Communication in Physical Sciences, 2021, 7(1):31-41

Optimization of investment strategies for a Defined Contribution (DC) plan member with Couple Risky Assets, Tax and Proportional Administrative Fee

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Received 30 January 2021/Accepted 06 March 2021/Published Online 08 March 2021

Abstract: *The aim of this paper is to study the optimal investment plans of a member in a defined contribution (DC) pension scheme with proportional administrative fee and tax on invested funds under logarithm utility function and Ornstein-Uhlenbeck (O-U) model. This is done by considering a portfolio consisting of a risk free asset (bank security) and two risky assets (stocks) where the stock market prices are driven by the Ornstein-Uhlenbeck (O-U) process. An optimization problem known as the Hamilton Jacobi Bellman (HJB) equation is obtained by maximizing the expected utility of the member's terminal wealth. Since the HJB equation is a non linear partial differential equation (PDE) and could be complex to solve, we use the Legendre transformation method and dual theory to reduce it to a linear PDE. By method of variable change and separation of variable, closed form solutions of the optimal investment plans are obtained using logarithm utility function. More so, sensitivity analysis of some parameters are carried out theoretically on the optimal investment plans with observations that apart from the changes experienced in the stock market prices caused by the O-U process, the optimal investment plans for the risky assets are inversely proportional to contribution rate, tax rate imposed on the invested fund , proportional administration fee, investment time* t*, but directly proportional to the appreciation rate of the risky assets*.

Key Words: *Ornstein-Uhlenbeck process, optimal investment plan, proportional administrative fee, Legendre transforms, logarithm utility*.

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1.0 Introduction

In determining the optimal investment plan for a member in a defined contribution pension scheme, the concept of stochastic volatility is informative in guiding decision making with respect to suitable and best investment strategy. Consequently, there is need to establish in depth of the behaviour of the different volatility models available in the financial market in order to model the behaviour of the risky assets arising from its fluctuating nature that is associated with the various information available in the financial market. The basic types of volatility models are the constant elasticity of variance (CEV), the Heston volatility model, the jump diffusion model, the Ornstein-Uhlenbeck (O-U) model etc Gao (2009), Sheng and Rong (2014), Ihedioha *et al*. (2017).

The defined contribution pension scheme is a type of retirement plan in which member's contribute a certain percentage of their income into a retirement saving account for the purpose of planning for their old age income. Funds accumulating from the saving are invested in the financial market since their retirement benefits depend mostly on the returns of their investments Deelstra *et al.* (2003) and Gao (2008). Due to the volatile nature of the risky assets and the risk involvement in those assets, there is need to develop an efficient and robust investment plan which will serve as a guide during investment period. This has led to the study of optimal investment plan. There are numerous work done in this area which include Xiao *et al.* (2007), Gao (2009) who studied utility maximization under constant elasticity model in DC pension scheme. The optimal investment and reinsurance problem of utility maximization under CEV model was also studied by Gu *et al.* (2010). Li *et al*. (2013), studied optimal investment problem with taxes, dividend and transaction cost using CEV model and logarithm utility function. The optimal portfolio strategy with multiple contributors in a DC pension fund using Legendre transformation method was studied by Osu *et al*. (2017). Li *et al.* (2017) solved the optimal portfolio problem with default risk and refund of premium clause in a DC plan; in their work, the stock market price followed the CEV model. Akpanibah and Oghenero (2018) investigated the effect of additional voluntary contribution on the investment strategies under CEV model using power transformation method in solving their problem. Boulier *et al.* (2001) studied the optimal portfolio management with stochastic interest rate for a protected case of DC fund. Deelstra *et al.* (2003) and Gao (2008) adopted stochastic interest rate model to obtain optimal investment plan in a DC fund. Also, Battocchio and Menoncin (2004) as well as Cairns *et al.* (2006) considered investment strategy with interest rate of Vasicek type while Zhang and Rong (2013), Njoku *et al.* (2017) and Akpanibah *et al*. (2017) studied the optimal portfolio problem when the interest rate is of affine interest.

According to Xiao and Yonggui (2020), the O-U process can be used to model both interest rate and stock market price since it reflects the fluctuation of the interest rates and asset prices. Also, the O-U process is closer to the change in interest rate.Ihedioha *et al.* (2020) studied the optimal investment plan for an investor with exponential utility under the modified CEV model. In their work, they used the O-U process to model their interest rate. Akpanibah and Ini (2020) studied an investor's investment plan with stochastic interest rate under the CEV model and the Ornstein-Uhlenbeck Process. They used the O-U process to model their interest rate and considered a portfolio consisting of one risk free asset and two risky assets modelled by the CEV model. The optimal investment plan for a DC plan under the O-U process was studied by Xiao and Yonggui (2020) in which a single risky asset modelled by the O-U process was combined with a risk-free asset and also investment in loan. Also, the strategic portfolio management for a pension plan member with Couple risky assets and transaction cost under the O-U Process was studied by Ini and Akpanibah (2020). In their work, they used the O-U process to model their risky assets and the exponential utility function to obtain their investment strategies.

In this work, we maximize the expected utility of a DC member's wealth under logarithm utility by studying the optimal investment plan whose risky assets are modelled by the O-U process. Also, we consider cases where there is proportional administrative fee and tax imposed on the invested funds. Furthermore, the Legendre transformation and change of variable method are used to derive the optimal investment plans for the three assets. We present some sensitivity analysis of the impact of some parameters on the investment strategies.

1.1 Preliminaries

Let us consider a member whose portfolio is made up of bank security and two stocks which are modelled by the O-U process. We also consider a financial market that is continuously open over an interval $t \in [0, T]$ such that T represents the date of expiration of such investment. Let $\{\mathcal{Z}_1(t), \mathcal{Z}_2(t): t \geq 0\}$ be standard Brownian motion defined on a complete probability space (Ω, F, P) where Ω is a real space, P is a probability measure and F is the filtration which represents the information generated by the two correlated Brownian motions, $\mathcal{Z}_1(t)$ and $\mathcal{Z}_2(t)$.

Let $\mathcal{B}_0(t)$ denote the price of the risk free asset at time t and the model is given as follows

$$\frac{d\mathcal{B}_0(t)}{\mathcal{B}_0(t)} = r dt \quad \mathcal{B}_0(0) = b_0 > 0 \tag{1}$$

where $r > 0$ is the risk free interest rate

Let $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ denote the prices of two different stocks which are described by the O-U process which describes the fluctuation in the stock market prices and the dynamics of the stock Let $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ denote the prices of two different stocks which are described by the O-U process which describes the fluctuation in the

stock market prices and the dynamics of the stock market prices are described by the stochastic differential equations as follows

$$d\begin{pmatrix}\mathcal{B}_1(t)\\ \mathcal{B}_2(t)\end{pmatrix} = \begin{pmatrix} \begin{pmatrix}k_1(n_1-\mathcal{B}_1)\\ k_2(n_2-\mathcal{B}_2)\end{pmatrix}dt \\ +\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}\begin{pmatrix}d\mathcal{Z}_1(t)\\ d\mathcal{Z}_2(t)\end{pmatrix}\end{pmatrix} \qquad (2)$$

where $k_1 > 0$ and $k_2 > 0$ are the recovery rates of the risky assets n_1 and n_2 are the appreciation rates of the two risky assets, σ_{11}, σ_{12}, σ_{21} and σ_{22} are instantaneous volatilities which forms a 2×2 matrix $\sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}$ such that $\sigma\sigma^T$ is positive definite Zhao and Rong (2012).

2.0 Model Formulation and Methodology

2.1 Hamilton Jacobi Bellman Equation

Let π be the optimal investment strategy, when the utility attained is defined by the investor from a given state x at time t as

$$\mathcal{Q}_\pi(t, b_1, b_2, x) = \left(E_\pi\left[U(\mathcal{X}(T)) \,\middle|\, \begin{matrix}\mathcal{B}_1(t) = b_1, \\ \mathcal{B}_2(t) = b_2,\ \mathcal{X}(t) = x\end{matrix}\right]\right), \qquad (3)$$

where t is the time, r is the risk free interest rate and x is the wealth, b_1 and b_2 are the stock market prices of the two risky assets

The objective here is to determine the optimal investment plan and the optimal value function of the investor given as

$$\pi^* \text{ and } \mathcal{Q}(t, b_1, b_2, x) = \sup_{\pi} \mathcal{Q}_\pi(t, b_1, b_2, x)$$

respectively such that

$$\mathcal{Q}_{\pi^*}(t, b_1, b_2, x) = \mathcal{Q}(t, b_1, b_2, x). \qquad (4)$$

Let $\mathcal{X}(t)$ represents the surplus wealth of an investor at time t and let c, ϑ and a represent the tax rate in the financial market, the investor's contributions rate at any given time and the administrative fee respectively. Also we assume that $\frac{k_1(n_1-b_1)}{b_1} - \frac{a}{2} - r > 0$ and $\frac{k_2(n_2-b_2)}{b_2} - \frac{a}{2} - r > 0$ .Therefore, the investor's surplus wealth can be expressed in differential form according to equation 5,

$$d\mathcal{X}(t) = \begin{pmatrix}\mathcal{X}(t)\begin{pmatrix}\pi_0\frac{d\mathcal{B}_0(t)}{\mathcal{B}_0(t)} + \pi_1\frac{d\mathcal{B}_1(t)}{\mathcal{B}_1(t)}\\ +\pi_2\frac{d\mathcal{B}_2(t)}{\mathcal{B}_2(t)}\end{pmatrix}\\ -\vartheta\mathcal{X}(t)dt + cdt\end{pmatrix} \qquad (5)$$

Putting (1) and (2) into (5), we have

$$d\mathcal{X}(t) = \begin{pmatrix}\begin{pmatrix}\mathcal{X}(t)\begin{pmatrix}\pi_1\begin{pmatrix}\frac{k_1(n_1-b_1)}{b_1}\\ -\frac{a}{2}-r\end{pmatrix}\\ +\pi_2\begin{pmatrix}\frac{k_2(n_2-b_2)}{b_2}\\ -\frac{a}{2}-r\end{pmatrix}\end{pmatrix}\\ +(\mathcal{X}(t)(r-\vartheta))+c\end{pmatrix}dt\\ +\mathcal{X}(t)\begin{pmatrix}\left(\frac{\pi_1\sigma_{11}}{b_1}+\frac{\pi_2\sigma_{21}}{b_2}\right)d\mathcal{Z}_1\\ +\left(\frac{\pi_1\sigma_{12}}{b_1}+\frac{\pi_2\sigma_{22}}{b_2}\right)d\mathcal{Z}_2\end{pmatrix}\\ \mathcal{X}(0) = \mathcal{X}_0\end{pmatrix} \qquad (6)$$

where π_0, π_1 and π_2 are the optimal investment plans for the risk-free asset and the two risky assets respectively, such that $\pi_0 = 1 - \pi_1 - \pi_2$.

According to Ihedioha (2020) and applying the Ito's lemma and maximum principle, the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (6) is obtained by maximizing $\mathcal{Q}_{\varphi^*}(t, r, b_1, b_2, x)$ subject to the insurer's wealth as follows

$$\left.\begin{matrix}\mathcal{Q}_t + k_1(n_1-b_1)\mathcal{Q}_{b_1}\\ +k_2(n_2-b_2)\mathcal{Q}_{b_2}\\ +((r-\vartheta)x+c)\mathcal{Q}_x\\ +\frac{1}{2}\mathcal{J}_1\mathcal{Q}_{b_1b_1}+\frac{1}{2}(\mathcal{J}_3)\mathcal{Q}_{b_2b_2}+\mathcal{J}_2\mathcal{Q}_{b_1b_2}\\ +\sup\limits_{\pi_1,\pi_2}\left\{\begin{matrix}\frac{x^2}{2}\begin{pmatrix}\pi_1^2\left(\frac{\mathcal{J}_1}{b_1^2}\right)+\left(\frac{\mathcal{J}_2}{b_1b_2}\right)\pi_1\pi_2\\ +\pi_2^2\left(\frac{\mathcal{J}_3}{b_2^2}\right)\end{pmatrix}\mathcal{Q}_{xx}\\ +x\begin{pmatrix}\begin{pmatrix}\frac{k_1(n_1-b_1)}{b_1}\\ -\frac{a}{2}-r\end{pmatrix}\pi_1\\ +\begin{pmatrix}\frac{k_2(n_2-b_2)}{b_2}\\ -\frac{a}{2}-r\end{pmatrix}\pi_2\end{pmatrix}\mathcal{Q}_x\\ x\left(\left(\frac{\mathcal{J}_1}{b_1}\right)\pi_1+\left(\frac{\mathcal{J}_2}{b_2}\right)\pi_2\right)\mathcal{Q}_{xb_1}\\ x\left(\left(\frac{\mathcal{J}_2}{b_2}\right)\pi_1+\left(\frac{\mathcal{J}_3}{b_2}\right)\pi_2\right)\mathcal{Q}_{xb_2}\end{matrix}\right.\end{matrix}\right\} = 0 \qquad (7)$$

where $\mathcal{J}_1 = (\sigma_{11}^2 + \sigma_{12}^2), \mathcal{J}_2 = (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}), \mathcal{J}_3 = (\sigma_{21}^2 + \sigma_{22}^2)$

Differentiating (7) with respect to π_1 *and* π_2, we obtain the first order maximizing condition for the equation as follows,

$$\pi_1^* = \left(\frac{\begin{bmatrix} \mathcal{J}_2\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right) \\ -\mathcal{J}_3\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\end{bmatrix}}{x(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} \frac{b_1\mathcal{Q}_x}{\mathcal{Q}_{xx}} - \frac{b_1\mathcal{Q}_{xb_1}}{x\mathcal{Q}_{xx}} \right) \quad (8)$$

$$\pi_2^* = \left(\frac{\begin{bmatrix} \mathcal{J}_2\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right) \\ -\mathcal{J}_1\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right)\end{bmatrix}}{x(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} \frac{b_2\mathcal{Q}_x}{\mathcal{Q}_{xx}} - \frac{b_2\mathcal{Q}_{xs_2}}{x\mathcal{Q}_{xx}} \right) \quad (9)$$

Substituting (8) and (9) into (7), yields equation 10,

$$\left.\begin{matrix} \mathcal{Q}_t + k_1(n_1-b_1)\mathcal{Q}_{s_1} + k_2(n_2-b_2)\mathcal{Q}_{b_2} \\ +\left((r-\vartheta)x+c\right)\mathcal{Q}_x + \frac{1}{2}\mathcal{J}_1\mathcal{Q}_{b_1b_1} \\ +\frac{1}{2}\mathcal{J}_3\mathcal{Q}_{b_2b_2} + \mathcal{J}_2\mathcal{Q}_{b_1b_2} \\ +\frac{1}{2}(\mathcal{J}_4-\mathcal{J}_5-\mathcal{J}_6)\frac{\mathcal{Q}_x^2}{\mathcal{Q}_{xx}} \\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\frac{\mathcal{Q}_x\mathcal{Q}_{xb_1}}{\mathcal{Q}_{xx}} \\ -\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right)\frac{\mathcal{Q}_x\mathcal{Q}_{xb_2}}{\mathcal{Q}_{xx}} \\ -\frac{1}{2}\mathcal{J}_1\frac{\mathcal{Q}_{xb_1}^2}{\mathcal{Q}_{xx}} - \frac{1}{2}\mathcal{J}_3\frac{\mathcal{Q}_{xb_2}^2}{\mathcal{Q}_{xx}} - \mathcal{J}_2\frac{\mathcal{Q}_{xb_1}\mathcal{Q}_{xb_2}}{\mathcal{Q}_{xx}} \end{matrix}\right\} = 0 \quad (10)$$

where

$$\begin{cases} \mathcal{J}_1 = (\sigma_{11}^2+\sigma_{12}^2), \mathcal{J}_2 = (\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}), \\ \mathcal{J}_3 = (\sigma_{21}^2+\sigma_{22}^2), \\ \mathcal{J}_4 = \frac{2\mathcal{J}_2\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right)}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}, \\ \mathcal{J}_5 = \frac{\mathcal{J}_3\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)^2}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}, \\ \mathcal{J}_6 = \frac{\mathcal{J}_1\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right)^2}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}, \end{cases}$$

2.2 Legendre Transformation and Dual theory

The differential equation obtained in (10) is a non linear PDE and is somehow complex to solve. In this section, we will introduce the Legendre transformation and dual theory and use it to transform the non linear PDE to a linear PDE.

Theorem 2.1: Let $f: R^n \to R$ be a convex function for $z > 0$, define the Legendre transform

$$W(z) = \max_x\{g(x) - zx\}, \quad (11)$$

The function$W(z)$ is the Legendre dual of the function $g(x)$. Jonsson and Sircir (2002)

Since $g(x)$ is convex, from theorem 3.1, the Legendre transform for the value function $\mathcal{Q}(t, b_1, b_2, x)$ can be defined as follows

$$\hat{\mathcal{Q}}(t, b_1, b_2, z) = \left(\sup\left\{\begin{matrix}\mathcal{Q}(t, b_1, b_2, x) \\ -zx\end{matrix} \,\middle|\, 0 < x < \infty\right\} \\ 0 < t < T\right) \quad (12)$$

where $\hat{\mathcal{Q}}$ is the dual of $\mathcal{Q}$ and $z > 0$ is the dual variable of x.

The value of x where this optimum is achieved is represented by $\ell(t, b_1, b_2, z)$, such that

$$\ell(t, b_1, b_2, z) = \left(\inf\left\{x \,\middle|\, \begin{matrix}\mathcal{Q}(t, b_1, b_2, x) \\ \geq zx + \hat{\mathcal{Q}}(t, b_1, b_2, z)\end{matrix}\right\} \\ 0 < t < T.\right) \quad (13)$$

From equation (13), the function ℓ and $\hat{\mathcal{Q}}$ are very much related and can be refers to as the dual of $\mathcal{Q}$ and are related thus

$$\hat{\mathcal{Q}}(t, b_1, b_2, z) = \mathcal{Q}(t, b_1, b_2, \ell) - z\ell. \quad (14)$$

where

$$\ell(t, b, b_2, z) = x, \mathcal{Q}_x = z, \ \ell = -\hat{\mathcal{Q}}_{z.} \quad (15)$$

differentiating equation 14 with respect to $t, , b_1, b_2$ and x leads to the following,

$$\begin{cases} \mathcal{Q}_t = \hat{\mathcal{Q}}_t, \ \mathcal{Q}_{b_1} = \hat{\mathcal{Q}}_{b_1}, \mathcal{Q}_{b_2} = \hat{\mathcal{Q}}_{b_2}, \\ \mathcal{Q}_x = z, \mathcal{Q}_{b_1x} = \frac{-\hat{\mathcal{Q}}_{b_1z}}{\hat{\mathcal{Q}}_{zz}}, \mathcal{Q}_{b_2x} = \frac{-\hat{\mathcal{Q}}_{b_2z}}{\hat{\mathcal{Q}}_{zz}}, \\ \mathcal{Q}_{xx} = \frac{-1}{\hat{\mathcal{Q}}_{zz}}, \ \mathcal{Q}_{b_1b_1} = \hat{\mathcal{Q}}_{b_1b_1} - \frac{\hat{\mathcal{Q}}_{b_1z}{}^2}{\hat{\mathcal{Q}}_{zz}}, \\ \mathcal{Q}_{b_2b_2} = \hat{\mathcal{Q}}_{b_2b_2} - \frac{\hat{\mathcal{Q}}_{b_2z}{}^2}{\hat{\mathcal{Q}}_{zz}} \end{cases} \quad (16)$$

At terminal time T, we define the dual utility in terms of the original utility function $U(x)$ as

$$\hat{U}(z) = sup\{U(x) - zx \mid 0 < x < \infty\},$$

and

$$G(z) = sup\ \{x \mid U(x) \geq zx + \hat{U}(z).$$

as a result $\hat{\mathcal{Q}}(t, r, b_1, b_2, z) = \mathcal{Q}(t, r, b_1, b_2, \ell) - z\ell$.

$$G(z) = (U')^{-1}(z), \quad (17)$$

where G is the inverse of the marginal utility U and note that $\mathcal{Q}(T, r, b_1, b_2, x) = U(x)$

At terminal time T, we can define

$$\ell(T,r,s_1,s_2,z) = \inf_{x>0}\{x \mid U(x) \ge zx + \hat{\mathcal{Q}}(t,r,b_1,b_2,z)\}\ and$$

$$\hat{\mathcal{Q}}(t,r,b_1,b_2,z) = \sup_{x>0}\{U(x) - zx\}$$

such that

$$\ell(T,r,b_1,b_2,z) = (U')^{-1}(z). \quad (18)$$

Substituting equation 16 into (8), (9) and (10), we have

$$\left.\begin{array}{c} \hat{\mathcal{Q}}_t + k_1(n_1 - b_1)\hat{\mathcal{Q}}_{b_1} \\ +k_2(n_2 - b_2)\hat{\mathcal{Q}}_{b_2} + \big((r-\vartheta)x + c\big)z \\ +\frac{1}{2}\mathcal{J}_1\hat{\mathcal{Q}}_{b_1b_1} + \frac{1}{2}\mathcal{J}_3\hat{\mathcal{Q}}_{b_2b_2} + \mathcal{J}_2\hat{\mathcal{Q}}_{b_1b_2} \\ -\frac{1}{2}(\mathcal{J}_4 - \mathcal{J}_5 - \mathcal{J}_6)z^2\hat{\mathcal{Q}}_{zz} \\ -\left(k_1(n_1 - b_1) - \frac{a}{2}b_1 - rb_1\right)z\hat{\mathcal{Q}}_{b_1z} \\ -\left(k_2(n_2 - b_2) - \frac{a}{2}b_2 - rb_2\right)z\hat{\mathcal{Q}}_{b_2z} \end{array}\right\} = 0 \quad (19)$$

$$\pi_1^* = \left(\frac{\begin{bmatrix}\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_2 \\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_3\end{bmatrix}}{x(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}b_1z\hat{\mathcal{Q}}_{zz} - \frac{b_1\hat{\mathcal{Q}}_{b_1z}}{x}\right) \quad (20)$$

$$\pi_2^* = \left(\frac{\begin{bmatrix}\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2 \\ -\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\end{bmatrix}}{x(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}b_2z\hat{\mathcal{Q}}_{zz} - \frac{b_2\hat{\mathcal{Q}}_{s_2z}}{x}\right) \quad (21)$$

From equation 15 and differentiating equations 19, 20 and 21 with respect to z, we have

$$\left.\begin{array}{c} \ell_t + rb_1\ell_{b_1} + rb_2\ell_{b_2} \\ -\big((r-\vartheta)\ell + c\big) - (r-\tau)z\,\ell_z \\ +\frac{1}{2}\mathcal{J}_1\ell_{b_1b_1} + \frac{1}{2}\mathcal{J}_3\ell_{b_2b_2} + \mathcal{J}_2\ell_{b_1b_2} \\ +\frac{1}{2}(\mathcal{J}_4 - \mathcal{J}_5 - \mathcal{J}_6)z^2\ell_{zz} \\ +\frac{1}{2}(\mathcal{J}_4 - \mathcal{J}_5 - \mathcal{J}_6)z\ell_z \\ -\left(k_1(n_1 - b_1) - \frac{a}{2}b_1 - rb_1\right)z\ell_{b_1z} \\ -\left(k_2(n_2 - b_2) - \frac{a}{2}b_2 - rb_2\right)z\ell_{b_2z} \end{array}\right\} = 0 \quad (22)$$

$$\pi_1^* = \left(\frac{\begin{bmatrix}\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_2 \\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_3\end{bmatrix}}{x(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}b_1z\,\ell_z - \frac{b_1\ell_{b_1}}{x}\right) \quad (23)$$

$$\pi_2^* = \left(\frac{\begin{bmatrix}\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2 \\ -\left(k_2(n_2-b_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\end{bmatrix}}{x(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}b_2z\,\ell_z - \frac{b_2\ell_{b_2}}{x}\right) \quad (24)$$

Our next interest here is to solve equation 22, for ℓ for an investor with logarithm utility, after which we substitute the solution into equations 23 and 24 to obtain the optimal investment strategies for the two risky assets.

3. 0 Optimal Investment Plans for a DC member with Logarithm Utility

From Li *et al.* (2013), Xiao and Yonggui (2020), the logarithm utility function is given as

$$U(x) = \ln x \ , \qquad x > 0$$

Recall from (18),

$$\ell(T,b_1,b_2,z) = (U')^{-1}(z) = \frac{1}{z} \quad (25)$$

Furthermore, we construct a solution for (22) similar to Yonggui (2020), in the form:

$$\begin{cases} \ell(t,b_1,b_2,z) = \frac{1}{z}[e(t,b_1) + f(t,b_2)] \\ \qquad + h(t) \\ e(T,b_1) = \frac{1}{2}, f(T,b_2) = \frac{1}{2},\ h(T) = 0, \end{cases} \quad (26)$$

$$\left.\begin{array}{c} \ell_t = \frac{1}{z}[e_t + f_t] + h_t,\ \ell_{b_1} = \frac{1}{z}e_{b_1}, \\ \ell_{b_2} = \frac{1}{z}f_{b_2}\ell_{b_1b_1} = \frac{1}{z}e_{b_1b_1}, \\ \ell_{b_2b_2} = \frac{1}{z}f_{b_2b_2}, \ell_z = -\frac{1}{z^2}[e + f], \\ \ell_{zz} = \frac{2}{z^3}[e + f], \\ \ell_{b_1z} = -\frac{1}{z^2}e_{s_1}, \ell_{b_2z} = -\frac{1}{z^2}f_{s_2} \end{array}\right\} \quad (27)$$

Substituting (27) into (22), we have

$$\left.\begin{array}{c} [h_t - (r-\tau)h - c] \\ +\frac{1}{z}\begin{bmatrix} e_t + \left(k_1n_1 - \left(k_1 + \frac{a}{2}\right)b_1\right)e_{b_1} \\ +\frac{1}{2}\mathcal{J}_1e_{b_1b_1} \end{bmatrix} \\ +\frac{1}{z}\begin{bmatrix} f_t + \left(k_2n_2 - \left(k_2 + \frac{a}{2}\right)b_2\right)f_{b_2} \\ +\frac{1}{2}\mathcal{J}_3f_{b_2b_2} \end{bmatrix} \end{array}\right\} = 0 \quad (28)$$

Splitting (28), we have

$$\begin{cases} h_t - (r-\vartheta)h - c = 0 \\ \quad h(T) = 0 \end{cases} \quad (29)$$

$$\begin{cases} e_t + \left(\begin{array}{c} k_1n_1 \\ -\left(k_1 + \frac{a}{2}\right)b_1 \end{array}\right)e_{b_1} + \frac{1}{2}\mathcal{J}_1e_{b_1b_1} = 0 \\ \qquad e(T,b_1) = \frac{1}{2} \end{cases} \quad (30)$$

$$\begin{cases} f_t + \left(\begin{array}{c} k_2n_2 \\ -\left(k_1 + \frac{a}{2}\right)b_2 \end{array}\right)f_{b_2} + \frac{1}{2}\mathcal{J}_3f_{b_2b_2} = 0 \\ \qquad f(T,b_2) = \frac{1}{2} \end{cases} \quad (31)$$

Solving equation (29) for h, we obtain

$$h = \frac{c}{\vartheta - r}\left[1 - e^{(r-\tau)(t-T)}\right] \quad (32)$$

Next, we assume a solution for (30) and (31) as follows

$$\begin{cases} e(t, s_1) = A(t) + b_1 B(t), \\ A(T) = \frac{1}{2}, B(T) = 0 \end{cases} \quad (33)$$

$$\begin{cases} f(t, s_2) = M(t) + b_2 N(t), \\ M(T) = \frac{1}{2}, N(T) = 0 \end{cases} \quad (34)$$

Differentiating (33) and (34), we have

$$\begin{cases} e_t = A_t + b_1 B_t, e_{b_1} = B, e_{b_1 b_1} = 0, \\ f_t = M_t + b_2 N_t, f_{b_2} = N, f_{b_2 b_2} = 0, \end{cases} \quad (35)$$

Substituting (35) into (30) and (31), we have

$$A_t + k_1 n_1 + \left(B_t - \left(k_1 + \frac{a}{2}\right) B\right) b_1 = 0 \quad (36)$$

$$M_t + k_1 n_2 + \left(N_t - \left(k_2 + \frac{a}{2}\right) N\right) b_2 = 0 \quad (37)$$

Splitting (36) and (37), we have

$$\begin{cases} A_t + k_1 n_1 = 0 \\ A(T) = \frac{1}{2}, \end{cases} \quad (38)$$

$$\begin{cases} B_t - \left(k_1 + \frac{a}{2}\right) B = 0 \\ B(T) = 0, \end{cases} \quad (39)$$

$$\begin{cases} M_t + k_2 n_2 = 0 \\ M(T) = \frac{1}{2}, \end{cases} \quad (40)$$

$$\begin{cases} N_t - \left(k_2 + \frac{a}{2}\right) N = 0 \\ N(T) = 0, \end{cases} \quad (41)$$

Solving (38), (39), (40) and (41), we have

$$A(t) = \frac{1}{2}, B(t) = 0, M(t) = \frac{1}{2}, N(t) = 0 \quad (42)$$

Substituting (42) into (33) and (34), we have

$$e(t, b_1) = \frac{1}{2} \quad (43)$$

$$f(t, b_2) = \frac{1}{2} \quad (44)$$

Substituting (32), (43) and (44) into (26), we have

$$\ell(t, b_1, b_2, z) = \frac{1}{z} + \frac{c}{\vartheta - r}\left[1 - e^{(r-\tau)(t-T)}\right] \quad (45)$$

Proposition 1 The optimal portfolio strategies for the three assets are given as

$$\pi_1^* = \left(\frac{b_1 \begin{bmatrix} \begin{pmatrix} k_1(n_1 - b_1) \\ -\frac{a}{2} b_1 - r b_1 \end{pmatrix} \mathcal{J}_3 \\ -\begin{pmatrix} k_2(n_2 - b_2) \\ -\frac{a}{2} b_2 - r b_2 \end{pmatrix} \mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1 \mathcal{J}_3 - \mathcal{J}_2^2)} \times \left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) \right) \quad (46)$$

$$\pi_2^* = \left(\frac{b_2 \begin{bmatrix} \begin{pmatrix} k_2(n_2 - s_2) \\ -\frac{a}{2} b_2 - r b_2 \end{pmatrix} \mathcal{J}_1 \\ -\begin{pmatrix} k_1(n_1 - b_1) \\ -\frac{a}{2} b_1 - r b_1 \end{pmatrix} \mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1 \mathcal{J}_3 - \mathcal{J}_2^2)} \times \left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) \right) \quad (47)$$

Proof

From (45), we have

$$\begin{cases} \ell_{b_1} = \ell_{b_2} = 0, \ell_z = -\frac{1}{z^2}, \\ \frac{1}{z} = x - \frac{c}{\vartheta - r}\left[1 - e^{(r-\vartheta)(t-T)}\right] \end{cases} \quad (48)$$

Substituting (48) into (23) and (24), proposition 1 is proved.

4.0. Sensitivity Analysis

In this section, we present some lemmas to demonstrate the impact of some parameters on the optimal investment plan.

Proposition 2 Suppose $a > 0, n_1 > 0$, $r > 0, \vartheta > 0, t \in [0, T], k_1 > 0, k_2 > 0, , b_1 > 0$, $b_2 > 0$, $x > 0$, $(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2) > 0$, and $T - t > 0$ then

(i) $\frac{\partial \pi_1^*}{\partial c} < 0$ (ii) $\frac{\partial \pi_1^*}{\partial a} < 0$ (iii) $\frac{\partial \pi_1^*}{\partial \vartheta} < 0$ (iv) $\frac{\partial \pi_1^*}{\partial n_1} > 0$ (v) $\frac{\partial \pi_1^*}{\partial t} < 0$

Proof

Recall that

$$\pi_1^* = \left(\frac{b_1 \begin{bmatrix} \begin{pmatrix} k_1(n_1 - b_1) \\ -\frac{a}{2} b_1 - r b_1 \end{pmatrix} \mathcal{J}_3 \\ -\begin{pmatrix} k_2(n_2 - b_2) \\ -\frac{a}{2} b_2 - r b_2 \end{pmatrix} \mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1 \mathcal{J}_3 - \mathcal{J}_2^2)} \times \left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) \right)$$

(i) $$\frac{\partial \pi_1^*}{\partial c} = \left(-\frac{1}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right] \times \frac{b_1 \begin{bmatrix} \left(k_1(n_1 - b_1) - \frac{a}{2} b_1 - r b_1\right) \mathcal{J}_3 \\ -\left(k_2(n_2 - b_2) - \frac{a}{2} b_2 - r b_2\right) \mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1 \mathcal{J}_3 - \mathcal{J}_2^2)} \right)$$

Since $\frac{1}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right] > 0$ and $\frac{b_1 \begin{bmatrix} \left(k_1(n_1 - b_1) - \frac{a}{2} b_1 - r b_1\right) \mathcal{J}_3 \\ -\left(k_2(n_2 - b_2) - \frac{a}{2} b_2 - r b_2\right) \mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1 \mathcal{J}_3 - \mathcal{J}_2^2)} > 0$

Then

$$\frac{\partial \pi_1^*}{\partial c} = \left(-\frac{1}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right] \times \frac{b_1 \begin{bmatrix} \left(k_1(n_1 - b_1) - \frac{a}{2} b_1 - r b_1\right) \mathcal{J}_3 \\ -\left(k_2(n_2 - b_2) - \frac{a}{2} b_2 - r b_2\right) \mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1 \mathcal{J}_3 - \mathcal{J}_2^2)} \right) < 0$$

Therefore $\frac{\partial \pi_1^*}{\partial c} < 0$

(ii) $\frac{\partial \pi_1^*}{\partial a} = \begin{pmatrix} -\frac{[\mathscr{b}_1^2 \mathcal{J}_3 - \mathscr{b}_1 \mathscr{b}_2 \mathcal{J}_2]}{2(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \\ \times \left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) \end{pmatrix}$,

Since $[\mathscr{b}_1^2 \mathcal{J}_3 - \mathscr{b}_1 \mathscr{b}_2 \mathcal{J}_2] > 0$ and

$\left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) > 0$

Then

$$\frac{\partial \pi_1^*}{\partial a} = \begin{pmatrix} -\frac{[\mathscr{b}_1^2 \mathcal{J}_3 - \mathscr{b}_1 \mathscr{b}_2 \mathcal{J}_2]}{2(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \\ \times \left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) \end{pmatrix} < 0$$

Therefore $\frac{\partial \pi_1^*}{\partial a} < 0$

(iii) $$\frac{\partial \pi_1^*}{\partial \vartheta} = \begin{pmatrix} -\begin{pmatrix} \frac{c}{x\vartheta^2} \\ + \frac{c(T-t)}{(r-\vartheta)x} \end{pmatrix}\left[1 + e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{\mathscr{b}_1\begin{bmatrix} \begin{pmatrix} k_1(n_1 - \mathscr{b}_1) \\ -\frac{a}{2}\mathscr{b}_1 - r\mathscr{b}_1 \end{pmatrix}\mathcal{J}_3 \\ -\begin{pmatrix} k_2(n_2 - \mathscr{b}_2) \\ -\frac{a}{2}\mathscr{b}_2 - r\mathscr{b}_2 \end{pmatrix}\mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \end{pmatrix}$$

Since $\left(\frac{c}{x\vartheta^2} + \frac{c(T-t)}{(r-\vartheta)x}\right)\left[1 + e^{(r-\vartheta)(t-T)}\right] > 0$ and $\frac{\mathscr{b}_1\begin{bmatrix} \left(k_1(n_1 - \mathscr{b}_1) - \frac{a}{2}\mathscr{b}_1 - r\mathscr{b}_1\right)\mathcal{J}_3 \\ -\left(k_2(n_2 - \mathscr{b}_2) - \frac{a}{2}\mathscr{b}_2 - r\mathscr{b}_2\right)\mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} > 0$

Then

$$\frac{\partial \pi_1^*}{\partial \vartheta} = \begin{pmatrix} -\begin{pmatrix} \frac{c}{x\vartheta^2} \\ + \frac{c(T-t)}{(r-\vartheta)x} \end{pmatrix}\left[1 + e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{\mathscr{b}_1\begin{bmatrix} \begin{pmatrix} k_1(n_1 - \mathscr{b}_1) \\ -\frac{a}{2}\mathscr{b}_1 - r\mathscr{b}_1 \end{pmatrix}\mathcal{J}_3 \\ -\begin{pmatrix} k_2(n_2 - \mathscr{b}_2) \\ -\frac{a}{2}\mathscr{b}_2 - r\mathscr{b}_2 \end{pmatrix}\mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \end{pmatrix} < 0$$

Therefore $\frac{\partial \pi_1^*}{\partial \vartheta} < 0$

(iv) $\frac{\partial \pi_1^*}{\partial n_1} = \begin{pmatrix} \frac{[\mathscr{b}_1 k_1 \mathcal{J}_3]}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \\ \times \left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) \end{pmatrix}$

Since $[\mathscr{b}_1 k_1 \mathcal{J}_3] > 0$ and

$\left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) > 0$

Then

$$\frac{\partial \pi_1^*}{\partial n_1} = \begin{pmatrix} \frac{[\mathscr{b}_1 k_1 \mathcal{J}_3]}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \\ \times \left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) \end{pmatrix}$$

Therefore $\frac{\partial \pi_1^*}{\partial n_1} > 0$

(v) $$\frac{\partial \pi_1^*}{\partial t} = \begin{pmatrix} -\frac{c}{x}\left[e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{\mathscr{b}_1\begin{bmatrix} \left(k_1(n_1 - \mathscr{b}_1) - \frac{a}{2}\mathscr{b}_1 - r\mathscr{b}_1\right)\mathcal{J}_3 \\ -\left(k_2(n_2 - \mathscr{b}_2) - \frac{a}{2}\mathscr{b}_2 - r\mathscr{b}_2\right)\mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \end{pmatrix}$$

Since $\frac{c}{x}\left[e^{(r-\vartheta)(t-T)}\right] > 0$ and $\frac{\mathscr{b}_1\begin{bmatrix} \left(k_1(n_1 - \mathscr{b}_1) - \frac{a}{2}\mathscr{b}_1 - r\mathscr{b}_1\right)\mathcal{J}_3 \\ -\left(k_2(n_2 - \mathscr{b}_2) - \frac{a}{2}\mathscr{b}_2 - r\mathscr{b}_2\right)\mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} > 0$

Then

$$\frac{\partial \pi_1^*}{\partial t} = \begin{pmatrix} -\frac{c}{x}\left[e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{\mathscr{b}_1\begin{bmatrix} \left(k_1(n_1 - \mathscr{b}_1) - \frac{a}{2}\mathscr{b}_1 - r\mathscr{b}_1\right)\mathcal{J}_3 \\ -\left(k_2(n_2 - \mathscr{b}_2) - \frac{a}{2}\mathscr{b}_2 - r\mathscr{b}_2\right)\mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \end{pmatrix} < 0$$

Therefore $\frac{\partial \pi_1^*}{\partial t} < 0$

Proposition 3 Suppose $a > 0, n_1 > 0, n_2 > 0, r > 0, \vartheta > 0, t \in [0, T], k_1 > 0, k_2 > 0,, \mathscr{b}_1 > 0, \mathscr{b}_2 > 0, x > 0, (\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2) > 0$, and $T - t > 0$ then

(i) $\frac{\partial \pi_1^*}{\partial c} < 0$ (ii) $\frac{\partial \pi_1^*}{\partial a} < 0$ (iii) $\frac{\partial \pi_1^*}{\partial \vartheta} < 0$ (iv) $\frac{\partial \pi_1^*}{\partial n_1} > 0$ (v) $\frac{\partial \pi_1^*}{\partial t} < 0$

Proof

Recall that

$$\pi_2^* = \begin{pmatrix} \frac{\mathscr{b}_2\begin{bmatrix} \begin{pmatrix} k_2(n_2 - \mathscr{s}_2) \\ -\frac{a}{2}\mathscr{b}_2 - r\mathscr{b}_2 \end{pmatrix}\mathcal{J}_1 \\ -\begin{pmatrix} k_1(n_1 - \mathscr{b}_1) \\ -\frac{a}{2}\mathscr{b}_1 - r\mathscr{b}_1 \end{pmatrix}\mathcal{J}_2 \end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3 - \mathcal{J}_2^2)} \\ \times \left(1 - \frac{c}{(r-\vartheta)x}\left[1 + e^{(r-\vartheta)(t-T)}\right]\right) \end{pmatrix}$$

(i) $$\frac{\partial \pi_2^*}{\partial c} = \begin{pmatrix} -\frac{1}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{b_2\begin{bmatrix}\left(k_2(n_2-s_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2\end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}\end{pmatrix}$$

Since $\frac{1}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right] > 0$ and $\frac{b_2\begin{bmatrix}\left(k_2(n_2-s_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2\end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} > 0$

Then

$$\frac{\partial \pi_2^*}{\partial c} = \begin{pmatrix} -\frac{1}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{b_2\begin{bmatrix}\left(k_2(n_2-s_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2\end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}\end{pmatrix} < 0$$

Therefore $\frac{\partial \pi_2^*}{\partial c} < 0$

(ii) $$\frac{\partial \pi_2^*}{\partial a} = \begin{pmatrix} -\frac{[b_2^2\mathcal{J}_1-b_1b_2\mathcal{J}_2]}{2(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} \\ \times\left(1-\frac{c}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right]\right)\end{pmatrix}$$

Since $[b_2^2\mathcal{J}_1 - b_1b_2\mathcal{J}_2] > 0$ and

$\left(1-\frac{c}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right]\right) > 0$

Then

$$\frac{\partial \pi_2^*}{\partial a} = \begin{pmatrix} -\frac{[b_2^2\mathcal{J}_1-b_1b_2\mathcal{J}_2]}{2(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} \\ \times\left(1-\frac{c}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right]\right)\end{pmatrix} < 0$$

Therefore $\frac{\partial \pi_2^*}{\partial a} < 0$

(iii) $$\frac{\partial \pi_2^*}{\partial \vartheta} = \begin{pmatrix} -\left(\frac{c}{x\vartheta^2}+\frac{c(T-t)}{(r-\vartheta)x}\right)\left[1+e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{b_2\begin{bmatrix}\left(k_2(n_2-s_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2\end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}\end{pmatrix}$$

Since $\left(\frac{c}{x\vartheta^2}+\frac{c(T-t)}{(r-\vartheta)x}\right)\left[1+e^{(r-\vartheta)(t-T)}\right] > 0$ and $\frac{b_2\begin{bmatrix}\left(k_2(n_2-s_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2\end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} > 0$

Then

$$\frac{\partial \pi_2^*}{\partial \vartheta} = \begin{pmatrix} -\left(\frac{c}{x\vartheta^2}+\frac{c(T-t)}{(r-\vartheta)x}\right)\left[1+e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{b_2\begin{bmatrix}\left(k_2(n_2-s_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2\end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}\end{pmatrix} < 0$$

Therefore $\frac{\partial \pi_2^*}{\partial \vartheta} < 0$

(iv) $$\frac{\partial \pi_2^*}{\partial n_2} = \begin{pmatrix} \frac{[b_2k_2\mathcal{J}_1]}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} \\ \times\left(1-\frac{c}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right]\right)\end{pmatrix}$$

Since $[b_2k_2\mathcal{J}_1] > 0$ and

$$\left(1-\frac{c}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right]\right) > 0$$

Then

$$\frac{\partial \pi_2^*}{\partial n_2} = \begin{pmatrix} \frac{[b_2k_2\mathcal{J}_1]}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} \\ \times\left(1-\frac{c}{(r-\vartheta)x}\left[1+e^{(r-\vartheta)(t-T)}\right]\right)\end{pmatrix} > 0$$

Therefore $\frac{\partial \pi_2^*}{\partial n_2} > 0$

(v) $$\frac{\partial \pi_2^*}{\partial t} = \begin{pmatrix} -\frac{c}{x}\left[e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{b_2\begin{bmatrix}\left(k_2(n_2-s_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2\end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)}\end{pmatrix}$$

Since $\frac{c}{x}\left[e^{(r-\vartheta)(t-T)}\right] > 0$ and $\frac{b_2\begin{bmatrix}\left(k_2(n_2-s_2)-\frac{a}{2}b_2-rb_2\right)\mathcal{J}_1\\ -\left(k_1(n_1-b_1)-\frac{a}{2}b_1-rb_1\right)\mathcal{J}_2\end{bmatrix}}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} > 0$

Then

$$\frac{\partial \pi_2^*}{\partial t} = \left(\begin{array}{c} -\frac{c}{x}\left[e^{(r-\vartheta)(t-T)}\right] \\ \times \frac{\mathscr{b}_2\left[\begin{array}{c}\left(k_2(\mathscr{n}_2-\mathscr{s}_2)-\frac{a}{2}\mathscr{b}_2-r\mathscr{b}_2\right)\mathcal{J}_1 \\ -\left(k_1(\mathscr{n}_1-\mathscr{b}_1)-\frac{a}{2}\mathscr{b}_1-r\mathscr{b}_1\right)\mathcal{J}_2\end{array}\right]}{(\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2)} \end{array} \right) < 0$$

Therefore $\frac{\partial \pi_2^*}{\partial t} < 0$

Here, we examine the effect of some sensitive parameters on the optimal investment strategies. From proposition 2 and 3, it is observed that the optimal investment strategies for the risky assets are inversely proportional to contribution rate c, tax rate imposed on the invested fund ϑ, proportional administration fee a, investment time t, and directly proportional to the appreciation rate of the risky assets.We also observed that the proportion of investment in the two risky assets increases as the risky asset value appreciate; this is so since we know that as the value of stock market prices appreciate, the investor will love to invest more to make more dividend and such investments are attractive and lucrative at the same time hence plays a vital role in the mind of the investor while taking his or her decision. On the contrary, the optimal investment plans are decreasing functions of taxes imposed on investment in the risky assets; this is because high tax rate discourages investment hence the investor may be discouraged in investing in assets with high taxation rate and may likely move on to invest more in a lesser or non-taxable asset. Also, it is observed that the as the administration fee increases, the proportion of the investor's wealth to be invested in the risky assets decreases; this so because most pension administrators demand some fees for portfolio management and since investment in stock is highly volatile, the administrative fee may also be high and, in some cases, discouraging for the investor to continue investing in the risky assets. Similarly, we observed as the contribution rate of the member increases the fund manager prefer invest more in the risk free asset and less in the risky assets. This is so because with more money in the system, the fund manager may be unwilling to take more risk hence a decrease in investment in the risky assets. Finally, we observed that as retirement time draw closer, the member shows unwillingness to invest in the two risky assets.

5.0 Conclusion

In general, the optimal investment plans of a member with logarithm utility in defined pension (DC) scheme with proportional administrative fee and tax on invested funds was studied. A portfolio of one risk free asset and two risky assets was considered where the market prices of the two risky assets were modelled by the O-U process. The Legendre transformation and dual theory with asymptotic expansion technique was used to find closed form solutions of the optimal investment plans. Finally, the impact of some sensitive parameters of the optimal investment plans was presented with observations that aside from the fluctuation in the stock market prices which can be seen in equation (2), the optimal investment plans for the risky assets are inversely proportional to contribution rate c, tax rate imposed on the invested fund ϑ, proportional administration fee a, investment time t, and directly proportional to the appreciation rate of the risky assets. in conclusion, this research will enable fund managers in a DC pension scheme to make a relatively right choice of investment strategies when dealing with markets modelled by stochastic volatilities such the O-U process which shows that there exist fluctuations in the stock market prices.

6.0 Acknowledgements

The authors are very grateful to anonymous referees for their careful and diligent reading of the paper and helpful suggestions.

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Conflict of Interest

The authors declared no conflict of interest