Convergence of Preconditioned Gauss-Seidel Iterative Method For L –Matrices

Abdulrahman Ndanusa

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Abstract: A great many real-life situations are often modeled as linear system of equations, $Ax =$. Direct methods of solution of such systems are not always realistic, especially where the coefficient matrix A is very large and sparse, hence the recourse to iterative solution methods. The Gauss-Seidel, a basic iterative method for linear systems, is one such method. Although convergence is rarely guaranteed for all cases, it is established that the method converges for some situations depending on properties of the entries of the coefficient matrix and, by implication, on the algebraic structure of the method. However, as with all basic iterative methods, when it does converge, convergence could be slow. In this research, a preconditioned version of the Gauss-Seidel method is proposed in order to improve upon its convergence and robustness. For this purpose, convergence theorems are advanced and established. Numerical experiments are undertaken to validate results of the proved theorems.

Key Words: Gauss-Seidel iterative method, $Preconditioning, \quad L$ --matrix, Splitting, Nonnegative matrix

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1.0 Introduction

In order to employ iterative solution method for the linear system of algebraic equations $Ax = b$, where the coefficient matrix $A \in \mathbb{R}^{n,n}$ is an irreducible L – matrix, $b \in \mathbb{R}^n$, and x being the vector of unknowns, the generic linear iteration formula takes the form

 $x^{(n)} = Gx^{(n-1)} + c, \qquad n = 0,1,2,\cdots$ (1) where $G = M^{-1}N$, referred to as the iteration matrix, is a matrix depending upon A and x , and $c (= M^{-1}b)$ is a column vector. Both G_n and c_k are obtained from a regular splitting of the matrix A thus: $A = M - N$. We assume for simplicity, without loss of generality, that the coefficient matrix A has the usual triangular splitting of the form $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are the strictly lower and strictly upper triangular parts of A , respectively. By the foregoing, the Gauss-Seidel method is easily described by the relation

 $x^{(n)} = Gx^{(n-1)} + c$ $n = 0,1,2,...$ (2) where $G = (I - L)^{-1}U$ is the Gauss-Seidel iteration matrix, and $c = (I - L)^{-1}b$. The Gauss-Seidel method is known to converge for linear systems with strictly or irreducibly diagonally dominant matrices, invertible H – matrices (generalized strictly diagonally dominant matrices) and Hermitian positive definite matrices. However, as with all basic iterative methods, convergence could be slow; hence the idea of preconditioning.

Preconditioning is the application of a transformation (preconditioner) to a linear system that transforms the system into a form that is more suitable for numerical computation. When preconditioners are applied to linear systems, the associated iterative methods tend to converge asymptotically faster than the unpreconditioned ones. Preconditioning, in relation to classical iterative methods, aims to reduce the spectral radius of the iteration matrix so as to improve convergence. However, when applied to Conjugate Gradient or other Krylov subspace methods, the goal of preconditioning is to increase the condition number of the coefficient matrix A in order to improve convergence.

A diversity of preconditioned Gauss-Seidel iterative techniques has been advanced by various researchers and authors. Among these include the preconditioners of Allahviranloo et al. (2012), Gunawardena et al. (1991), Hadjidimos et al. (2003), Kohno et al. (1997), Li (2005), Li and Sun (2000), Milaszewicz (1987), Nazari and Borujeni (2012), Ndanusa and Adeboye (2012), Noutsos and Tzoumas (2006) , Zhang *et al.* (2015) and Zheng and Miao (2009). This present research

aims to investigate the applicability of the preconditioner of [9] to the classical Gauss-Seidel method in order to improve on its convergence.

2.0 Materials and Methods

2.1 Preliminaries

In order to use the successive overrelaxation (SOR) method to solve the preconditioned linear system, equation 3 is significant

$$
PAx = Pb \tag{3}
$$

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so to investigate the applicability of the preconditioned Gauss-Seidel iterative

conditioner of [9] to the classical Gauss-Seidel defined as

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aims to investigate the applicability of the preconditioned Gauss-Seidel iterative

preconditioner of [9] to the classical Gauss-Seidel defined as

method in orde where $P \in \mathbb{R}^{n \times n}$, called the preconditioner, is nonsingular. Ndanusa and Adeboye (2012) **Communication in Physical Sciences, 2020, 6(1): 803-808**

saims to investigate the applicability of the preconditioned Gauss-Scidel iterative scheme is

preconditioner of $[9]$ to the classical Gauss-Scidel defined as

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preconditioner of [9] to the classical Gauss-Seidel defined as

method in order to improve on its convergence.
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the conditioner of [9] to the classical Gauss-Scidel defined as

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or equivalently,
 $\gamma^{(n)} =$ Materials and Methods
 $\begin{array}{ll}\n\text{Matrix and Methods} & + (I - L)^{-1}b & (7) \\
\text{the linearized and Methods} & + (I - L)^{-1}b & (7) \\
\text{to use the successive overrelaxation} & \text{rowically} \\ \text{the two preconditioned linear} & \text{where the iterative matrix of the preconditioned} \\
\text{equation 3 is significant} & (3) & \text{G}_{12} = (I - T)^{-1} (I - D_1) \\
\text{equation 3 is significant} & (3) & \text{G}_{12} = (I - T)^{-1} (I - D_1) \\
\text{equation 4.3 is significant} & (3)$ ellminaries

ellminaries

consume of $x^{(0)} = G_1x^{(0)} + C = 0$

to use the successive overrelaxation $x^{(0)} = G_1x^{(0)} - 1 + C$ and the preconditioned

into the preconditioned linear
 $\begin{pmatrix}\nP_b \\
P_b\n\end{pmatrix}$ (3) $G_1 = (I - \overline{L})^{-1}(\over$ (SOR) method to solve the preconditioned linear

where the irrestive matrix of the preconditioned
 $PAx = Pb$, equality of the preconditioned $PAX = Pb$
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 $PAX = Pb$
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Ax = *Pb*

Ax = *Pb*

Ax = *Pb*

Ax = (3) and $G_1 = (I - \overline{L})^{-1}(\overline{U} - D_1)$ (9)

there *P* ∈ $R^{0.200}$, Roles (2012) Scielel iteration scheme can be defined Gauss-

Forey one diperconditioner *P* = *H* 5, $R^{m \times n}$, called the preconditioner, is Also, from (5), a second preconditioned Gauss-
 $R^{m \times n}$, called the preconditioner, is Also, from (5), a second preconditioned Gauss-
 $P = P + S$, where I is $\frac{x^{(n)} = (D - I) \cdot T J x$

$$
S = \begin{cases} -a_{i1}, & i = 2, \dots, n \\ -a_{i,i+1}, & i = 1, \dots, n-1 \\ 0, & \text{otherwise} \end{cases}
$$

The nonzero entries of S are the negatives of the Convergence Analysis to

$$
\bar{A}x = \bar{b}
$$
 (4)
From (4) we obtain

we obtain
\n
$$
\bar{A} = PA = (I + S)(I - L - U)
$$
\n
$$
-I - I - II + S - SI - SI
$$

where,

$$
S = -L_S - U_S - SL - SU = D_1 - L_1 - U_1
$$

Therefore,

$$
\bar{A} = I - L - U - L_S - U_S + D_1 - L_1 - U
$$

$$
= (I + D_1) - (L + L_S + L_1) - (U + U_S + U_S)
$$

It implies,

$$
\bar{A} = \bar{D} - \bar{L} - \bar{U} \tag{5}
$$

osed the preconditioner $P = I + S$, where I is $\chi^{(n)} = (D - I)^{-1}U\chi^{(n-1)} + (D - I)^{-1}U$
 \times X n identity matrix and S is a sparse matrix Ω or more compactly,
 $S = \begin{cases} -a_{i1}, & i = 1, ..., n-1 \\ -a_{i} & i = 1, ..., n-1 \end{cases}$ where
 $S =$ defined by
 $S = \begin{cases} -a_{i1}, & i = 2,...,n \\ -a_{i} & i = 1,...,n-1 \end{cases}$ where compactly,
 $S = \begin{cases} -a_{i1}, & i = 2,...,n \\ 0, & \text{otherwise} \end{cases}$ where $K_2 = (D - L)^{-1}U$

The nonzero entities of S are the negatives of the coefficient matrix A.

correspond $U + U_s + U_1$ constitute the diagonal, strictly $x, x \neq 0$, then $\alpha \leq \rho(A)$.
lower and strictly upper components of \overline{A} ii. If $Ax \leq \beta x$ for some positive vector x, respectively.

iteration scheme (2) is rewritten as

$$
x^{(n)} = (I - L)^{-1} U x^{(n-1)} + (I - L)^{-1} b
$$

= 0,1,2,... (6)

preconditioned coefficient matrix \overline{A} is a nonsingular M -matrix. considered and the resulting models are as Theorem 1 f_0 llow

$$
\overline{A} = (\overline{D} - \overline{L} - \overline{U}) = (I + D_1 - \overline{L} - \overline{U}) =
$$

(*I* - \overline{L}) - (\overline{U} - D₁)
Therefore,

is a regular splitting of A, where $M = (I - a_{i,i+1}a_{i+1,i} < 1, i = 2(1)n$, then G, G_1 and G_2
 \overline{L}) and $N = (\overline{U} - D_1)$. Therefore, the first are nonnegative and irreducible matrices.

preconditioned Gauss-Seidel iterative scheme is defined as

$$
x^{(n)} = (I - \bar{L})^{-1}(\bar{U} - D_1)x^{(n-1)} + (I - \bar{L})^{-1}b \tag{7}
$$

or equivalently,

(a) 503-808

(a) = (*I* − *D*)⁻¹(*U* − *D*₁) $x^{(n-1)}$
 $+(I - \overline{L})^{-1}b$ (7)

equivalently,

(a) = (*I* $+(I - \overline{L})^{-1}b$ (7)

equivalently,

(a) = (*G*₁ $x^{(n-1)} + c$ *n* = 0,1,2, ... (8)

encre the iterative matrix of t 804

Sauss-Seidel iterative scheme is
 $\begin{aligned}\n &{}^{1}(\overline{U} - D_{1})x^{(n-1)} \\
 &+ (I - \overline{L})^{-1}b \\
 &+ c & n = 0,1,2, \cdots (8) \\
 & \text{we matrix of the preconditioned} \\
 & \text{eme, } G_{1}, \text{ is represented as} \\
 &{}^{1}(\overline{U} - D_{1}) \\
 & \text{second preconditioned Gauss}.\n \end{aligned}$ $x^{(n)} = G_1 x^{(n-1)} + c$ $n = 0,1,2,...$ (8) **: 803-808** 804

econditioned Gauss-Seidel iterative scheme is

fined as

(n) = $(I - \overline{L})^{-1}(\overline{U} - D_1)x^{(n-1)}$
 $+ (I - \overline{L})^{-1}b$ (7)

equivalently,

(n) = $G_1x^{(n-1)} + c$ n = 0,1,2,... (8)

here the iterative matrix of the 804

ed Gauss-Seidel iterative scheme is
 $-L^{-1}(\overline{U} - D_1)x^{(n-1)}$
 $+(I - \overline{L})^{-1}b$ (7)

tly,
 $(n-1) + c$ $n = 0,1,2,...$ (8)

erative matrix of the preconditioned

el scheme, G_1 , is represented as
 $-L)^{-1}(\overline{U} - D_1)$ (9)

(5 where the iterative matrix of the preconditioned Gauss-Seidel scheme, G_1 , is represented as 803-808 804

onditioned Gauss-Seidel iterative scheme is

led as
 $\begin{aligned}\n &= (I - \bar{L})^{-1}(\bar{U} - D_1)x^{(n-1)} \\
 &+ (I - \bar{L})^{-1}b \\
 &= G_1x^{(n-1)} + c \qquad n = 0,1,2,\cdots \quad \text{(8)} \\
 &= c_1x^{(n-1)} + c \qquad n = 0,1,2,\cdots \quad \text{(9)} \\
 &= c_1x^{(n-1)} + c \qquad n = 0$ (continuous delays Seidel iterative scheme is

fined as
 $c^{(n)} = (I - \overline{L})^{-1}(\overline{U} - D_1)x^{(n-1)}$
 $+ (I - \overline{L})^{-1}b$ (7)
 $+ (I - \overline{L})^{-1}b$ (8)
 $+ (I - \overline{L})^{-1}$ 804

S-Seidel iterative scheme is
 $-D_1)x^{(n-1)}$
 $-\overline{L}$)⁻¹*b* (7)
 $n = 0,1,2,...$ (8)

aatrix of the preconditioned
 G_1 , is represented as
 $\overline{U} - D_1$ (9)

cond preconditioned Gauss-

e can be defined as
 $(n^{-1}) + (\overline{$: 803-808 804

econditioned Gauss-Seidel iterative scheme is

fined as
 $(n) = (I - \overline{L})^{-1}(\overline{U} - D_1)x^{(n-1)}$
 $+(I - \overline{L})^{-1}b$ (7)

equivalently,
 $(n) = G_1x^{(n-1)} + c$ $n = 0,1,2,...$ (8)

experiments of the preconditioned

tuss-Se (c) the diagonal strainer is \overline{L})⁻¹(\overline{U} - D_1) $x^{(n-1)}$
 $+(I-\overline{L})^{-1}b$ (7)
 \overline{u} + $(I-\overline{L})^{-1}b$ (7)

(h),
 $(n-1) + c$ $n = 0,1,2,...$ (8)

c) strainer of the preconditioned

strainer G_1 , is represented a 1): 803-808 804

preconditioned Gauss-Seidel iterative scheme is

defined as
 $x^{(n)} = (I - \overline{L})^{-1}(\overline{U} - D_1)x^{(n-1)}$

or equivalently,
 $x^{(n)} = G_1x^{(n-1)} + c$ $n = 0,1,2,...$ (8)

where the iterative matrix of the preconditioned defined as
 $x^{(n)} = (I - \overline{L})^{-1}(\overline{U} - D_1)x^{(n-1)}$
 $+ (I - \overline{L})^{-1}b$ (7)

or equivalently,
 $x^{(n)} = G_1x^{(n-1)} + c$ $n = 0,1,2,...$ (8)

where the iterative matrix of the preconditioned

Gauss-Seidel sheme, G_1 , is represented

$$
G_1 = (I - \bar{L})^{-1}(\bar{U} - D_1) \tag{9}
$$

Also, from (5), a second preconditioned Gauss-Seidel iteration scheme can be defined as

$$
x^{(n)} = (\overline{D} - \overline{L})^{-1} \overline{U} x^{(n-1)} + (\overline{D} - \overline{L})^{-1} b \quad (10)
$$

Or more compactly,

$$
x^{(n)} = G_2 x^{(n-1)} + c \qquad n = 0, 1, 2, \cdots \quad (11)
$$

where

$$
G_2 = (\overline{D} - \overline{L})^{-1}\overline{U}
$$
 (12)

is the Gauss-Seidel iteration matrix.

e preconditioner, $\frac{1}{16} - (l - L)$ ($0 - L_1$) = ($0 - L_1$) = (0) = $\frac{1}{2}$ (0) Sected iteration scheme can be defined as
 $d \text{ Adebye}$ (2012) Seidel iteration scheme can be defined as
 $l = 2, ..., n$ $x^{(n)} = (D - \overline{L})^{-1}Bx^{(n$ The following lemmas and theorems are advanced in order to establish convergence of the derived preconditioned iterative processes. there the iterative matrix of the preconditioned

diauss-Scidel scheme, G_1 , is represented as
 $G_1 = (I - \overline{L})^{-1}(\overline{U} - D_1)$ (9)
 G_2 , from (5), a second preconditioned Gauss-

eidel iteration scheme can be defined a

irreducible matrix. Then,

- $i.$ A has a positive real eigenvalue equal to its spectral radius.
- ii. For $\rho(A)$ there corresponds an
- increases.
-) iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Varga (1981)) Let A be a nonnegative matrix. Then

-
- the $n \times n$ identity matrix and S is a sparse matrix

defined by

defined by

defined by

defined by

defined by

defined by

defined by
 $S = \begin{pmatrix} -a_{i1}, & i = 2, \dots, n \\ -a_{i1+1}, & i = 2, \dots, n \\ 0, & \text{otherwise} \end{pmatrix}$

The nonzero entries o respectively.
The classical (unpreconditioned) Gauss-Seidel $\rho(A) \le \beta$. Moreover, if A is
irreducible and if $0 \ne \alpha x \le Ax \le \beta x$ for responding entries of the coefficient matrix A. Convergence and theorems are advanced
 $PA = \bar{A}$ and $Pb = \bar{b}$, system (3) is simplified in order to establish convergence of the derived
 $\bar{A} = PA = (I + S)(I - L - U)$ irreduction *Pb* = \overline{b} , system (3) is simplified in order to establish convergence of the derived

in order to establish convergence of the derived
 \overline{a} **i. A** has a positive processes.
 $L - U + S - SL - SU$
 $L - U + S - SL - SU$
 $L - U + S - SL$ incered to the different order of $C_2 = (\overline{D} - \overline{L})^{-1} \overline{U}$

(n) = $G_2 x^{(n-1)} + c$ n = 0,1,2, ... (11)

(regional control of the Gauss-Seidel iteration matrix.
 i. $G_2 = (\overline{D} - \overline{L})^{-1} \overline{U}$ (12)

declive in glamm then contributed in the different interaction and the dialytic (0, i) = $G_2 x^{(n-1)} + c$ n = 0,1,2, ... (11)
there the Gauss-Seidel iteration matrix.
onvergence Analysis
only the Gauss-Seidel iteration matrix.
onvergence incess the control of the same of the political states. Scidel iteration matrix.
 i (Le) contablish convergence of the derived to to establish convergence of the derived trioned iterative processes.

1 (Varga (1981)) sometric included in the space of the dentation in the space Analysis
some considerative processes.
1 (Varga (1981)) Let $A \ge 0$ be an
ble matrix. Then,
 A has a positive real eigenvalue equal to
its spectral radius.
For given Transiss

solution (and the set of the derived to establish convergence of the derived

tioned iterative processes.

1 (Varga (1981)) Let $A \ge 0$ be an

be matrix. Then,

A has a positive real eigenvalue equal to

i In order to establish convergence of the derived

in order to establish convergence of the derived

preconditioned iterative processes.

Lemma 1 (Varga (1981)) Let $A \ge 0$ be an

in complement in (201) there corresponds in exact to stationaristic of the splitting of $A \ge 0$ be an irreducible matrix. Then,

i. A has a positive real eigenvalue equal to

its spectral radius.

ii. For $\rho(A)$ there corresponds an eigenvactor $x > 0$.

iii. F

 $= 0,1,2,...$ (6) Lemma 3 (Li and Sun (2000)) Let $A = M - N$
To construct a preconditioned version of the began M -splitting of A . Then the splitting is iteration (6), consider a regular splitting of the convergent, i.e., $\rho(M^{-1}N < 1)$, if and only if A is

where,
 $\sum_{i=1}^{n} E_{-1} = 0.5U = D_1 - L_1 - U_1$
 $\sum_{i=1}^{n} E_{-1} = U_2 - U_3 + D_1 - L_1 - U_1$
 $\sum_{i=1}^{n} (A)$ increases when any entry of A
 $\sum_{i=1}^{n} E_{-1} = U_1 - U_2 - U_3 + D_1 - L_1 - U_1$
 $\sum_{i=1}^{n} (A)$ increases when any entry of A
 $S = -L_S - U_S - SL - SU = D_1 - L_1 - U_1$

Therefore,

Therefore,

The interpretent $A = I - L - U - L_S - U_S + D_1 - L_1 - U_1$
 $A = I - L - U - L_S - U_S + D_1 - L_1 - U_1$
 $A = I - L - U - L_S - U_S + D_1 - L_1 - U_1$

It implies,

It implies,
 $A = D - L - \overline{U}$
 $A = D - L - \overline{U}$
 $B = \over$ $I = L - U - L_S - U_S + D_1 - L_1 - U_I$
 $= D - L - U$
 $= D - L - U$
 $= D - L - U$
 $= (L - L)^2 + (L_S + L_1)$ interases.
 $= D - L - U$
 $= 0$
 $=$ $\overline{A} = M - N = (I - \overline{L}) - (\overline{U} - D_1)$ irreducible L - matrix with $0 \le a_{1i}a_{i1}$ + $= (I + D_1) - (L + L_8 + L_1) - (U + U_3 + U_1)$

iv. $p(A)$ is a simple eigenvalue of *A*.

if it mplies,
 $A = D = \overline{L} - \overline{U}$ (5) **clearing (Varig (1981)** Let *A* be a nonnegative

where $D = I + D_1$, $\overline{L} = L + L_8 + L_1$ and $U =$ i. If It implies,
 $\lim_{A \to D} E = \overline{L} - \overline{U}$
 $\overline{L} = \overline{L} + \overline{L}_5 + L_1$ and $\overline{L} = \overline{L}$
 $\overline{L} = \overline{L} + \overline{L}_5 + L_1$ and $\overline{L} = \overline{L}$
 $\overline{L} = \overline{L} + \overline{L}_5 + L_1$ and $\overline{L} = \overline{L}$
 $\overline{L} = \overline{L} + \overline{L}_5 + L_1$ and $\$ **Example 10**
 Example 10 (Varga (1981)) Let $A \ge 0$ be an irreducible matrix. Then,

i. A has a positive real eigenvalue equal to

its spectral radius.

ii. For $\rho(A)$ there corresponds an eigenvector $x > 0$.

iii. $\rho(A$ Eventralistic metric is spectral radius.

i. A has a positive real eigenvalue equal to

iis spectral radius.

For $\rho(A)$ there corresponds an

eigenvector x > 0.

iii. $\rho(A)$ increases when any entry of A

iv. $\rho(A)$ is i. A has a positive real eigenvalue equal to

is spectral radius.

ii. For $\rho(A)$ there corresponds an

eigenvector $x > 0$.

iii. $\rho(A)$ increases when any entry of A

iv. $\rho(A)$ increases.

iv. $\rho(A)$ increases when any is spectral radius.

in For $\rho(A)$ there corresponds an eigenvector $x > 0$.

iii. $\rho(A)$ increases when any entry of A increases.

iv. $\rho(A)$ is a simple eigenvalue of A.
 Lemma 2 (Varga (1981)) Let A be a nonnegative
 ectral radius.
 $\rho(A)$ there corresponds an wector $x > 0$.

increases when any entry of *A*

increases when any entry of *A*

asses.

is a simple eigenvalue of *A*.
 $\leq Ax$ for some nonnegative vector x ,
 $\rho(A) \leq \beta$. Gauss-Seidel, the first preconditioned Gauss-Seidel and the second preconditioned Gauss-Seidel iteration matrices respectively. If A is an increases.

iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Varga (1981)) Let A be a nonnegative

matrix. Then

i. If $ax \le Ax$ for some nonnegative vector
 $x, x \ne 0$, then $\alpha \le \rho(A)$.

ii. If $Ax \le Bx$ for some positive vec iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Varga (1981)) Let A be a nonnegative

matrix. Then

i. If $\alpha x \le Ax$ for some nonnegative vector
 $x, x \ne 0$, then $\alpha \le \rho(A)$.

ii. If $Ax \le \beta x$ for some positive vector x ,
 are nonnegative and irreducible matrices.

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Proof For *A* being an *L* –matrix, it implies that $U = \lambda(I - L)$ (13)
 $L \ge 0$ and $U \ge 0$. Then $(I - L)^{-1} = I + L +$ And for this $x > 0$,
 $L^2 + \cdots + L^{n-1} \ge 0$. Thu **Communication in Physical Sciences, 2020, 6(1): 803-808** 805
 Proof For *A* being an *L* -matrix, it implies that $U = \lambda(I - L)$ (13)
 $L \ge 0$ and $U \ge 0$. Then $(I - L)^{-1} = I + L +$ And for this $x > 0$,
 $L^2 + \cdots + L^{n-1} \ge 0$. T It can also be shown that

irreducible for irreducible A . Hence, G is an irreducible matrix.

The first preconditioned iteration matrix G_1 is

examined as follows.
 $G_1 = (I - \overline{L})^{-1}(\overline{U} - D_1)$

$$
G_1 = (I + L + L^2 + \dots + L^{n-1})(\overline{U} - D_1)
$$

= $(\overline{U} - D_1) + L(\overline{U} - D_1) + L^2(\overline{U} - D_1) + \dots + L^{n-1}(\overline{U} - D_1) \ge 0$

So G_1 is a nonnegative. We can also get that $(\bar{U} - D_1) + L(\bar{U} - D_1) + L^2(\bar{U} - D_1) + \cdots$ $+L^{n-1}(\overline{U}-D_1)$ is irreducible since A is irreducible, hence G_1 is irreducible. Similarly, we consider

$$
G_2 = (\overline{D} - \overline{L})^{-1}\overline{U}
$$

= $[\overline{D}(I - \overline{D}^{-1}\overline{L})]^{-1}\overline{U}$
= $(I - \overline{D}^{-1}\overline{L})^{-1}\overline{D}^{-1}\overline{U}$
= $[I + \overline{D}^{-1}\overline{L} + (\overline{D}^{-1}\overline{L})^2 + \cdots + (\overline{D}^{-1}\overline{L})^{n-1}]\overline{D}^{-1}\overline{U}$
= $\overline{D}^{-1}\overline{U} + (\overline{D}^{-1})^2\overline{L}\overline{U} + (\overline{D}^{-1})^3\overline{L}^2\overline{U}$
+ nonnegative terms

Using similar arguments it is conclusive that $G_2 = (\overline{D} - \overline{L})^{-1}\overline{U}$ is a nonnegative and irreducible matrix.

preconditioned Gauss-Seidel iteration matrices

(i)
$$
\rho(G_1) < \rho(G)
$$
, if $\rho(G) < 1$;
\n(ii) $\rho(G_1) = \rho(G)$, if $\rho(G) = 1$;

(iii)
$$
\rho(G_1) > \rho(G)
$$
 if $\rho(G) > 1$

Proof Theorem 1 established *G* and G_1 as L_s) + SU } ≥ 0 . nonnegative and irreducible matrices. Suppose \overline{L} ⁻¹{ λ ($L_1 + L_5$) + SU} $x \ge 0$, since $x > 0$. $(x_1, x_2, \cdots, x_n)^T$, such that

$$
Gx=\lambda x
$$

That is,

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\n**Proof** For A being an L-matrix, it implies that
$$
U = \lambda (L - L)
$$

\n $L \ge 0$ and $U \ge 0$. Then $(I - L)^{-1} = I + L +$ And for this $x > 0$,
\nHence, G is a nonnegative matrix.
\nHence, G is a nonnegative matrix.
\n $G = [I + L + L^2 + \cdots + L^m] \cup U$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + LU + L^2U + \cdots$
\n $= U + L^2U + L^2U + \cdots$
\n $= U + L^2U + L^2U + \cdots$
\n $= U + L^2U + L^2U + \cdots$
\nH can also be shown that $U + L^2U$ is
\n $G_1 = (I - \bar{L})^{-1} \{I - \bar{L} + \bar{L} + U + \bar{L} + \bar{$**

 $\{SU\} \geq 0$, since $\lambda(L_1 + L_S) \geq 0$, and $SU \geq 0$. Also, $(I - \overline{L})^{-1} = I + \overline{L} + \overline{L}^2 + \cdots + \overline{L}^{n-1} \ge 0$, $\rho(G_1) > \rho(G)$, if $\rho(G) > 1$. since $\overline{L} \geq 0$. Therefore, $J = (I - \overline{L})^{-1} \{ \lambda (L_1 +$ $) +$ Consequently, $H = (I -$

 $)$ < \leq

- 2, we have $\rho(G_1) = \lambda = \rho(G)$. From equation (13), $(I L) = U/\lambda$
If $\lambda > 1$, then $G_1x \lambda x \ge 0$ but not $= (\lambda 1)(\overline{D} \overline{L})^{-1}\{(-D_1 + \lambda) | \overline{D}\}$
- **nmunication in Physical Sciences, 2020, 6(1): 803-808** 806

(ii) If $\lambda = 1$, then $G_1 x \lambda x = 0$. $= (\overline{D} \overline{L})^{-1} \{(\lambda 1)(-D_1 + L_1 + L_S) + (\lambda 1)(D_2 + L_1)\}$

2, we have $\rho(G_1) = \lambda = \rho(G)$. From equation (13), $(I L) = U/\lambda$

(iii **nmunication in Physical Sciences, 2020, 6(1): 803-808** 806

(ii) If $\lambda = 1$, then $G_1x - \lambda x = 0$. $= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)(-D_1 + L_1 + L_S)$

2, we have $\rho(G_1) = \lambda = \rho(G$

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(ii) If $\lambda = 1$, then $G_1x - \lambda x = 0$. $= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-\overline{D}_1 + \overline{R})^{-1}\}$

2, we have $\rho(G_1) = \lambda = \rho(G)$. From equation (13), $(I - L) = 1$

(iii) If $\lambda > 1$, t preconditioned Gauss-Seidel iteration matrices $\lambda L_s + SU \ge 0$, since $SU \ge 0$, $-\lambda D_1 \ge 0$, $\lambda L_1 \ge$

-
- (ii) $\rho(G_2) = \rho(G)$, if $\rho(G) = 1$;
- (iii) $\rho(G_2) > \rho(G)$, if $\rho(G) > 1$.

nonnegative and irreducible matrices. Suppose $(x_1, x_2, \dots, x_n)^T$, such that (13) holds.
 \overline{L} is an M -splitting and $\rho(\overline{D}^{-1}\overline{L}) < 1$, we $=(\overline{D}-\overline{L})^{-1}\{(U+U_{S}+U_{1})-\lambda(I+D_{1})$ equal to 0. Therefore, $G_{2}x \leq \lambda x$. $+\lambda(L+L_s+L_1)x$
 $=(\overline{D}-\overline{L})^{-1}\{-\lambda D_1+\lambda L_1+\lambda L_s+\lambda L+U_s$

From Lemma 2, we have $\rho(G_2) < \lambda = \rho(G_{SOR})$. $\rho(G_0) > \rho(G_1)$. $f(G_1) > 1$. implication its eigenvalues lie on its main

from Theorem 1, G and G₂ are diagonal; this ease the give all zeros. Therefore,

alterne exists a positive vector $x = \arccos{(\log x + \log x)}$. The form of d irreducible matrices. Suppose $\rho(\overline{D}^{-1}\overline{L}) = 0$. Since $\rho(\overline{D}^{-1}\overline{L}) < 1$, $K = \overline{D}$

there exists a positive vector $x =$ a convergent splitting By the froregoing, $K = \overline{L}$, such that (13) holds.

is $x > 0$, s $G_1 = \lambda$, then there exists a positive vector $x = \frac{1}{2}$ is a convergent splitting. By the foregoing, $K = \overline{D} - \frac{1}{2}$, λ , we are
from c, for this $x > 0$, L is an $M = \text{split}$ Summa 3 to establish that K is an x , such that (13) holds.
 \vec{L} is an M -splitting and $\rho(\vec{D}^{-1}\vec{L}) < 1$, we
 $- L$)- $T(x - \vec{D} - L)$
 $- L$)- $T(x - \vec{D} - L)$
 $- L$)- $T(x - \vec{D} - L)$
 $- L$)- $T(x - \vec{D} - L)$
 $- L$)- $T(x - \vec{D} - L)$
 $- L$)- $T(x - \vec{D} - L)$
 $- L$)- $T(x - \vec{$ erefore, for this $x > 0$,
 $x - \lambda x = (D - L)^{-1}(D - L) + (D - L)$
 $= (D - L)^{-1}(D - L) + (D - L)$
 $= (D - L)^{-1}(D - L) + (D - L)$
 $= (D - L)^{-1}(D - L) + (D - L)$
 $= (D - L)^{-1}(D + \lambda(D - L))$
 $= (D - L)^{-1} + (D + (D + L) + \lambda + L) + L$
 $= (D - L)^{-1} + (D - L) + (D - L)$
 $= (D - L)^{-1} + (D + L) + L + L$ $xx - \lambda x = (D - D^{-1} \{D - D + (D - D + \lambda)x\}$ $= (D - D)^{-1} \{D - \lambda + D + \lambda x\}$ $= (D - D)^{-1} \{D - \lambda + D + \lambda x\}$ $= (D - D)^{-1} \{D - \lambda + D + \lambda x\}$ $= (D - D)^{-1} \{D - \lambda + D + \lambda x\}$ $= (D - D)^{-1} \{D - \lambda + D + \lambda x\}$ $= (D - D)^{-1} \{D - \lambda + D + \lambda x\}$ $= (D - D)^{-1} \{D - \lambda + D + \lambda x\}$ $= (D - D$ = $(\overline{D} - \overline{L})^{-1} \{(\overline{D} - \overline{L}) + \overline{L} + \overline{L} + \overline{L}\}$ and R ≥ 0.

= $(\overline{D} - \overline{L})^{-1} \{(\overline{D} + \overline{L} + \overline{L}\} + \overline{L}\}$ (i) If $\lambda < 1$, then $G_2 x - \lambda x \le 0$ but not
 $-(\overline{D} - \overline{L})^{-1} \{(\overline{L} + \overline{L} + \overline{L}\} + \overline{L}\}$ F

$$
G_2x - \lambda x = (D - L)^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + U_S + U_{S}
$$

+ U_1 - \lambda L) \}x
= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1) + (\lambda - 1)L_1 - (D_1 - L_1 - U_1) + \lambda L_S - L_S\}

$$
= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1) - (- (SL + SU)) + (\lambda - 1)L_S + (L_S + U_S)\}x
$$

$$
= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-\overline{D}_1 + L_1 + L_S) + SL + SU + (L_S + U_S)\}x
$$

$$
= (\overline{D} - \overline{L})^{-1} \{ (\lambda - 1)(-D_1 + L_1 + L_S) + SL \} - S + SL \} x
$$

$$
= (D - L)^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + SL - S(I - L)\}x
$$

$$
= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + S[U - (I - L)]\}x
$$

$$
G_2x - \lambda x = (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_5) + S[\lambda(I - L) - (I - L)]\}x
$$

$$
= (\overline{D} - \overline{L})^{-1} \{ (\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L) \} x
$$

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\n= 0.
$$
= (\overline{D} - \overline{L})^{-1} \{ (\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L) \} x
$$
\nFrom equation (13), $(I - L) = U/\lambda$
\nnot $= (\lambda - 1)(\overline{D} - \overline{L})^{-1} \{ (-D_1 + L_1 + L_S) \}$
\n λx . $+ SU/\lambda \} x$
\n $+ SU/\lambda \} x$
\n $= [(\lambda - 1)/\lambda] (\overline{D} - \overline{L})^{-1} \{ -\lambda D_1 + \lambda L_1 + \lambda L_S + SU \} x$
\n $+ SU \} x$
\n2 = Suppose $R = Qx$, with $Q = (\overline{D} - \overline{L})^{-1} \{ -\lambda D_1 + \lambda L_1 + \lambda L_2 + SU \}.$ Obviously, $-\lambda D_1 + \lambda L_1 + \lambda L_S + SU > 0$, since $SU > 0$. $-\lambda D_2 > 0$. $\lambda L_1 > 0$

exation in Physical Sciences, 2020, 6(1): 803-808

If $\lambda = 1$, then $G_1x - \lambda x = 0$. $= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-\overline{D}_1 + L_1 + L_S) + (\lambda - 1)(\overline{D}_1 + L_2 + L_S)\}$
 λ , we have $\rho(G_1) = \lambda = \rho(G)$. From equation (13) , $(I - L) = U/\lambda$
 $\lambda^2 +$ **Communication in Physical Sciences, 2020, 6(1): 803-808** 806

(ii) If $\lambda = 1$, then $G_1x - \lambda x = 0$. $= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)(\overline{L} - L_1)\}$

Therefore, $G_1x = \lambda x$. From Lemma

(iii) If $\lambda > 1$, then $G_$ **Communication in Physical Sciences, 2020, 6(1): 803-808** 806

(ii) If $\lambda = 1$, then $G_1x = \lambda x$. From Lemma -1 $S(I - 1)(-D_1 + L_1 + L_S) + (\lambda$

Therefore, $G_1x = \lambda x$. From Lemma -1 $S(I - L)$) x

(ii) $I'_1 \lambda > 1$, then $G_1x - \lambda x$ **Communication in Physical Sciences, 2020, 6(1): 803-808**

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(ii) If $\lambda = 1$, then $G_1x - \lambda x = 0$. $= (\overline{D} - \overline{L})^{-1}[(\lambda - 1)(-D_1 + L_1 + L_3) + (\lambda - 1)(-D_1 + L_1 + L_2) + (\lambda - 1)(-D_1 + L_1 + L_3) + (\lambda - 1)(-D_1 + L_1 + L_2 + L_3 + L_3 + L_4 + L_5 + L_5 + L_$ (i) $\rho(G_2) < \rho(G)$, if $\rho(G) < 1$; Thus, $K = \overline{D} - \overline{L}$ is an M -splitting. Now, $\overline{D}^{-1}\overline{L}$ **n** in Physical Sciences, 2020, 6(1): 803-808
 $\lambda = 1$, then $G_1x = \lambda x = 0$. $=(\overline{D}-\overline{L})^{-1}\{(\lambda - 1)(-\overline{D}_1 + L_1 + L_5) + (\lambda - 1)(-\overline{D}_2 + L_6) + (\lambda - 1)(-\overline{D}_3 + L_7)$

thave $\rho(G_1) = \lambda = \rho(G)$. From equation (13), $(I - L) = U/A$
 $\lambda = \lambda$, **n** in Physical Sciences, 2020, 6(1): 803-808
 $\lambda = 1$, then $G_1x - \lambda x = 0$. $=(\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)(-D_2 + L_1 + L_S) + (\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)(-D_1 + L_1 + L_S)$
 $\lambda = 1$, then $\varphi(\Omega_1) = \lambda = \varrho(\Omega)$. From equatio **n** in Physical Sciences, 2020, 6(1): 803-808
 $\lambda = 1$, then $G_1x - \lambda x = 0$. $= (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)(-D_2 + L_1 + L_S) + (\lambda - 1)(-D_1 + L_1 + L_S)$

then $G_1x - \lambda x \ge 0$ but not $= (\lambda - 1)(\overline{D} - \overline{L})^{-1}\{ (-D_1 + L_1 + L_S)$ **Proof** From Theorem 1, G and G_2 are diagonal; in this case they are all zeros. Therefore, nonnegative and irreducible matrices. Suppose $\rho(\overline{D}^{-1}\overline{L}) = 0$. Since $\rho(\overline{D}^{-1}\overline{L}) < 1$, $K = \overline{D} - \overline{L}$ is Columentation in Fig. statistics, $2x\Delta x$, $y(x)$, $6x\Delta x = 0$, $y(x) = 1$ ($\Delta x = 1$, $\Delta x = 0$, $y(x) = 1$, $\Delta x = 1$, (ii) If $\lambda = 1$, then $G_1x - \lambda x = 0$. $= (\overline{D} - \overline{L})^{-1} \{(\lambda - 1)(-D_1 + L_1 + \lambda) \}$
 $= \text{Therefore, } G_1x = \lambda x$. From Lemma $-1)S(I - L) \}\times$

(iii) If $\lambda > 1$, then $G_1x - \lambda x \ge 0$ but not $= (\lambda - 1)(\overline{I} - \overline{L})^{-1} \{(-D_1 + L_1 + \lambda) \}$
 $= \text{equal$ (ii) If $\lambda = 1$, then $\Gamma_{\text{A}} = 0$, $\lambda =$ ve have $ρ(G_1) > = [(λ - 1)/λ](\overline{D} - \overline{L})^{-1}{{-λD_1 + λL_1 + λL_5}}$

(c)⁻¹*U* and G_2 = Suppose $R = Qx$, with $Q = (\overline{D} - \overline{L})^{-1}{{-λD_1 + λL_1 + λL_5 + 5U}}$. Seidel and the $λL_1 + λL_5 + SU \ge 0$, since $\Sigma U \ge 0$, $-λD_1 + λL_1 + Ω_2 + SU \ge 0$ $\rho(G)$

Let $G = (I - L)^{-1}U$ and $G_2 = \text{Suppose } R = \rho x$, with $Q = (D - L)^{-1}(-\lambda D_1 + \lambda L_1 + \lambda L_2 + \lambda L_1)$

causs-Scidel iteration matrices $\lambda L_2 + \lambda L_3 + \lambda L_2$

(anaroction let $\lambda L_1 + \lambda L_2 + \lambda L_3 = 0$, $\lambda L_1 \ge 0$, $\lambda L_1 \ge 0$, $\lambda L_1 \ge$ **EOPERENT and \overline{G} = \begin{cases} \sin(-L)^{-1}U & \text{and } G_2 = \text{Supp}(R) = \overline{D} - \overline** the Gauss-Scidel and the $\lambda L_1 + \lambda L_5 + S\overline{U}$. Obviously, $-\lambda D_1 + \lambda L_1 + \lambda E_2$

sus smirreducible *L* mentrix $\lambda L_5 + S\overline{U}$ 2 0, since $SU \geq 0$, show $\overline{D} = \overline{D}$ and $\lambda L_1 \geq 2$

san irreducible *L* mentrix with 0 preconditioned Gauss-Scidel iteration matrices $\lambda L_x + SU \geq 0$, since $SU \geq 0$, $-\lambda D_1 \geq 0$, $\lambda L_1 \geq 0$, $\lambda L_2 \geq 0$, $SU(4, 4R_{i,i+1}a_{i+1,i} < 1, i = 2(1)n$.

Then,
 $0 \leq a_{1i}a_{1i} + a_{i,i+1}a_{i+1,i} < 1, i = 2(1)n$.

We let $\$ is a in irreducible *L* —matrix with 0 and $L_x \ge 0$. Since *D* is a nonsingular matrix, i.e,
 $0 \le \alpha_1 \alpha_1 + a_{\ell_1+1} a_{\ell_1+1}$ < 1, $i = 2(1)$ we let $\overline{D} - \overline{L}$. Also, *D* is an *M* —matrix and *L* ≥ (i) $\rho(G_2) = \rho(G$ $a_1a_1 + a_{1i+1}a_{i+1i}$ < 1), $i = 2(1)n$, we let $\overline{D} - \overline{L}$. Note of $\overline{D} - \overline{L}$, then $M - \text{matrix } M = 0$

(i) $\rho(G_2) < \rho(G)$, if $\rho(G) < 1$; $K = \overline{D} - \overline{L}$ is a nH \rightarrow phitting; Now, $\overline{D}^{-1}L$

(ii) $\rho(G_2)$ G) < 1;

Thus, $K = \overline{D} - \overline{L}$ is an M -splitting. Now, $\overline{D}^{-1}\overline{L}$

G) = 1; is a strictly lower triangular matrix, and by

G) > 1. implication its eigenvalues lie on its main

G and G_2 are diagonal; in this c $\begin{aligned}\n\text{F}_{22} &= \rho(G), \text{ if } \rho(G) = 1; \\
\text{F}_{23} &= \rho(G), \text{ if } \rho(G) > 1. \\
\text{m Theorem 1, } G \text{ and } G_2 \text{ are diagonal; in this case they are all zeros. There are non-regularity, and the first term is given by the frequency, and the first term is given by:\n
$$
\begin{aligned}\n\text{F}_{31} &= \rho(G), \text{ if } G \text{ is odd, } G \text{ is even, } \rho(G^{-1}L) < 1, K = \overline{D} - \overline{D} - \overline{D} - \overline{D} - \
$$$ 806
 $[(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L)]x$
 $13)$, $(I - L) = U/\lambda$
 $\overline{D} - \overline{L}$, $[(-D_1 + L_1 + L_S) + SU/\lambda]x$
 $[(\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + SU\}x]$
 x , with $Q = (\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + S U\}x$
 \overline{D} , Obviously, $-\lambda D_1$ **: 803-808** 806

= $(\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L)\}x$

com equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\overline{D} - \overline{L})^{-1}\{(-D_1 + L_1 + L_S) + SU/\lambda\}x$

= $[(\lambda - 1)/\lambda](\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + SU\}x$

appose $R = Qx$, 806
 $[(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L)]x$
 $= 13$, $(I - L) = U/\lambda$
 $\overline{D} - \overline{L}$, $[-1)(-D_1 + L_1 + L_S)$
 $+ SU/\lambda)x$
 $(\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S$
 x , with $Q = (\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S\}$. Obviously, $-\lambda D_1 + \lambda L_1 + \lambda$ 1): 803-808

= $(\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-\overline{D}_1 + L_1 + L_S) + (\lambda - 1)S(I - L) \}x$

From equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\overline{D} - \overline{L})^{-1}([-D_1 + L_1 + L_S)$

+ $SU/\lambda 1\}x$

= $[(\lambda - 1)/\lambda](\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S$

+ $SU/\lambda x$

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= $(\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-\overline{D}_1 + L_1 + L_S) + (\lambda - 1)S(I - L)\}x$

From equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\overline{D} - \overline{L})^{-1}\{(-D_1 + L_1 + L_S)$

+ $SU/\lambda\}x$

= $[(\lambda - 1)/\lambda][(\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + SU\}x$

Supp 1): 803-808

= $(\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L)\}x$

From equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\bar{D} - \bar{L})^{-1}\{(-D_1 + L_1 + L_S)$

+ $SU/\lambda\}x$

= $[(\lambda - 1)/\lambda](\bar{D} - \bar{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S$

+ SU/λ

Suppose $R = Qx$, 1): 803-808
 $=(\overline{D}-\overline{L})^{-1}\{(\lambda-1)(-D_1+L_1+L_S)+(\lambda-1)S(I-L)\}\}\times$

From equation (13), $(I-L) = U/\lambda$
 $=(\lambda-1)(\overline{D}-\overline{L})^{-1}\{(-D_1+L_1+L_S) + S U/\lambda\}\}\times$
 $+ SU/\lambda\}\times$
 $=[(\lambda-1)/\lambda](\overline{D}-\overline{L})^{-1}\{-\lambda D_1+\lambda L_1+\lambda L_S$

Suppose $R = Qx$, with $Q = (\over$ 1): 803-808

= $(\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-\bar{D}_1 + L_1 + L_S) + (\lambda - 1)S(I - L)\}x$

From equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\bar{D} - \bar{L})^{-1}\{(-D_1 + L_1 + L_S)$

+ $SU_1/\lambda\}x$

= $[(\lambda - 1)/\lambda][\bar{D} - \bar{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + SU_1\}x$

Suppose $R = Qx$ 1): 803-808

= $(\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-\overline{D} + L_1 + L_S) + (\lambda - 1)S(I - L)\}$

From equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\overline{D} - \overline{L})^{-1}\{(-D_1 + L_1 + L_S)$

+ $SU/\lambda\}$

= $[(\lambda - 1)/\lambda](\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S$

Suppose $R = Qx$, wi is a strictly lower triangular matrix, and by implication its eigenvalues lie on its main diagonal; in this case they are all zeros. Therefore, 1): 803-808
 $\qquad 806$
 $\qquad = (\overline{D} - \overline{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L) \}$

From equation (13), $(I - L) = U/\lambda$
 $\qquad = (\lambda - 1)(\overline{D} - \overline{L})^{-1}\{(-D_1 + L_1 + L_S)$
 $\qquad + SU/\lambda\}$
 $\qquad = [(\lambda - 1)/\lambda](\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + SU\})$ a convergent splitting. $(D - L) + L$ + L splitting. By the form equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\overline{D} - \overline{L})^{-1}\{(-D_1 + L_1 + L_S)$

= $[(\lambda - 1)/\lambda](\overline{D} - \overline{L})^{-1}\{(-D_1 + L_1 + L_S)$

= $[(\lambda - 1)/\lambda](\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda$ = $(\overline{D} - \overline{L})^{-1} \{(A - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L)\}$

From equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\overline{D} - \overline{L})^{-1}\{(-D_1 + L_1 + L_S)$

+ SU/λ)x

+ SU/λ)x

+ SU/λ)x

+ SU/λ

+ SU/λ

Suppose $R = Qx$, with $Q = (\overline{D} - \overline{L})^{-1}\{$ employ Lemma 3 to establish that K is an −1)S($I - L$))x

From equation (13), $(I - L) = U/A$

= $(λ - 1)(D - L)$ ⁻¹ $(-D_1 + L_1 + L_S)$

+ SU/ λ x

+ SU/ λ

+ SU/ λ

+ SU/ λ

+ SU/ λ

= [(λ - 1)/λ]($\overline{D} - \overline{L}$)⁻¹(-λ $D_1 + \lambda L_1 + \lambda L_S$

Suppose $R = Qx$, with $Q = (\overline{$ From equation (13), $(I - L) = U/\lambda$

= $(\lambda - 1)(\overline{D} - \overline{L})^{-1}\{(-D_1 + L_1 + L_S)$

= $[(\lambda - 1)/\lambda](\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S$
 $+ SU\}x$

Suppose $R = Qx$, with $Q = (\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + SU\}$.

Suppose $R = Qx$, with $Q = (\$ = (λ - 1)(\overline{D} - \overline{L})⁻¹{(-D₁ + L_1 + L_s)

= [(λ - 1)/λ](\overline{D} - \overline{L})⁻¹{-λD₁ + λ L_1 + λ L_s

+ *SU*/3x

Suppose $R = Qx$, with $Q = (\overline{D} - \overline{L})^{-1}\{-\overline{A}D_1 + \lambda L_1 + \lambda L_s + \lambda L_s + SU\}$.

Suppose R $f(X|\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_5$
 $+ SU_X$ with $Q = (\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda S$
 $-. SU_X$ with $Q = (\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda S$
 0 , since $SU \geq 0$, $-\lambda D_1 \geq 0$, $\lambda L_1 \geq$
 0 . Since \overline{D} is an M -matri pose $R = Qx$, with $Q = (\overline{D} - \overline{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_2 + SU\}$. (biviously, $-\lambda D_1 + \lambda L_1 + \lambda L_2 \le 0$, since $SU \ge 0$, $-\lambda D_1 \ge 0$, $\lambda L_1 \ge 0$, $SU_2 \ge 0$. Since \overline{D} is a nonsingular matrix, $\overline{D} - \overline{L}$ be a splittin 1 nonsingular matrix,

f some matrix *K*, i.e.,
 $-\text{matrix and } \overline{L} \geq 0$.

splitting. Now, $\overline{D}^{-1}\overline{L}$

lar matrix, and by

s lie on its main

all zeros. Therefore,
 \overline{L}) $\lt 1$, $K = \overline{D} - \overline{L}$ is

foregoing, $K =$ ($\overline{B} - \overline{L}$ be a splitting of some matrix *K*, i.e.,
 $\overline{D} - \overline{L}$. Also, \overline{D} is an *M* - matrix and $\overline{L} \ge 0$.
 \overline{S} , $K = \overline{D} - \overline{L}$ is an *M* - splitting. Now, $\overline{D}^{-1}\overline{L}$

it strictly lowe Also, \overline{D} is an M -matrix and $\overline{L} \ge 0$.
 $\overline{D} - \overline{L}$ is an M -splitting. Now, $\overline{D}^{-1}\overline{L}$

to lover triangular matrix, and by

this eigenvalues lie on its main

this case they are all zeros. Therefor is a strictly lower triangular matrix, and by
implication its eigenvalues lie on its main
diagonal; in this case they are all zeros. Therefore,
 $\theta(\overline{D}^{-1}\overline{L}) = 0$. Since $\rho(\overline{D}^{-1}\overline{L}) < 1$, $K = \overline{D} - \overline{L}$ is
a co

- $)$ < \leq
-
- $+ L_S + U_S$)}x $\lambda = \rho(G_{SOR})$. 2, we have $\rho(G_2) = \lambda = \rho(G_{SOR})$
If $\lambda > 1$, then $G_2x - \lambda x \ge 0$ but not $) >$

 $\{x\}$ section, the preconditioned Gauss-Seidel methods $\int_{\mathcal{X}}$ and 2. The spectral radii of iteration matrices of employ Lemma 3 to establish that K is an
 M -matrix. Since K is an M -matrix, by

definition, $K^{-1} = (\overline{D} - \overline{L})^{-1} \ge 0$. Thus, $Q \ge 0$

and $R \ge 0$.

(i) If $\lambda < 1$, then $G_2x - \lambda x \le 0$ but not
 $+D_1$

qual to 0. The L) λx definition, $K^{-1} = (\overline{D} - \overline{L})^{-1} \ge 0$. Thus, $Q \ge 0$

and $R \ge 0$.

(i) If $\lambda < 1$, then $G_2 x - \lambda x \le 0$ but not

equal to 0. Therefore, $G_2 x \le \lambda x$.

From Lemma 2, we have $\rho(G_2) < L + U_5$
 $\lambda = \rho(G_{SOR})$.

(ii) If $-[\overline{L}]^{-1}([\overline{L} \rightarrow \overline{L} + L_3 + L_4 + L_5 + L_1)]x$
 $-[\overline{L}(\overline{L} \rightarrow \overline{L} + L_4 + L_5 + L_1)]x$ (i) If $\lambda \le 1$ to the force, $\overline{L_2} \times \overline{L} \times L_1 + \lambda (L_4 + L_5 + L_1)]x$ (ii) the angle to Cherefore, $\overline{L_2} \times \overline{L} \times L_1 + \lambda (L_4 + L_5 + L_$ $+ \lambda(L + L_5 + L_1)x$

From Lemma 2, we have $\rho(G_2) <$
 $-(D_1 + L_1 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 + L_9 + L_1 + L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 + L_9 + L_1 + L_1 + L_2 + L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 + L_9 + L_1 + L_1 + L_2 + L_1 + L_2 + L_2 + L_3 + L_4 + L_5 + L_7 + L_8 +$ ଶ − = (ഥ − ത)ିଵ{(− 1)(−ଵ + ଵ - $\lambda I + U = -\lambda L$
 $- U - \lambda D_1^{-1} + \lambda L_1 + \lambda L_s + U_s$
 $- U_1 - \lambda L_1 + \lambda L_2 + U_s$
 $+ U_2 - \lambda L_1 + \lambda L_2 + U_s$
 $+ U_3 - \lambda L_1 + U_2 - L_2$
 $+ U_4 - U_1 + U_1 + \lambda L_s - L_s$
 $+ U_5 + U_6$
 $+ U_5 + U_8$
 $+ U_7 + U_8 + U_9$
 $+ U_8 + U_9$
 $+ U_9 + U_9$
 $+ U_1 + U_1 + U_2 + U_3$
 - $\lambda I + \hat{U} = -\lambda L$
 $I + \hat{U} = -\lambda L$
 $I = -\lambda D^{-1}(-\lambda D) + \lambda \lambda I + \lambda I_S + U_S$
 $\{H(\lambda - 1)(-D_1) + (\lambda - 1)L_1\}$
 $\{H, \lambda > 1, \text{ then } G_2 x - \lambda x \ge 0 \text{ but not } (G_2) = \lambda = \rho(G_{G\omega B})$
 $-(D_1 - L_1 - U_1) + \lambda I_S - L_S$
 $\{H_2 + H_2\}$
 $\{H_2 + H_2\}$
 $\{H_2 + H_2\}$
 In order to validate the results of the preceding introduced in this work are applied to Problems 1 the two methods are obtained and compared to those of some other methods. Are matter. $\sin A = 0$.

The matter is $\cos A = 0$ and $R \ge 0$.

(i) If $\lambda < 1$, then $G_2x - \lambda x \le 0$ but not

equal to 0. Therefore, $G_2x \le \lambda x$.

From Lemma 2, we have $\rho(G_2) < \lambda = \rho(G_{SOR})$.

(ii) If $\lambda = 1$, then $G_2x - \lambda x = 0$ (i) If $\lambda < 1$, then $G_2x - \lambda x \le 0$ but not
equal to 0. Therefore, $G_2x \le \lambda x$.
From Lemma 2, we have $\rho(G_2) < \lambda = \rho(G_{50R})$.
(ii) If $\lambda = 1$, then $G_2x - \lambda x = 0$.
Therefore, $G_2x = \lambda x$. From Lemma
2, we have $\rho(G_2) = \lambda = \rho(G_{$

$$
A = \begin{pmatrix} 1 & -0.172 - 0.234 & 0 \\ -0.365 & 1 & 0 & -0.204 \\ -0.165 & 0 & 1 & -0.215 \\ 0 & -0.236 - 0.372 & 1 \end{pmatrix}
$$

By letting G, G_1 and G_2 be the iteration matrices **Communication in Physical Sciences, 2020, 6(1): 803-808**
 Problem 2 Consider a 6× 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M of the classical Gauss-Seidel method, preconditioned Gauss-Seidel methods of (9) and (12) respectively, the spectral radii of these matrices are computed for Examples 1 and 2 and the results presented in Tables I and II.

3.0 Results and Discussion

Tables I and II depict the results of Problems 1 and 2 respectively. In the Tables, G , G_1 , G_2 , G_{SOR} , G_{GN} , G_M , and $G_{M \& N}$ represent the iteration matrices of the Gauss-Seidel, our first preconditioned Gauss-Seidel, our second preconditioned Gauss-Seidel, SOR, Gunawardena *et al.* (1991), Milaszewicz (1987) and Mayaki and 5.0 Ndanusa (2019) respectively.

Table I: Comparison of spectral radii of G_1 and G_2 with various iteration matrices for Problem 1

| Iteration matrix | Spectral radius | References 6.0 |
|-------------------------|------------------------|---|
| G_1 | 0.1601241711 | Allahviranloo, T., Moghaddam, R. G. & Afshar, |
| G_2 | 0.06681777737 | M. (2012). Comparison theorem with modified |
| G | 0.2277905779 | Gauss-Seidel and modified Jacobi methods by |
| G_{SOR} | 0.1000000002 | M – matrix. Journal of Interpolation and |
| G_{GN} | 0.08177303033 | Approximation in Scientific Computing, pp.1- |
| G_M | 0.1682312333 | 8, doi:10.5899/2012/jiasc-00017. |
| $G_{M\ \&\ N}$ | 0.08177303033 | Gunawardena, A. D., Jain, S. K. & Snyder, L. |

Table 2: Comparison of spectral radii of G_1 and G_2 with various iteration matrices for Problem 2

It is well known that the spectral radius of the iteration matrix of an iterative method for linear systems is sufficient for convergence of the method. The method is known to converge when the spectral radius is less than 1 in absolute value; the closer it is towards 0 the faster the convergence. In Table I, the spectral radius of G_2 is seen to be smaller than that of the unpreconditioned Gauss-Seidel G . It is shown to

Communication in Physical Sciences, 2020, 6(1): 803-808 807
 Problem 2 Consider a 6× 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M and
 $A = \begin{pmatrix} 1.0 & -0.1 & -0.1 & -0.4 & -0.2 & -0.1 \\ -0.5 & 1 & 0 & 0 & 0$ $\begin{bmatrix} -0.3 & -0.1 & 1 & -0.2 & -0.1 & 0 \\ -0.2 & 0 & -0.1 & 1 & -0.1 & -0.3 \end{bmatrix}$ those of G_2 , G_{SOR} , G_{GN} and $G_{M \& N}$. Similar trend $\begin{pmatrix} -0.5 & 1 & 0 & 0 & 0 & 0 \\ -0.3 & -0.1 & 1 & -0.2 & -0.1 & 0 \end{pmatrix}$ G₁ outperforms those of G and G_M, it lags behind **nunication in Physical Sciences, 2020, 6(1): 803-808**
 nm 2 Consider a 6× 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M & N

1.0 −0.1 −0.1 −0.4 −0.2 −0.1

−0.3 1 0 0 0 0

−0.3 −0.1 1 −0.2 − **nunication in Physical Sciences, 2020, 6(1): 803-808**
 em 2 Consider a 6× 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M & N and $1.0 -0.1 -0.1 -0.4 -0.2 -0.1$ even that of G_{SOR} . Although the **munication in Physical Sciences, 2020, 6(1): 803-808** 8
 em 2 Consider a 6× 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M a N and $1.0 -0.1 -0.4 -0.2 -0.1$ even that of G_{SOR} . Although the sp **unication in Physical Sciences, 2020, 6(1): 803-808**
 n 2 Consider a 6× 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M a $N = 0.1$ −0.1 −0.4 −0.2 −0.1 ⁰ −0.2 −0.1 ^{−0.2} −0.1 ^{−0.2} ^{−0.1} [−] **unication in Physical Sciences, 2020, 6(1): 803-808**
 n2 Consider a 6× 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M & A .
 -0.2 -0.1 -0.1 -0.2 -0.1 0
 -0.2 $0 - 0.1$ -0.1 -0 **unication in Physical Sciences, 2020, 6(1): 803-808**
 n2 Consider a 6× 6 matrix of the form be smaller that those of G_1 , G_G , G_M Physical Sciences, 2020, 6(1): 803-808

a 6× 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M a N and
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6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M & N and

0.4 -0.2 -0.1 over that of G_{SOR} . Although the spectral radius of

0 0 0 0 G_1 outperforms tho **Solution** Sciences, 2020, 6(1): 803-808
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 -0.4 −0.2 −0.1 or even that of G_{SOR} . Although the spectral radius of
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6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M and
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 \times 6 matrix of the form be smaller that those of G_1 , G_G_N , G_M , G_M & N and
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 \times 6 matrix of the form be smaller that those of G_1 , G_{GN} , G_M , G_M & M and
 -0.4 -0.2 -0.1 over that of G_{SOR} . Although the spectral radius of
 0 0 0 0 0 0 be smaller that those of G_1 , G_{GN} , G_M , G_M and even that of G_{SOR} . Although the spectral radius of is witnessed in Table II, with G_2 in the lead, followed by G_M , G_{SOR} , G_1 , $G_{M & N}$, G_{GN} and G , in that order.

Conclusion

Two preconditioned schemes of the Gauss-Seidel iterative method for solving linear systems are introduced, analysed and their convergence established. Numerical experiments reaffirmed their superiority over the unpreconditioned Gauss-Seidel method. More so, the performance of these methods, when compared to some other preconditioned methods in literature, showed significant improvement in the rate of convergence of the new methods over the existing ones. **Concusion**
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Conflict of Interest

The author declare no conflict of interest

