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Generalized Odd Gompertz-G Family of Distributions: Statistical Properties and Applications

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Abstract: *In this study, we introduce a novel generator known as the Generalized Odd Gompertz distribution, which includes an extra shape parameter. We examine various mathematical properties of this new generator and explicitly derive its characteristics such as moments, moment-generating function, survival function, hazard function, entropies, quantile function, and the distribution of order statistics. Within this family of distributions, we focus on one member, the Generalized Odd Gompertz-Exponential distribution, defining and analyzing its properties. To assess the flexibility and performance of the model's parameters, we employ Monte Carlo simulations. We further evaluate the versatility of the Generalized Odd Gompertz- Exponential distribution by applying it to real-world datasets and comparing its performance with other existing models. Additionally, we explore the estimation of model parameters using the maximum likelihood method, demonstrating the potential applicability of this distribution family to real-life data analysis.*

Keywords: *Gompertz distribution, Generalized odd Gompertz-G Family, exponential distribution, moments, Quantile function, maximum likelihood.*

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1.0 Introduction

The Gompertz distribution has garnered significant attention and has been extensively employed by researchers across diverse fields in the past few decades. Its applications span various domains, including reliability analysis, actuarial sciences, engineering, biological studies, environmental science, demography, medical sciences, economics, and finance Alizadeh *et al.,* (2017a). This distribution, which is a generalization of the exponential distribution, finds common use in a wide range of lifetime analyses and applications (Sanku *et al.,* 2018). In the quest for more versatile models capable of representing complex datasets, researchers have developed numerous statistical distributions. This effort has led to the creation of additional distributions by generalizing the parent distributions, as demonstrated in the work by Oguntunde *et al.,* (2015).

The innovation of developing a generalized family of probability distribution drew the devotion of researchers and dedicated statisticians to the flexibility possessions of the generalized distributions. The flexibility

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of the probability distribution is of interest because more flexible models can provide more information than the less flexible models, but the tractability of the probability distributions makes it easier for the researcher, especially when it comes to the simulation of random samples (Usman *et al*., 2020). Numerous scholars have broadened the application of this distribution, resulting in the creation of several related distributions. Among these extensions are the Beta-G developed by Eugene *et al.,* (2002), the Gamma-G developed by Zografos and Balakrishnan (2009), Gamma-G type-3 by Torabi and Montazari (2012) and the Exponentiated-G family by Gupta *et al.,* (1998). The T-X family of distributions by Alzaatreh *et al.,* (2013), the exponentiated T-X family of distributions by Alzaghal *et al.,* (2013) where Bourguignon *et al.,* (2014) introduced the Weibull-G family, Odd Log-logistic by Cordeiro *et al.,* (2017), New Weibull-X by Ahmad *et al.,* (2018), the Beta Weibull-G by Yousof *et al.,* (2017) , the Generalized Odd Generalized Exponential-G by Alizadeh *et al.,* (2017b), Type I Half-Logistic Exponentiated-G Family by Bello *et al.,* (2021) and New Generalized Odd Frechet-G by Abubakar Sadiq *et al.,* (2023). This paper explores the development and application of the Generalized Odd Gompertz distribution (GOG-G) to analyze lifetime data. The motivation for expanding distribution models arises from the need to accommodate diverse behaviour patterns in lifetime data, including increasing, decreasing, constant, and bathtub-shaped failure rates. The new distribution family addresses practical goals such as introducing asymmetry, adapting kurtosis, and tailoring models to better represent real-world datasets. The paper is organized into four sections, covering the introduction, definition of the GOG-G family, expansion of density for cumulative distribution function, derivation of statistical characteristics, discussion of sub-model, and parameter estimation through maximum likelihood, demonstration of practical utility through simulations and real datasets, and concluding remarks. The Generalized Odd Gompertz distribution proves to be a versatile tool for analyzing diverse lifetime data patterns.

2. 0 Methodology

2.1. Generalized odd Gompertz-G (GOG-G) family

The probability density function (pdf) and cumulative distribution function (cdf) of the Gompertz distribution, as outlined by Lenart (2012), represent the scale parameter and γ a symbol for the shape parameter which is mathematically formulated as follows:

$$G(t;\theta,\gamma) = 1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma t}-1\right)};\quad 0 < t < \infty,\quad \theta,\gamma > 0 \qquad (1)$$

$$g(t;\theta,\gamma) = \theta e^{\gamma t} e^{-\frac{\theta}{\gamma}\left(e^{\gamma t}-1\right)};\ 0 < t < \infty,\ \theta,\gamma > 0 \qquad (2)$$

The cumulative distribution function, denoted as $G(t,\Phi)^{\alpha}$, along with the survival function $\overline{G}(t;\Phi)^{\alpha} = 1 - G(t;\Phi)^{\alpha}$ of the parent distribution, is contingent on a parameter vector Φ . When considering a random variable "T" related to a system characterized by a baseline $G(t;\Phi)^{\alpha}$ distribution, the probability that the system will not function within a specific time interval is represented as $G(t;\Phi)^{\alpha} \big/ \overline{G}(t;\Phi)^{\alpha}$. The random variable "T" in the context of the Gompertz model is expressed as follows:

$$p(T \le t) = p(T \le G(t;\Phi)^{\alpha} \big/ \overline{G}(t;\Phi)^{\alpha}) = F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)$$

If we substitute t in the Gompertz cumulative distribution function with the odds ratio $G(t;\Phi)^{\alpha} \big/ \overline{G}(t;\Phi)^{\alpha}$, we propose a new family of distribution called the Generalized Odd Gompertz-G (GOG-G) family by integrating the density function in equation (2) to obtain the CDF given as follows:

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$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)=1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}-1\right)} \tag{3}$$

The corresponding pdf from the CDF in equation (3) is given by

$$f_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)=\alpha\theta g(x;\Phi)G(x;\Phi)^{\alpha-1}(1-G(x;\Phi)^{\alpha})^{-2}e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}e^{-\frac{\theta}{\gamma}\left(e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}-1\right)} \tag{4}$$

where $\theta>0$ is the scale parameter, $\alpha>0$, $\gamma>0$ are the shape parameters, $g(x,\Phi)$, $G(x;\Phi)$ represents the pdf and CDF of any parent distribution and Φ is the parameter vector, we, therefore, represent a random variable "X" with the distribution function and density function described in equations (3) and (4) as $X \sim GOG-G(\alpha,\theta,\gamma,\Phi)$.

2.2. Survival and Hazard Rate function of the GOG-G family

The survival function and hazard rate function are provided as follows, respectively:

$$S_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)=e^{-\frac{\theta}{\gamma}\left(e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}-1\right)} \tag{5}$$

$$h_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)=\alpha\theta g(x;\Phi)G(x;\Phi)^{\alpha-1}(1-G(x;\Phi)^{\alpha})^{-2}e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}} \tag{6}$$

2.3. Quantile function of GOG-G family

The quantile function of the GOG-G family is derived by reversing the cumulative distribution function (CDF) provided in equation (3). It is expressed as:

$$x=G(x;\Phi)^{-1}\left[\frac{\frac{1}{\gamma}\log\left[1-\frac{\gamma}{\theta}\log(1-u)\right]}{1+\left[\frac{1}{\gamma}\log\left[1-\frac{\gamma}{\theta}\log(1-u)\right]\right]}\right]^{\frac{1}{\alpha}} \tag{7}$$

In this context, $G(x;\Phi)^{-1}$ represents any future parent distribution and "u" is treated as a random variable following a uniform distribution in the range of *(0, 1)*.

2.4. Suitable expansion for the density of GOG-G family

In this section, we will explore a detailed expansion of the distribution functions for the GOG-G family.

Proposition: The equation that provides a linear depiction of the GOG-G family of distributions can be stated as follows:

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)=1-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}Z_{k,l}H_{\alpha(k+l)}(x;\Phi) \tag{8}$$

Proof

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)=1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}-1\right)}$$

By power series expansion

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$$e^{-\frac{\theta}{\gamma}\left(e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}-1\right)} = \sum_{i=0}^{\infty} \frac{(-1)^{i}}{i!}\left(\frac{\theta}{\gamma}\right)^{i}\left(e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}-1\right)^{i}$$

and $\left(e^{\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)}-1\right)^{i} = \left(\frac{1}{e^{-\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)}}-1\right)^{i} \Rightarrow \left(\frac{1-e^{-\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)}}{e^{-\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)}}\right)^{i}$

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\sum_{i=0}^{\infty} \frac{(-1)^{i}}{i!}\left(\frac{\theta}{\gamma}\right)^{i}\left(1-e^{-\gamma\left(\frac{G(x)^{\alpha}}{1-G(x)^{\alpha}}\right)}\right)^{i}\left(e^{\gamma\left(\frac{G(x)^{\alpha}}{1-G(x)^{\alpha}}\right)}\right)^{i} \tag{9}$$

Appling binomial expansion,

$$\left(1-e^{-\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)}\right)^{i} = \sum_{j=0}^{\infty} (-1)^{j}\binom{i}{j} e^{-\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)^{j}}$$

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i}}{i!}\left(\frac{\theta}{\gamma}\right)^{i}(-1)^{j}\binom{i}{j}\left(e^{\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)}\right)^{i}\left(e^{-\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)}\right)^{j}$$

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{i!}\left(\frac{\theta}{\gamma}\right)^{i}\binom{i}{j} e^{(i-j)\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)} \tag{10}$$

From power series expansion,

$$e^{(i-j)\gamma\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)} = \sum_{k=0}^{\infty} \frac{(i-j)^{k}\gamma^{k}\left(\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right)^{k}}{k!}$$

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\theta^{i}\gamma^{k-i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j}(i-j)^{k}}{i!k!}\binom{i}{j}\left(G(x;\Phi)^{\alpha}\right)^{k}\left(1-G(x;\Phi)^{\alpha}\right)^{-k} \tag{11}$$

By generalize binomial expansion,

$$\left(1-G(x;\Phi)^{\alpha}\right)^{-k} = \sum_{l=0}^{\infty} \frac{\Gamma k+l(G(x;\Phi)^{\alpha})^{l}}{l!\Gamma k}$$

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\theta^{i}\gamma^{k-i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j}(i-j)^{k}}{i!k!l!}\binom{i}{j}\frac{\Gamma k+l}{\Gamma k}G(x;\Phi)^{\alpha(k+l)} \tag{12}$$

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}G(x;\Phi)^{\alpha(k+l)}$$

$$F_{GOG-G}(x;\alpha,\theta,\gamma,\Phi) = 1-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}H_{\alpha(k+l)}(x;\Phi) \tag{13}$$

Where $Z_{k,l} = \theta^{i}\gamma^{k-i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i-j)^{k}}{i!k!l!}\binom{i}{j}\frac{\Gamma(k+l)}{\Gamma k}$ and $H_{\alpha(k+l)}(x;\Phi) = G(x;\Phi)^{\alpha(k+l)}$ denotes the CDF of the Exponentiated-G distribution with power parameter $\alpha(k+l) > 0$. Differentiating equation (13) w.r.t x, we have the corresponding pdf as;

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$$f_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}h_{\alpha(k+l)}(x;\Phi) \quad (14)$$

Where $h_{\alpha(k+l)}(x;\Phi)=\alpha(k+l)g(x;\Phi)G(x;\Phi)^{\alpha(k+l)-1}$

2.5. Moments of the GOG-G family

Moments play a vital role in the application of statistics and most necessary probability distribution features are studied via moments. The expression for the r^th^ moment of a random variable X, which adheres to the Generalized Odd Gompertz-G (GOG-G) family is as follows:

$$E(X^r)=\int_0^{\infty} x^r f_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)dx=\int_0^{\infty} x^r\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}h_{\alpha(k+l)}(x;\Phi)dx=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}E(X^{r}{}_{k,l}) \quad (15)$$

Where $E(X^{r}{}_{k,l})=\int_0^{\infty} x^r h_{\alpha(k+l)}(x;\Phi)dx$

2.6. Moment-generating function of the GOG-G family

The moment-generating function for a random variable X, which belongs to the GOG-G family, is expressed as:

$$M_x^{GOG-G}(t)=E(e^{tx})=\int_0^{\infty} e^{tx} f_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)dx=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}E(e^{tx_{k,l}}) \quad (16)$$

where $E(e^{tx_{k,l}})=\int_0^{\infty} e^{tx} h_{\alpha(k+l)}(x;\Phi)dx$

2.7. Entropy of the GOG-G Family

Entropy serves as a metric to gauge the level of diversity or unpredictability in a random variable X Renyi, (1961). Statistically, the entropy of the GOG-G family is defined as follows:

$$I_R(\Omega)=\frac{1}{1-\Omega}\log\int_0^{\infty} f_{GOG-G}(x;\alpha,\theta,\gamma,\Phi)^{\Omega}dx=\frac{1}{1-\Omega}\log\int_0^{\infty} f\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}\int_0^{\infty} h_{\alpha(k+l)}(x;\Phi)dx\right)^{\Omega}dx$$

$$I_R(\Omega)=\frac{1}{1-\Omega}\log\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}\right)^{\Omega}\int_0^{\infty}\left(h_{\alpha(k+l+1)}(x;\Phi)\right)^{\Omega}dx \quad (17)$$

Where $\Omega>0$ and $\Omega\neq 1$

2.8. Order Statistics of the GOG-G Family

Suppose we have a random sample with values $X_1,X_2,...,X_n$ from the GOG-G distribution, and we denote the corresponding order statistics as $X_{1:n}\leq X_{2:n}\leq ...X_{n:n}$. In this context, the expression for the i^{th} order statistic can be stated as:

$$f_{i:n}(x;\alpha,\theta,\gamma,\Phi)=\frac{n!}{(i-1)(n-i)!}\left[f(x;\alpha,\theta,\gamma,\Phi)\right]\left[F(x;\alpha,\theta,\gamma,\Phi)\right]^{i-1}\left[1-F(x;\alpha,\theta,\gamma,\Phi)\right]^{n-i}$$

$$=\frac{n!}{(i-1)(n-i)!}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}h_{\alpha(k+l)}(x;\Phi)\right]\left[1-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}H_{\alpha(k+l)}(x;\Phi)\right]^{i-1}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} Z_{k,l}H_{\alpha(k+l)}(x;\Phi)\right]^{n-i} \quad (18)$$

2.9. Estimation of parameters of the GOG-G family

Assuming we have $x_1,x_2,x_3,...,x_n$ observed values from the proposed GOG-G family with α,θ,γ parameters, and we have a $[m\times 1]$ parameter vector. The log-likelihood function, denoted as Ω, is formulated as follows:

$$L(\Omega)=\log\prod_{i=1}^{n}f(x)=n\log\alpha+n\log\theta+\sum_{i=1}^{n}\log(g(x;\Phi))+(\alpha-1)\sum_{i=1}^{n}\log G(x;\Phi)-2\sum_{i=1}^{n}\log(1-G(x;\Phi))$$

$$+\gamma\sum_{i=1}^{n}\left[\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}\right]-\frac{\theta}{\gamma}\sum_{i=1}^{n}\left[e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}-1\right] \qquad (19)$$

The partial derivatives of equation (19) concerning the $(\alpha,\theta,\gamma,\Phi)$ parameters are provided as follows

$$\frac{\partial L(\Omega)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log(G(x;\Phi))+2\sum_{i=1}^{n}\frac{G(x;\Phi)^{\alpha}\ln G(x;\Phi)}{\left(1-G(x;\Phi)^{\alpha}\right)}$$

$$+\gamma\sum_{i=1}^{n}\left[\frac{G(x;\Phi)^{\alpha}\ln G(x;\Phi)}{(1-G(x;\Phi)^{\alpha})^{2}}\right]-\theta\sum_{i=1}^{n}\left[\frac{e^{\gamma\frac{G(x;\Phi)^{\alpha}}{(1-G(x;\Phi)^{\alpha})}}G(x;\Phi)^{\alpha}\ln G(x;\Phi)}{(1-G(x;\Phi)^{\alpha})^{2}}\right] \qquad (20)$$

$$\frac{\partial L(\Omega)}{\partial\theta}=\frac{n}{\theta}+\frac{1}{\gamma}\sum_{i=1}^{n}\left[e^{\gamma\frac{G(x;\Phi)^{\alpha}}{(1-G(x;\Phi)^{\alpha})}}-1\right] \qquad (21)$$

$$\frac{\partial L(\Omega)}{\partial\gamma}=\sum_{i=1}^{n}\left[\frac{G(x;\Phi)^{\alpha}}{(1-G(x;\Phi)^{\alpha})}\right]+\frac{\theta}{\gamma^{2}}\sum_{i=1}^{n}\left[e^{\gamma\frac{G(x;\Phi)^{\alpha}}{(1-G(x;\Phi)^{\alpha})}}-1\right]-\frac{\theta}{\gamma}\sum_{i=1}^{n}\left[\frac{G(x;\Phi)^{\alpha}}{(1-G(x;\Phi)^{\alpha})}e^{\gamma\frac{G(x;\Phi)^{\alpha}}{(1-G(x;\Phi)^{\alpha})}}\right] \qquad (22)$$

$$\frac{\partial L(\Omega)}{\partial\Phi}=\sum_{i=1}^{n}\frac{g'(x;\Phi)}{g(x;\Phi)}+(\alpha-1)\sum_{i=1}^{n}\frac{g(x;\Phi)}{G(x;\Phi)}+2\sum_{i=1}^{n}\frac{\alpha g(x;\Phi)G(x;\Phi)^{\alpha-1}}{(1-G(x;\Phi)^{\alpha})}$$

$$+\gamma\sum_{i=1}^{n}\left[\frac{\alpha g(x;\Phi)G(x;\Phi)^{\alpha-1}}{(1-G(x;\Phi)^{\alpha})^{2}}\right]-\theta\sum_{i=1}^{n}\left[\frac{\alpha g(x;\Phi)G(x;\Phi)^{\alpha-1}e^{\gamma\frac{G(x;\Phi)^{\alpha}}{1-G(x;\Phi)^{\alpha}}}}{(1-G(x;\Phi)^{\alpha})^{2}}\right] \qquad (23)$$

2.10. Sub-model of the GOG-G family

When incorporating an Exponential distribution into the GOG-G family, a novel distribution emerges. The cumulative distribution function (CDF) and probability density function (PDF) of the Exponential distribution, which serves as the foundational distribution with a parameter denoted as δ, can be mathematically described as follows:

$$B(x;\delta)=1-e^{-\delta x} \qquad (24)$$

$$b(x;\delta)=\delta e^{-\delta x}\quad x>0,\delta>0 \qquad (25)$$

By incorporating equations (24) and (25) into equations (3), (4), (5), (6), and (7), will provide the cumulative distribution function $F(x)$, probability density function $f(x)$, survival function $S(x)$, hazard function $h(x)$, and the quantile function $Q(u)$ of the Generalized Odd Gompertz-Exponential (GOG-E) distribution.

2.11. Graph of the special sub-model of the GOG-G family

The plots of the density function, distribution function, survival function and hazard function of the GOG- Exponential distribution at different parameter values is as follows;

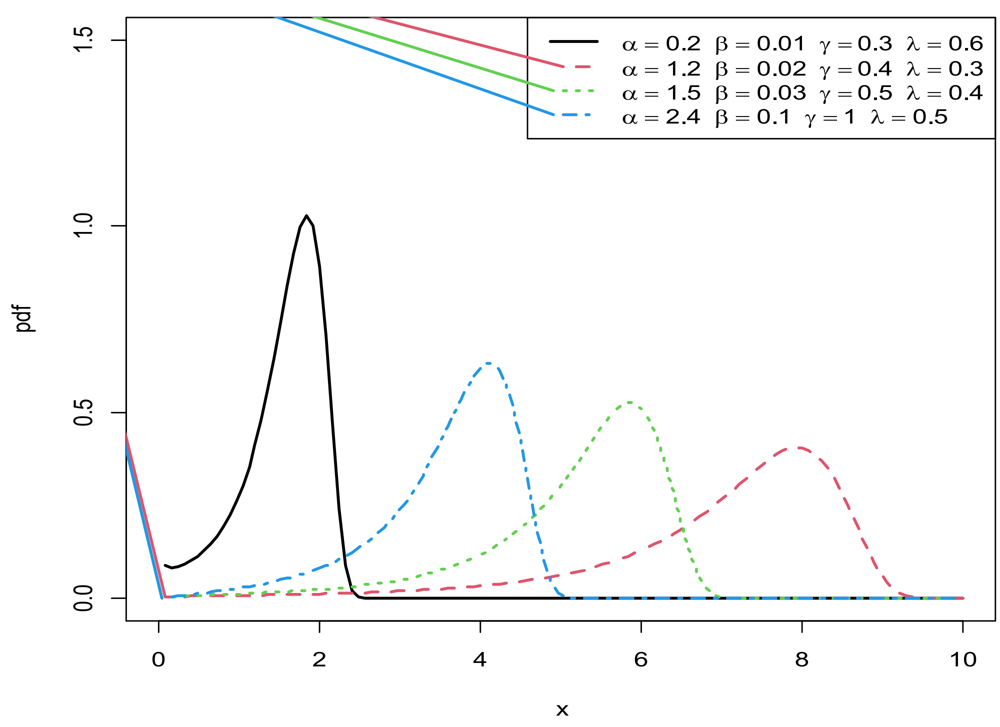

Fig. 1: pdf of the Generalized Odd Gompertz-Exponential Distribution

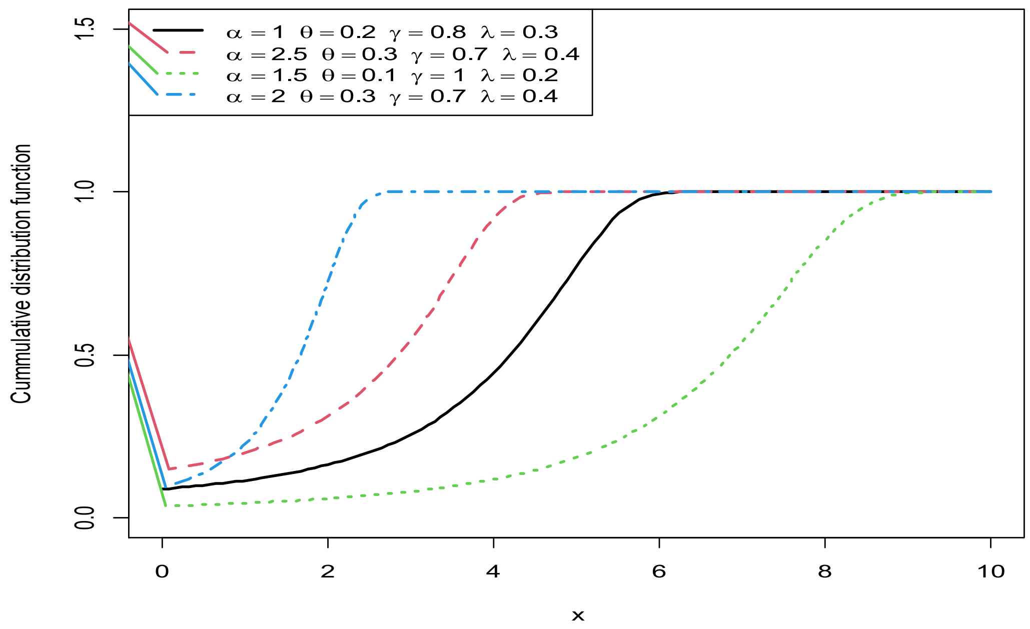

Fig. 2: cdf of the Generalized Odd Gompertz-Exponential Distribution

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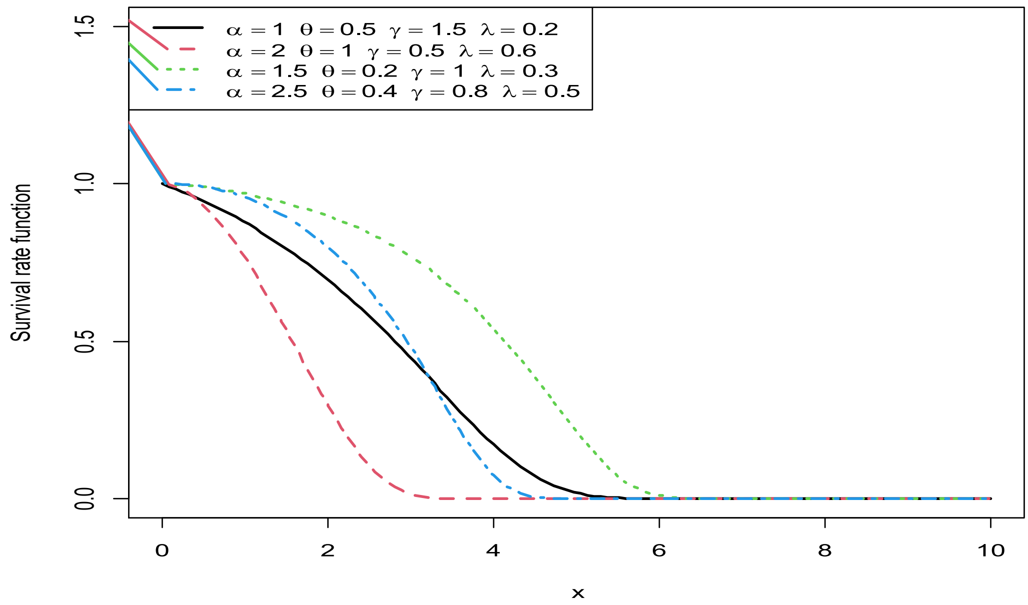

Fig. 3: Survival function of the Generalized Odd Gompertz-Exponential Distribution

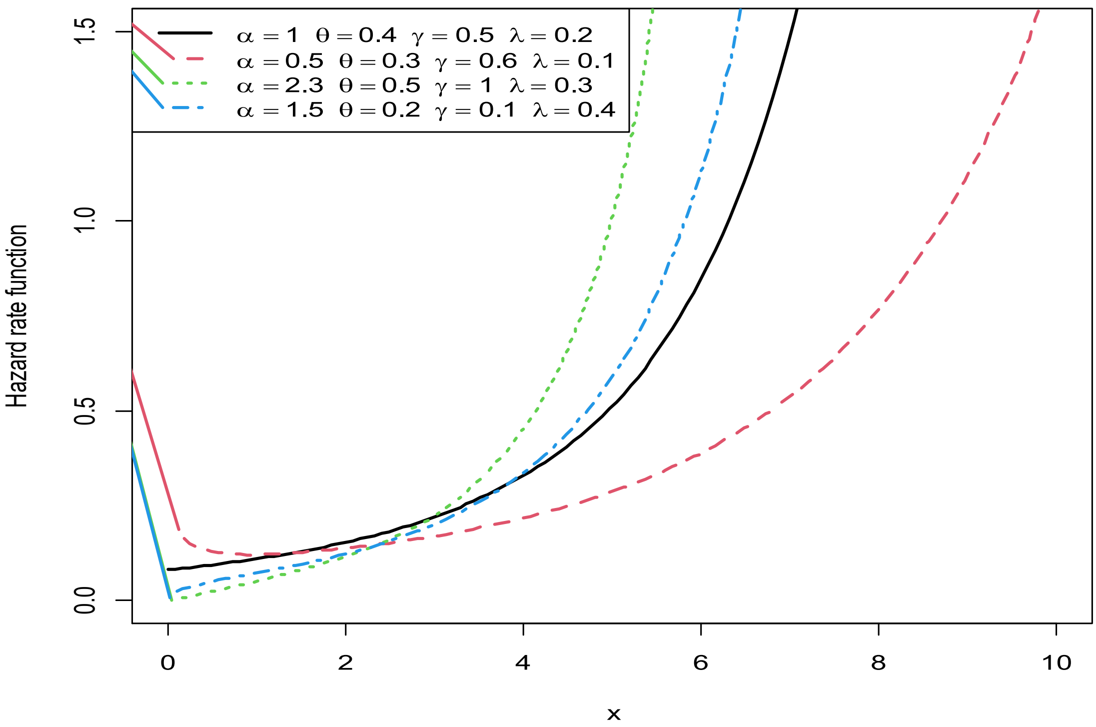

Fig. 4: Hazard function of the Generalized Odd Gompertz-Exponential Distribution

3.0 Monte Carlo Simulation and Application

3.1. Monte Carlo Simulation

The commonly employed set of computational methods known as "Monte Carlo simulation" is applied to generate numerical results based on replicated random samples. This methodology is utilized to address the challenge of

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evaluating risk when modelling lifetime data.

3.1.1. Simulation study

To evaluate the trustworthiness of the GOG-E model, a simulation study was carried out using the Monte Carlo Simulation method. This study aimed to calculate the mean, bias, and root mean square error of the model parameters estimated through maximum likelihood estimation. Simulated data was generated by applying the quantile function defined in equation (7), and this process was repeated 1,000 times for various sample sizes: n = 20, 50, 100, 250, 500, and 1,000. In each of these simulation runs, the parameters were held constant at specific values.

Table 1: MLEs, bias and RMSE for some values of parameters

		(θ=1, α=1.5, λ=0.5, γ=2)		
n	**Parameters**	**Estimated Values**	**Bais**	**RMSE**
20	θ	1.0997	0.0997	0.5931
	α	1.6876	0.1876	0.5227
	λ	0.5608	0.0608	0.1494
	γ	1.9986	-0.0014	0.6026
50	θ	1.0877	0.0877	0.4790
	α	1.5859	0.0859	0.3170
	λ	0.5186	0.0186	0.0846
	γ	2.0378	0.0378	0.4904
100	θ	1.0859	0.0859	0.3369
	α	1.5725	0.0725	0.2334
	λ	0.5100	0.0100	0.0607
	γ	2.0431	0.0431	0.3594
250	θ	1.0691	0.0691	0.2620
	α	1.5407	0.0407	0.1526
	λ	0.4998	-0.0002	0.0302
	γ	2.0617	0.0617	0.2483
500	θ	1.0606	0.0606	0.1959
	α	1.5327	0.0327	0.1147
	λ	0.4991	-0.0009	0.0222
	γ	2.0423	0.0423	0.1860
1000	θ	1.0416	0.0416	0.1371
	α	1.5204	0.0204	0.0742
	λ	0.4988	-0.0012	0.0132
	γ	2.0303	0.0303	0.1442

Table 1 illustrates that the bias and root mean square errors (RMSEs) diminish as the sample size grows larger. This trend shows that the estimates are converging towards the actual values, signifying improved accuracy and reliability. It highlights that the estimates are both consistent and efficiency as the sample size increases.

3.2. Application

In this context, we demonstrate the applicability of the Generalized Odd Gompertz-Exponential Distribution (GOG-ED) using a real dataset obtained from Arshad *et al.,* (2021) and was previously used by Kotz and Dorp (2004). We computed the maximum likelihood estimates and assessed the goodness-of-fit measures

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using R software. We then compared the results with those of other distributions, namely the Odd Gompertz Exponential (OGE), Gompertz Exponential (GE), Kumaraswamy Exponential (KE), Exponentiated Weibull-Exponential (EW-E), and Exponential (Ex) distributions.

In determining the best out of the competing models, the Akaike Information Criterion, Akaike (1974) "AIC" was employed and is statistically expressed as:

$AIC = -2LL + 2K$. Where "LL" stands for log-likelihood function and K is the number of model parameters in the model.

The data narrates the 85 hailing times of civil engineering dataset obtained from Arshad *et al.,* (2021) as follows;

4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80

Table 2: Parameters Estimates and Goodness of Fit Measures for civil engineering data with 85 hailing times

Model	Parameter Estimates and Goodness of Fit						
	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\phi}$		-LL	AIC
GOGE	10.3578	2.3295	0.3513	0.6166	-	136.0699	**280.1399**
OGE	0.1168	0.1358	2.2030	-	-	147.5815	301.1631
Ex	0.1757	-	-	-	-	232.7956	467.5913
GE	0.0163	0.8348	0.7680	-	-	145.4489	296.8978
KE	5.7391	3.3890	0.2543	-	-	155.4583	316.9165
EWE	0.8696	0.0068	1.0046	0.7831	-	143.6837	295.3674

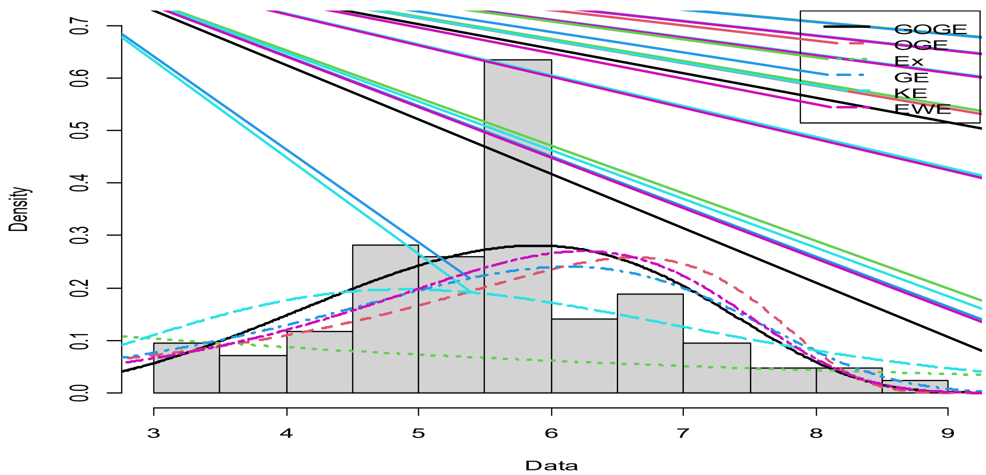

Fig. 5: Histogram Plots of the Distributions on the civil engineering data with 85 hailing times

Table 2 presents the results of the maximum likelihood estimation for the parameters of the novel distribution and five other reference distributions. In evaluating the goodness of fit, it was noted that the proposed distribution demonstrated the lowest AIC value, closely followed by EWE. A visual inspection of the fit, as illustrated in Fig. 5, provides additional confirmation that the proposed distribution surpasses the comparison distributions. Therefore, among the considered distributions, the GOG-ED is

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identified as the most appropriate for modelling the data related to civil engineering, specifically the 85 hailing times.

4.0 Conclusion

In this article, we introduced and delved into an innovative continuous probability distribution called the Generalized Odd Gompertz-G Family of Distribution. We thoroughly examined various statistical aspects associated with this novel distribution, including the explicit moment, quantile function, entropy, reliability function, hazard function, and order statistics. The parameters of the distribution were estimated using the maximum likelihood technique. Simulation results were presented to evaluate the performance of this new distribution. Additionally, we applied the distribution to analyze a real dataset, emphasizing the significance and versatility of this novel distribution. The results show that the Generalized odd Gompertz Exponential Distribution outperforms the existing models considered, highlighting its potential applicability in a broad spectrum of practical scenarios for data modelling.

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Compliance with Ethical Standards Declarations

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data Public.

Competing interests

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