in the literature, none were able to show that Cramer's formula is a mere corollary to the result they established.

1.1 Definitions and Notations

Let $R^n(C^n)$ denote the set of real (complex) n-tuple vectors (column vectors) such that

$$R^{n\times 1} = \{\bar{x} : \bar{x} = (x_1, x_2, \cdots, x_n)^T, x_j \in R \ \forall\, j = 1,2,\cdots,n\}$$

x_i is the i-th component of the vector $\bar{x}$. By convention $R^{n\times 1} = R^n$ denotes the set of column vectors while $R^{1\times n}$ denotes the set of row vectors, that is

$$R^{1\times n} = \{\bar{x} : \bar{x} = (x_1, x_2, \cdots, x_n), x_i \in R \ \forall\, i = 1,2,\cdots,n\}$$

$R^{m\times n}\,(C^{m\times n})$ denote the set of all real (complex) $m \times n$ matrices then

$$R^{m\times n} = \{A : A = (a_{ij}), a_{ij} \in R \ \forall\, i = 1,2,\cdots,m\,, j = 1,2,\cdots,n\}$$

a_{ij} is the (i,j)-th component (entry) of the matrice A, $i,j \in [n] = \{1,2,3,\cdots,n\}$.

An n square system (n equations and n unknowns) is said to be linear if for $a_{ij} \in R$; $i,j \in [n] = \{1,2,3,\cdots,n\}$, $x_j \in R \ \forall\, j = 1,2,\cdots,n$ the following equations holds

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \qquad (1)$$

Its equivalent matrix equation is given by

$$A\bar{x} = B \qquad (2)$$

Where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \vdots & \vdots & \vdots \ \ddots \ \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{pmatrix}; \ \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \ B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Definition 1. A determinant of order n, or size $n \times n$, see (Hamiti, 2002; Barnard and Child, 1959; Scott, 1904; Ferrar, 1957)) is the sum

$$D = \det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \vdots & \vdots & \vdots \ \ddots \ \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix} = \sum_{j_1 j_2 j_3 \cdots j_n \in S_n} \mu_{j_1 j_2 j_3 \cdots j_n} a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n} \quad (3)$$

Summing over the permutation set (symmetric permutation group) S_n, where

$$\mu_{j_1 j_2 j_3 \cdots j_n} = \begin{cases} +1; \ if\ j_1 j_2 j_3 \cdots j_n \ is\ an\ even\ permutation \\ -1; \ if\ j_1 j_2 j_3 \cdots j_n \ is\ an\ odd\ permutation \end{cases}$$

1.2 Properties of determinants

Let A and B be any $n \times n$ matrices.

1. If A is a triangular matrix, i.e. $a_{ij} = 0$ whenever $i > j$ or, whenever $i < j$, then $det(A) = a_{11}a_{22}a_{33} \cdots a_{nn}$.
2. If B results from A by interchanging two rows or columns, then $det(B) = -det(A)$.

3. If B results from A by multiplying one row or column with a number c, then $det(B) = c\, det(A)$.
4. If B results from A by adding a multiple of one row to another row, or a multiple of one column to another column, then $det(B) = det(A)$.

These four properties can be used to compute the determinant of any matrix, using Gaussian elimination (Eves, 1990; Bunch and Hopcroft, 1974; Gjonbala and Salihu, 2010). This is an algorithm that transforms any given matrix to a triangular matrix, only by using the operations from the last three items above. Since the effect of these operations on the determinant can be traced, the determinant of the original matrix is known, once Gaussian elimination is performed. It is also possible to expand a determinant along a row or column using Laplace's formula, which is efficient for relatively small matrices. To do this along the row i, say, we write

$$\det(A) = \begin{cases} \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij} \ \forall\, i\ (row\ wise) \\ \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}M_{ij} \ \forall\, j\ (column\ wise) \end{cases}.$$

where the C_{ij} represent the matrix cofactors, i.e., C_{ij} is $(-1)^{i+j}$ times the minor M_{ij}, which is the determinant of the matrix that results from A by removing the i-th row and the j-th column, and n is the size of the matrix.

From the definition above one easily observe using Laplace's (theorem) formula that the corollary follows

Corollary 1.1 Let k and p be positive integers in $[n]$, then we have

$$\sum_{i=1}^{n} a_{ik}C_{ip} = \sum_{j=1}^{n} a_{kj}C_{pj} = \begin{cases} \det(A) \quad if\ \ k = p \text{ (first Laplace theorem)} \\ 0 \qquad if\ \ k \neq p \text{ (second Laplace theorem)} \end{cases}.$$

2. 0 Review of Some Previous Proofs

Let

$$A_{r'|B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,r-1} & b_1 & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,r-1} & b_2 & a_{2,r+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,r-1} & b_3 & a_{3,r+1} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4,r-1} & b_4 & a_{4,r+1} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,r-1} & b_n & a_{n,r+1} & \cdots & a_{nn} \end{pmatrix}$$

be the coefficient matrix of the system whose r^{th} the column is replaced by (restricted to) the column vector B of the coefficient matrix of the system with the corresponding determinant denoted as $D_{r'|B}$. (Cramer, 1750; Ufuoma, 2013; Okoli and Nsiegbe, 2021, 2022), proposes the following theorem for solving a square system of linear equations.

Theorem 2.1 (Cramer's Theorem) If the matrix of coefficients A of a system (1.1), (4.1) is non-singular, then the unique solution $(x_1, x_2, x_3, \cdots, x_n)$ to the system is given by

$$x_r = \frac{D_{r'|B}}{|A|} \quad \forall\, r = 1,2,3,\cdots,n \qquad (4)$$

The first proof of Cramer's Rule was 1841 and appeared in a paper by Carl Gustav Jacob Jacobi (1804-1851). This is not the oldest proof ever published. In 1825, for instance, Heinrich Ferdinand Scherk (1798-1885) published a 17 page long proof by induction on the number of unknowns sketched in (Muir, 1960). Because of its poor informative content and lengthiness, researchers seem to pay less attention to it (Maurizio, 2014).

Proof 1 (Jacobi): For arbitrary but fixed $p \in [n]$, using the i-th equation of (1);

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = b_n$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij}x_j = b_i$$

$$\Rightarrow C_{ip}\sum_{j=1}^{n} a_{ij}x_j = C_{ip}b_i\,; \Rightarrow \sum_{j=1}^{n} a_{ij}\,C_{ip}\,x_j = C_{ip}b_i; \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,C_{ip}\,x_j = \sum_{i=1}^{n} C_{ip}b_i$$

By corollary 1.1, it follows that for $j = p$ we shall have

$$\sum_{i=1}^{n} a_{ip}C_{ip}\,x_p = C_{ip}b_i; \Rightarrow x_p\left(\sum_{i=1}^{n} a_{ip}C_{ip}\right) = C_{ip}b_i; \Rightarrow x_p \det(A) = D_{p'|B}$$

$$\Rightarrow x_p = \frac{D_{p'|B}}{\det(A)}$$

The next proof that followed soon after was by Nicola Trudi (1811-1884), a professor of infinitesimal calculus at the University of Naples in his work Teoria de' determinants e loro applicazioni. Using the idea of Trudi we shall give a proof which is slightly different from Trudi's method of proof as contained in (Brunetti, 2014). In fact, this seems to marry Jacobi's and Trudi's method of proof.

Proof 2 (Trudi): Now suppose we put the system in equation (1.1) as

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_p\begin{pmatrix} a_{1p} \\ a_{2p} \\ \vdots \\ a_{np} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Rightarrow \sum_{j=1}^{n}\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} x_j = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Rightarrow C_{ip}\sum_{j=1}^{n}\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} x_j = C_{ip}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}; \Rightarrow \sum_{j=1}^{n} C_{ip}\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} x_j = C_{ip}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$; \Longrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ip}\begin{pmatrix}a_{1j}\\ a_{2j}\\ \vdots\\ a_{nj}\end{pmatrix}x_j = \sum_{i=1}^{n} C_{ip}\begin{pmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{pmatrix}$$

By corollary 1, it follows that for $j = p$ we shall have

$$\sum_{i=1}^{n} C_{ip}\begin{pmatrix}a_{1p}\\ a_{2p}\\ \vdots\\ a_{np}\end{pmatrix}x_p = C_{ip}b_i; \Longrightarrow x_p\left(\sum_{i=1}^{n} a_{ip}C_{ip}\right) = C_{ip}b_i; \Longrightarrow x_p \det(A) = D_{p'|B}$$

$$\Longrightarrow x_p = \frac{D_{p'|B}}{\det(A)}$$

It was noted in (Brunetti, 2014) that Trudi's proof has been rediscovered in (Whitford and Klamkin, 1953) and included in some modern widespread textbooks (e.g. (Lang, 2004) and (Cohen et al, 2010)). It also appears on the Italian Wikipedia page devoted to Cramer's Rule (Regola, 2014). Nevertheless, most textbooks on linear algebra (we just mention the classic (Curtis, 1963), (Robinson, 2006), and the recently published Italian textbook (Lomonaco, 2013)) choose to prove Cramer's Rule via the adjoint matrix, which is in Proof 3.

Proof 3 Since A is non-singular, the matrix equation $AX = B$ is equivalent to

$$X = A^{-1}B\ ; \Longrightarrow X = \frac{adj(A)}{\det(A)}B$$

For a fixed $p \in [n]$, then the p-th component of $adj(A)B$ is $D_{p'|B}$, so that

$$x_p = \frac{D_{p'|B}}{\det(A)}$$

The next proof as contained in the paper (Robinson, 1970), is the oldest source found in print as contained in (Brunetti, 2014). Such proof has been adopted by (Horn and Johnson, 1985). As noted in (Carison et al, 1992), it gives practice in the important skill of exploiting the structure of sparse matrices, hence it should be worthy of more consideration.

Proof 4 Let I be the identity matrix. Observe that for a fixed $p \in [n]$,

$$x_p = \det(I_{p'|B})$$

Thus,

$$A(I_{p'|B}) = (A_{p'|B})$$

By the properties of determinant, it follows that

$$\det(A)\, det(I_{p'|B}) = det(A_{p'|B})$$

$$\Longrightarrow det(I_{p'|B}) = \frac{det(A_{p'|B})}{\det(A)}\ ; \Longrightarrow x_p = \frac{D_{p'|B}}{\det(A)}$$

For the remaining two proofs which is due to Richard Ehrenborg (Ehrenborg, 2004) and (Brunetti, 2014) one may wish to see (Brunetti, 2014).

3.0 Results

It is important to note that the Cramer's rule measures the ratio of two determinants (quantities), where one has a running variable (the numerator part) while the other has not (the denominator part). In a more general setting, unlike that of Cramer's, we will define a parameter that measures the ratio of two determinants such that they are both indexed by a running variable as follows. Let for $p, r \in [n]$ we define

$$v_{p,r} = \frac{D_r}{D_p} = \frac{D_{r'|B}}{D_{p'|B}} \tag{5}$$

We may simply replace $D_{r'|B}$ by D_r for any positive integer r. So by the definition of this parameter, we have not assumed the Cramer's rule since $p \neq 0$ (i.e. $p \in [n]$). To see the generic nature of this parameter, which also accommodates the result of Cramer's we write

$$v_{p,r} = \begin{cases} \dfrac{D_r}{D_0} & if \quad r \neq p; p = 0 \\ \dfrac{D_r}{D_p} & if \quad r \neq p; p, r > 0 \\ 1 & if \qquad p = r \end{cases} \tag{6}$$

So that now $p \in [n] \cup \{0\}$ and $r \in [n]$. Observe if we restrict p to the singleton set $\{0\}$ (i.e. $p = 0$), $v_{p,r}$ immediately yield the Cramer's rule, that is

$$v_{0,r} = \frac{D_r}{D_0} = x_r \tag{7}$$

But if we restrict p to the set $[n]$ (i.e. $p \in [n]$), then $v_{p,r}$ defines an entirely different quantity that is independent of the determinant of the coefficient matrix for the given system which is not the case for Cramer's rule. In the sequel, we shall give additional implications of this parameter $v_{p,r}$ as we now proceed to prove a modified version of Cramer's formula and show that the Cramer's formula is a corollary (can easily be deduced) from the modified version of our result.

Theorem 3.1 If the coefficients matrix A of the system (1) is non-singular, then the unique solution $(x_1, x_2, x_3, \cdots, x_n)$ to the system is given by

$$x_p = \frac{b_i D_p}{\sum_{j=1}^{n} a_{ij} D_j} \quad \forall\, i \in [n], p = 1,2,3, \cdots, n \tag{8}$$

Proof:

For $p, j \in [n]$, let $v_{p,j} = \frac{D_j}{D_p}$ and $x_j = v_{p,j} x_p$, using the i-th equation of (1);

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = b_n$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} v_{p,j} x_p = b_i\, ; \Rightarrow \sum_{j=1}^{n} a_{ij} \frac{D_j}{D_p} x_p = b_i\, ; \Rightarrow x_p \sum_{j=1}^{n} a_{ij} D_j = b_i D_p$$

$$\Rightarrow x_p = \frac{b_i D_p}{\sum_{j=1}^{n} a_{ij} D_j} \quad \forall\, i \in [n], p = 1,2,3, \cdots, n$$

This completes the proof of the theorem.

Lemma 3.2 let the entries of the column matrix $\left(D_{1'|B}, D_{2'|B}, \cdots, D_{n'|B}\right)^T$ represent the determinant defined above then the determinant $\det(A)$ ($|A|$ $or D_0$) of the coefficient matrix A is given by

$$D_0 = \frac{1}{b_i}\left(\sum_{j=1}^{n} a_{ij} D_{j'|B}\right) : \forall\ i \in [n] \tag{9}$$

Proof.
Recall that the system (1) above can be written as

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \; ; \quad i = 1,2,3, \dots \; , n$$

Now if $p = 0$, by definition $v_{0,r} = \frac{D_{r'|B}}{D_0} = x_r$, so we shall have that

$$b_i = \sum_{j=1}^{n} a_{ij}x_j = \frac{1}{D_0}\sum_{j=1}^{n} a_{ij}D_{j'|B} \; ; \Longrightarrow D_0 = \frac{1}{b_i}\sum_{j=1}^{n} a_{ij}D_{j'|B}$$

This completes the proof of the lemma.

Corollary 3.3 let the entries of the column matrix $\left(D_{1'|B}, D_{2'|B}, \cdots, D_{n'|B}\right)^T$ represent the determinant defined above, if $p \in \{0\}$ (i.e. $p = 0$) then theorem 3.1 reduces to theorem 1.1.
Now observe that using the result of theorem 3.1 and lemma 3.2, it follows that

$$x_r = \frac{b_i D_r}{\sum_{j=1}^{n} a_{ij}D_j} = \frac{D_r}{\frac{1}{b_i}\sum_{j=1}^{n} a_{ij}D_j} = \frac{D_r}{D_0} = \frac{D_r}{\det(A)}$$

4.0 Conclusion

So far we have proved Cramer's formula for solving square linear system of equations in a manner which is of independent interest when compared with the existing proofs of the authors mentioned above as contained in the literature. Also we showed that Cramer's formula is a mere corollary to the result in theorem 3.1.
The importance of this lemma 3.3 is that it eliminates the computation of D_0 from the matrix of coefficients A of the system (1), thus reducing the amount of time required to complete the computation and guaranteeing the fact that for $j = 1,2,3, \dots \; , n$ the collection $\{ D_{j'|B} \}$ completely determine the solution set $\{ x_j : j = 1,2,3, \dots \; , n \}$ of the linear system (1) respectively, unlike in the case of (Cramer 1750; Babarinsa and Kamarulhaili, 2017, 2019; Kitaro , 2018; Ufuoma, 2013, 2019) respectively, which requires the direct computation of D_0 from the matrix of coefficients A of the system unlike that of (Okoli and Nsiegbe, 2021, 2022).
We provided an alternative proof to Cramer's formula for solving square linear system of equations which is of independent interest when compared with the existing proofs in the literature. In particular, amongst the authors that proved this formula in the literature, none were able to show that Cramer's formula is a mere corollary to the result they established.

5.0 References

Babarinsa, O. & Kamarulhaili, H. (2017). Modification of cramer's rule. *Journal. Fundam. Appl. Sci.*,9, pp. 556-567.

Babarinsa, O. & Kamarulhaili, H. (2019). Modified Cramer's Rule and its Application to Solve Linear Systems in WZ Factorization MATEMATIKA, 35, 1, pp. 25–38.

Barnard S. & Child, J (1959). *Higher Algebra*. London Macmillan LTD, New York, ST Martin* s Press, 131.

Boyer, C. B. (1996). Colin Maclaurin and Cramers Rule. *Scripta Math*. 27, pp 377-379.

Brunetti, M. (2014). Old & new proofs of Cramer's Rule. *Applied Mathematical Sciences*, 8, 133, pp. 6689 – 6697

Bunch, J. R. & Hopcroft, J. E. (1974). Triangular factorization and inversion by fast matrix multiplication, Mathematics of Computation, 28, pp.. 231 – 236.

Cramer, G. (1750). *Introduction _a l'analyse des lignes courbes alg_ebriques*, Geneva: Freres Cramer & Cl. Philbert.

Curtis, C. W. (1963). Linear Algebra, Boston: Allyn and Bacon.

Debnath, L. (2013). A brief historical introduction to matrices and their applications. *International Journal of Mathematical Education in Science and Technology*, 45, 3, pp. 360-377.

Debnath, L (2013). A brief historical introduction to matrices and their applications. *International Journal of Mathematical Education in Science and Technology*,45, 3, pp. 360-377.

Ehrenborg, R. A. (2004). Conceptual Proof of Cramer's Rule, Math. Mag. 77 no. 4, 308. Old and new proofs of Cramer's rule 66-97.

Eves, H. (1990). *An introduction to the History of Mathematics*,– 493, Saunders College Publishing.

Ferrar W. L.: (1957). *Algebra. A Text-Book of determinants, matrices, and algebraic forms, Second edition*, Fellow and tutor of Hertford college Oxford, 7

Gjonbalaj, Q. and Salihu, A. (2010). Computing The Determinants by reducing the orders by four. *Applied Mathematics E – Notes*, 10, pp. 151 – 158.

Hamiti, E (2002) Matematika I, Universiteti I Prishtines: Fakulteti Elektroteknik, Prishtine, pp. 163 -164

Hedman, B. A. (1999). An earlier date for `Cramer's rule. *Historia Math*. 26, 4, pp. 365{368.

Horn, R. A. & Johnson, C. A. (1985). *Matrix analysis*, New York: Cambridge University Press.

Jacobi, C. G. J. (1884). *De Formatione et Proprietatibus Determinantium, in Idem, Gesammelte* Werke vol. III, Berlin: G. Reimer, pp. 355-392.

Kosinsky, A. A. (2001). Cramer's Rule is due to Cramer, *Math. Mag*. 74, 4, 310{312.

Lang, S. (2004). *Linear Algebra*, 3rd ed., New York: Springer-Verlag.

Lomonaco, L. A. (2013). *Geometria e algebra. Vettori, equazionie curve* elemen- tari, Roma: Aracne.

MacLaurin, C. (1748) A treatise of algebra. London: A. Millar and J. Nourse.

Muir, T. (1960). *The theory of determinants in the historical order of development*, vol. 1, New York: Dover Publications.

Okoli O. C. & Nsiegbe N. A. (2022) *Undetermined parameter Method; a Dodgson's condensation-base applicat-ion of cramer's rule for solving linear systems*, 3rd International Scientific Conference by FS, COOU (FAPSCON) , Vol 5 (1); 410 - 423.

Okoli, O. C. & Nsiegbe N. A., (2021). Correction to "A new and simple method of solving large linear systems based on Cramer's rule but employing Dodgson's condensation", COOU Journal of Physical Sciences (CJPS), 1, 4, pp. 219 - 225.

Robinson, D. J. S. (2006). *A Course in Linear Algebra with Applications*, New Jersey: World Scientific.

Robinson, S. M. (1970). A Short Proof of Cramer's Rule. *Math. Mag*. 43, 2, pp. 44-45.

Scott, R. F (1904). *The theory of determinants and their applications, Ithaca, New York:* Cornell University Library, Cambridge: University Press, 3 – 5.

Trudi, N. (1862). *Teoria de determinanti e loro applicazioni*, Napoli: Pellerano.

Ufuoma, O. (2013). *A new and simple method of solving large linear systems: based on cramer' s rule but employing dodgson' s condensation*. In the Proceedings of the World Congress on Engineering and Computer Science, San Francisco

Ufuoma, O. (2019). A New and Simple Condensation Method of Computing

Determinants of Large Matrices and Solving LargeLinear Systems, *Asian Research Journal of Mathematics* 15, 4, pp. 1-13; Article no. ARJOM.51185

Vandermonde, A. 772). Memoire sur elimination, Historie de l'Acade royale *des Sciences, Paris, pt*. 2, pp. 516-532.

Whitford, D. E. and Klamkin, M. S. (1953). On an elementary derivation of Cramer's rule, *Amer. Math. Monthly* 60, pp. 186-187.

Compliance with Ethical Standards Declarations

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data public.

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Authors' Contributions

Conceptualization of the title vis-à-vis identification of knowledge gap, design of the research objectives, development and interpretation of results were done by Njoku, K. N. C, while Okoli, O. C. developed both the literature and the research methodology.