The Weibull-Power Lomax Distribution: Properties and Application

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Abstract

The power lomax distribution is a very good model in modelling real life financial and reliability data. However, we extend the power lomax distribution with the Weibull G family in order to increase its flexibility and usage. Therefore, in this paper a new five-parameter distribution is introduced called the Weibull-Power Lomax distribution. The structural properties of the proposed distribution such as hazard function, moments, probability weighted moments, distribution of order statistics and quantile function are derived. The maximum likelihood estimation technique is employed to estimate the parameters of the proposed distribution. To also prove the increased flexibility and performance of the distribution, it is used to model 63 observations of strengths of 1.5cm glass fibers, along with its other competing distributions. The results indicate that the proposed distribution fit the glass fiber data and performs much better than its competitors.

Keywords: Weibull-G, Power-Lomax Distribution, Hazard function, Likelihood estimation.

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1 Introduction

The Lomax or Pareto II, distribution by Lomax (1954) was originally used for modeling business failure data, however, it has been widely applied in a variety of contexts. Hassan and Al-Ghamdi (2009) mentioned that it is used for reliability modelling and life testing. The distribution has been used for modeling different data studied by so many authors: - Harris (1968) used Lomax distribution for income and wealth data; Atkinson and Harrison (1978) used the distribution for modelling business failure data, while Corbelini et al. (2007) used it to model firm size and queuing problems. The lomax distribution has also found application in the biological sciences and in modelling the distribution of the sizes of computer files on

servers Holland et al. (2006). Some authors, such as Bryson (1974), have suggested the use of this distribution as an alternative to the exponential distribution when the data are heavy-tailed. This distribution has established wide application in numerous fields especially in engineering, actuarial science, medical and biological sciences, in lifetime and reliability modelling Tahir et al. (2015b). The Lomax distribution is suggested to be a good alternative to exponential, gamma, and Weibull distributions Tahir et al. (2015b).
Power Lomax distribution belongs to inverted family of distributions and known to be very flexible in fitting data sets, where the non-monotonicity of failure rate has been realized Rady et al. (2016). Power Lomax was applied to bladder Cancer data by Rady et al. (2016). The Weibull Generalized (Weibull-G) family, is very flexible and can be used for analysing life time data of different types Bourguignon et al. (2014). The main reasons for using the Weibull G family are to make the kurtosis more flexible (compared to the baseline model) and also to construct heavy-tailed distributions that are not long-tailed for modelling real data Tahir et al. (2015a).

In this paper we extend the Power Lomax distribution using Weibull-G family of probability distributions, in order to increase the flexibility of the baseline Power Lomax distribution. The rest of the paper is outlined as follows. In Section 2, we define the cumulative distribution function (cdf), probability density function (pdf), hazard function and quantile function of the proposed Weibull Power Lomax (W-PL) distribution. Section 3 provides the mixture distribution of the proposed distribution. Some statistical properties of the distribution are discussed in Section 4. Section 5 discusses the estimation methods. The glass fibers data are analyzed in Section 6, while Section 7 concludes the paper.

2 The Proposed Weibull-Power Lomax Distribution (WPLD)

2.1 The WG- Distribution

Given a continuous baseline distribution function $G(x,\xi)$ and probability density function $g(x,\xi)$, Bourguignon et al. (2014)Bourguignon, defined the cdf of the Weibull-G family of distributions, $F(x;\rho,\varphi,\xi)$ as

$$F(x;\rho,\varphi,\xi) == 1 - exp\left\{-\rho\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\varphi}\right\} \quad (1)$$
$$x \in D \subseteq \Re;\ \rho,\ \varphi > 0,$$

where baseline cdf depends on a parameter vector ξ. The family pdf $f(x;\rho,\varphi,\xi)$ is obtained from (1) as,

$$f(x;\rho, \quad \varphi \quad \xi) \quad = \quad \rho\varphi g(x;\xi)\frac{G(x;\xi)^{\varphi-1}}{\bar{G}(x;\xi)^{\varphi+1}}exp\left\{-\rho\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\varphi}\right\} \quad (2)$$

2.2 The Power Lomax Distribution as a Baseline Distribution

Abdul-Moniem (2017) define the probability density function and the cumulative distribution function of the Power Lomax Distribution as (3) and (4) respectively.

$$g(x;\alpha,\ \beta,\ \lambda)\ =\ \alpha\beta\lambda^{-1}x^{\beta-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+1)};x\ >\ 0,(\alpha,\beta,\lambda\ >\ 0) \quad (3)$$

$$G(x;\alpha,\ \beta,\ \lambda)\ = 1-\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha} \quad (4)$$

Where, α, β, are shape parameters, while λ, is a scale parameter.

3 The Weibull-Power Lomax Distribution (WPL)

This is the five parameter, distribution, written as WPLD (ξ) with the parameter vector $(\xi) = (\rho,\varphi,\alpha,\beta,\lambda)$. Now inserting(4) in (3) yields the WPL-cdf

$$F(x;\rho,\varphi,\alpha,\beta,\lambda,) = 1-\exp\left\{-\rho\left[\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}-1\right]^{\varphi}\right\} \quad (5)$$

The pdf corresponding to(5) is given by

$$f(x;\rho,\varphi,\alpha,\beta,\lambda) = \frac{\rho\varphi\alpha\beta}{\lambda}x^{\beta-1}\left[1+\frac{x^{\beta}}{\lambda}\right]^{(\alpha-1)}\left(\left[1+\frac{x^{\beta}}{\lambda}\right]^{\alpha}-1\right)^{\varphi-1} \times\exp\left\{-\rho\left(\left[1+\frac{x^{\beta}}{\lambda}\right]^{\alpha}-1\right)^{\varphi}\right\} \quad (6)$$

where ρ,φ, and $\alpha > 0, \beta > 0$ are the shape parameters while λ is the scale parameter. Henceforth, we denote a random variable X having pdf (6) by X$\sim$WpL(a,b, α,β,λ).

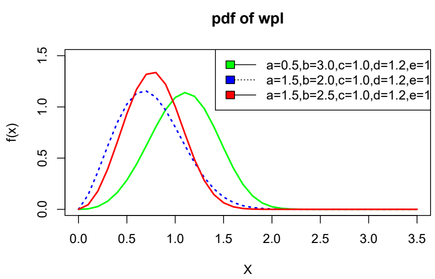

Figure 1 The Graph of the Probability density function of WPL distribution, at different values of the parameters

The graph of the pdf and cdf of Weibull-Power Lomax Distribution, with different parameter values are given in Fig (1) and Fig (2), respectively. The proposed distribution appears to be right Skewed and flexible heavy/fat tailed distribution as can be glanced by varying the shape parameter values.

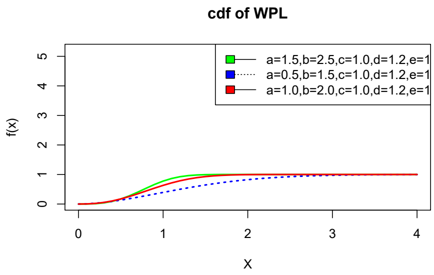

Figure 2 The Graph of the Cumulative distribution function of WPL distribution, at different values of the parameters

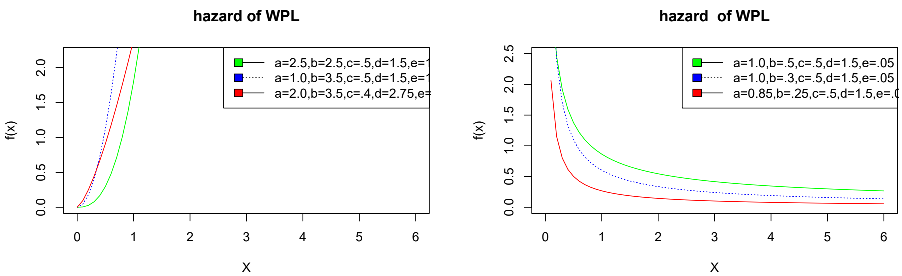

Figure 3 Hazard rate function of WPL distribution

3.1 Hazard Rate

The hazard function which has an important application in survival (reliability) analysis, and is defined by:

$$h(x) = \frac{g(x)}{1 - G(x)} \tag{7}$$

Hence, the hazard function for Weibull-Power Lomax Distribution is given by 8. The hazard function is the probability of failure in an infinitesimally small time period between$x$ and $x + \partial(x)$ given that the subject has survived up to time x . The graph of the hazard function of Weibull-Power Lomax Distribution is shown in fig 3.

$$h(x; \rho, \varphi, \alpha, \beta, \lambda) = \frac{\rho\varphi\alpha\beta}{\lambda} x^{\beta-1} \left[1 + \frac{x^{\beta}}{\lambda}\right]^{(\alpha-1)} \left(\left[1 + \frac{x^{\beta}}{\lambda}\right]^{\alpha} - 1\right)^{\varphi-1} \tag{8}$$

3.2 Quantile Function

The Quantile Function $Q(u)$, is used to partition probability distributions. It can also be used to obtain median of a distribution and for simulation of random samples of different sizes. Quantile Function can be obtained by taking the inverse of the cumulative distribution function of a distribution.

3.2.1 Lemma 1:

Let the random variable u be uniformly distributed on the interval $(0,1)$ and define another random variable as

$$y = \frac{1}{1 + \left\{-1\frac{1}{\rho}\ln(1-u)\right\}^{\frac{-1}{\varphi}}} \tag{9}$$

then the Quantile function $Q(u)$ of the Weibull Power Lomax Distribution WPLD $(\varphi, \rho, \alpha, \beta, \lambda)$ is given by

$$Q(u) = \left\{\lambda\left[(1-y)^{\frac{-1}{\alpha}} - 1\right]\right\}^{\frac{1}{\beta}} \tag{10}$$

3.2.2 Lemma 2:

Let the random variable u follow a uniform distribution on the interval $(0,1)$, then the Quantile function $Q(u)$ of the Weibull Power Lomax Distribution WPLD $(\varphi, \rho, \alpha, \beta, \lambda)$ is given by

$$Q(u) = \left\{\lambda\left[\left(1 + \left\{-\frac{1}{\rho}\ln(1-u)\right\}^{\frac{1}{\varphi}}\right)^{\frac{1}{\alpha}} - 1\right]\right\}^{\frac{1}{\beta}} \tag{11}$$

The median of Weibull Power Lomax Distribution WPLD can be obtained by substituting $u = \frac{1}{2}$ in equation (11) which gives

$$Q(u) = \left\{\lambda\left[\left(1 + \left\{-\frac{1}{\rho}\ln(1-\frac{1}{2})\right\}^{\frac{1}{\varphi}}\right)^{\frac{1}{\alpha}} - 1\right]\right\}^{\frac{1}{\beta}} \tag{12}$$

4 Mixture Representation

The WPL density can be expressed as

$$f(x;\rho,\varphi,\alpha,\beta,\lambda) = \rho\varphi g(x)\frac{G(x)^{\varphi-1}}{\bar{G}(x)^{\varphi+1}}\exp\left\{-\rho\left[\frac{G(x)}{\bar{G}(x)}\right]^{\beta}\right\} \tag{13}$$

Inserting (3) and (4) in (13), we obtain

$$f(x;\rho,\varphi,\alpha,\beta,\lambda) = \frac{\rho\varphi\alpha\beta}{\lambda}x^{\beta-1}\left[1+\frac{x^{\beta}}{\lambda}\right]^{(\alpha-1)}\left(\left[1+\frac{x^{\beta}}{\lambda}\right]^{\alpha}-1\right)^{\varphi-1} \times \exp\left\{-\rho\left(\left[1+\frac{x^{\beta}}{\lambda}\right]^{\alpha}-1\right)^{\varphi}\right\} \tag{14}$$

for us to obtain a simple form for the WPL pdf,we can expand (14) in power series.
Let $A = \exp\left\{-\rho\left(\left[1+\frac{x^\beta}{\lambda}\right]^\alpha - 1\right)^\varphi\right\}$

$$A = \sum_{k=0}^{\infty} \frac{(-1)^k \rho^k}{k!} \left\{\left(\left[1+\frac{x^\beta}{\lambda}\right]^\alpha - 1\right)^{\varphi k}\right\}$$

Inserting this expansion in (14) and, after some simplification, we obtain

$$f(\rho,\varphi,\alpha,\beta,\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{k+1}}{k!} \left(\varphi\alpha\beta\lambda^{-1}\right) x^{\beta-1} \times \left[1+\frac{x^{\beta-1}}{\lambda}\right]^{(\alpha-1)} \times \left(\left[1+\frac{x^\beta}{\lambda}\right]^\alpha - 1\right)^{\varphi(k+1)-1} \tag{15}$$

After a power series expansion the last term in equation (15) becomes

$$\sum_{j=0}^{(k+1)\varphi-1} (-1)^j \binom{[(k+1)\varphi-1]}{j} \left\{\left[1+(\frac{x^\beta}{\lambda})\right]\right\}^{\alpha j}$$

putting the two results together we have,

$$f(\rho,\varphi,\alpha,\beta\lambda) = \underbrace{\frac{\sum_{k=0}^{\infty}(-1)^k\rho^k \sum_{j=0}^{(k+1)\varphi-1}(-1)^j\binom{[(k+1)\varphi-1]}{j}}{k!}}_{v_{k,j}} \underbrace{\times \left(\rho\varphi\alpha\beta\lambda^{-1}\right) x^{\beta-1}\left[1+\frac{x^\beta}{\lambda}\right]^{[\alpha(j+1)-1]}}_{g[\alpha(j+1)-1](x)} \tag{16}$$

The above equation can be written as

$$f(\rho,\varphi,\alpha,\beta\lambda) = \sum_{k,=o}^{\infty} \sum_{j=0}^{(k+1)\varphi-1} V_k g\left[\alpha(j+1)-1\right](x) \tag{17}$$

Equation (17) shows that the WPL density function has a double mixture representation of Exponential densities. So, several of it's structural properties can derived from those of Exponential distribution. The coefficients $v_{k,j}$ depends on the generator parameters.

5 some statistical properties

5.1 moments

Most of the features and characteristics of a distribution can be studied through moments (for example Kurtosis, skewness and dispersion).

5.1.1 Lemma 3

If X $\sim$ Wpl(a,b,α,β,λ) then the r^{th} moments of X is given as

$$\mu_r^{'} = E(X^r) = \int_0^\infty X^r f(x)dx \tag{18}$$

From (17) the $f(x)$ of WPL is given as,

$$f(\rho,\varphi,\alpha,\beta\lambda) = \sum_{k,=o}^{\infty}\sum_{j=0}^{(k+1)\varphi-1} V_k g\left[\alpha(j+1)-1\right](x)$$

$$\mu_r^{'} = E(X^r) = \sum_{k,=0}^{\infty}\sum_{j=0}^{(k+1)\varphi-1} V_{k,j}\int_0^\infty X^r g\alpha[(j+1)-1](x)dx \tag{19}$$

$$(for\, r \leq \alpha)$$

$$\mu_r^{'} = \lambda^{\frac{r}{\beta}}\sum_{k=0}^{r}\sum_{j=0}^{(k+1)\varphi-1}(-1)^j V_{k,j} B\left[\frac{r}{\beta}+1, \left((-\alpha(j+1)) - \left(\frac{r}{\beta}\right)\right)\right] \tag{20}$$

Where $V_{k,j}$ is given in (16).

5.2 Moment Generating Function

Theorem 1. *If X $\sim$ Wpl$(\rho,\varphi,\alpha,\beta,\lambda)$ then the moment generating function(mgf) of X is given as,*

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^1 \tag{21}$$

Proof 1. *By definition, the mgf of a variable X with density f(x) is given as*

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{22}$$

By substituting the value of $f(x)$ from (17) in (22) we've

$$V_{k,j}\int_0^\infty x^{\beta-1}\left(1+\frac{x^\beta}{\lambda}\right)^{-[1-\alpha(j+1)]} e^{tx}dx \tag{23}$$

After some algebra and simplification (23) can be written as,

$$\sum_{n=0}^{\infty} V_{k,j}(-1)n!\frac{\binom{1-\alpha(j+1)}{n}}{\lambda^{n+1}}\int_0^\infty x^{\beta(n+1)-1}e^{tx}dx \tag{24}$$

$$M_x(t) = \sum_{n=0}^{\infty} V_{k,j}(-1)n!\frac{\binom{1-\alpha(j+1)}{n}}{\lambda^{n+1}}\left[(-t)^{-(n+1)\beta}, \Gamma\left(\beta\left(n+1\right)\right)\right] \tag{25}$$

$$for \beta(n+1) > 0, and\, t < 0.$$

$V_{k,j}$ is given in equation (17)

5.3 Probability Weighted Moments

The probability weighted moments (pwms) are very useful in driving estimators of the distribution parameters and quantiles of generalized distributions. They have low variance and non severe bias, and they are comparably favourable with maximum likelihood estimators.
the $(q,r)^{th}$ pwm of X for $(r \geq 1, q \geq 0)$ is formally defined by

$$P_{r,q} = E\left[x^r \, F(x)^q\right] = \int_0^\infty F(x)^q f(x)dx \tag{26}$$

So from (5) we can write,

$$F(x:\rho,\varphi,\alpha,\beta,\lambda)^q = \sum_{i=0}^{\infty}(-1)^i \binom{q}{i} \times \exp\left\{\rho i\left\{\left(1+\frac{x^\beta}{\lambda}\right)^\alpha - 1\right\}^\varphi\right\} \tag{27}$$

Then after some algebra from (5) and (6) we can express

$$P_{r,q} = \sum_{q,i=0}^{\infty}(-1)^i \frac{\binom{q}{i}}{i+1}\int_o^\infty x^r f(x:(i+1)\rho,\varphi,\alpha,\beta,\lambda)dx$$

by using (19) we can obtain for $(r < \alpha)$

$$p_{r,q} = \lambda^{\frac{r}{\beta}} \sum_{i=0}^{\infty}\sum_{j=0}^{(k+1)\varphi-1}(-1)^{i+j}\frac{\binom{q}{i}}{i+1}Q_{k,j}\sum_{k=0}^{r} B\left[\frac{r}{\beta}+1, \quad \left((-\alpha(j+1)) - \frac{r}{\beta}\right)\right] \tag{28}$$

where

$$Q_{k,j} = \frac{\sum_{k=0}^{\infty}(-1)^k \rho^k (1+i)^k \sum_{j=0}^{(k+1)\varphi-1}(-1)^j \binom{[(k+1)\varphi-1]}{j}}{k!}$$

5.4 Order Statistics

Order statistics are used in various areas of statistical theories, for example, identifying of outliers in quality control process. In this section we derive the the density of the i^{th} order statistics expression, for the pdf of the Weibull power Lomax distribution.
Let $X_1, X_2, ..., X_n$ be a random sample of size n from a continuous population having a pdf f(x) and distribution function (cdf) F(x) Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the corresponding order statistics (OS). The pdf of $X_{i:n}$ the i^{th} OS is given by Tahir et al. (2015b), as,

$$f_{i:n}(x) = \frac{f(x)}{B(j, n-i+1)}\sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} F(x)^{i+j-1} \tag{29}$$

Thus we can write

$$F(x)^{i+j-1} = \sum_{t=0}^{\infty}(-1)^t \binom{i+j-i}{t} \exp\left\{-t\rho\left\{\left(1+\frac{x^\beta}{\lambda}\right)^\alpha - 1\right\}^\varphi\right\} \tag{30}$$

Now inserting (6) in (29) the above equation we obtained

$$f_{i:n}(x) = \sum_{t=0}^{\infty} Z_{t+1} f(x:(t+1)\rho,\varphi,\alpha,\beta,\lambda) \tag{31}$$

where

$$Z_{t+1} = \frac{1}{B(j, n-i+1)(t+1)} \sum_{j=0}^{n-i} \sum_{t=0}^{\infty} (-1)^{j+t} \binom{n-i}{j} \binom{i+j-i}{t} \quad (32)$$

Where $(f(x : (t+1)\rho, \varphi, \alpha, \beta, \lambda))$ denotes the density of WPL with parameters $(t+1)\rho, \varphi, \alpha, \beta, \lambda$ so the density function of WPL order statistics is a mixture of WPL densities.

6 Estimation

6.1 Maximum Likelihood Estimation

In this section we make estimation of the parameters of WPL, by the maximum likelihood method, Let $X_1, X_2,$
$, X_n$ be a random sample of size n from the Wpl distribution given by

$$f(x; \rho, \varphi, \alpha, \beta, \lambda) = \frac{\rho\varphi\alpha\beta}{\lambda} x^{\beta-1} \left[1 + \frac{x^\beta}{\lambda}\right]^{(\alpha-1)} \times \left(\left[1 + \frac{x^\beta}{\lambda}\right]^\alpha - 1\right)^{\varphi-1} \times \exp\left\{-\rho\left(\left[1 + \frac{x^\beta}{\lambda}\right]^\alpha - 1\right)^\varphi\right\} \quad (33)$$

the log-likelihood function for the vector of parameters $\Theta = (\rho, \varphi, x, \beta, \lambda)^T$ can be expressed as

$$l = l(\Theta) = n\log(\rho\varphi\alpha\beta) - n\log\lambda + (\beta-1)\sum_{i=1}^{n}\log x_i + (\alpha-1)\sum_{i=1}^{n}\log\left(1 + \frac{x_i^\beta}{\lambda}\right) + (\varphi-1) \times \sum_{i=1}^{n}\log\left[\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha - 1\right] + \sum_{i=1}^{n}\left\{-\rho\left[\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha - 1\right]^\varphi\right\} \quad (34)$$

The log-likelihood function can be maximized by solving the non-linear likelihood equations below obtained by differentiating the above equation.

$$\frac{d(l(\Theta))}{d\rho} = U_\rho = \frac{n}{\rho} - \sum_{i=1}^{n}\left\{\left[\left(1 + \frac{x^\beta}{\lambda}\right)^\alpha - 1\right]^\varphi\right\} \quad (35)$$

$$\frac{\partial(l(\Theta))}{\partial\varphi} = U_\varphi = \frac{n}{\varphi} + \sum_{i=1}^{n}\log\left[\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha - 1\right] - \rho\sum_{i=1}^{n}\left\{\left[\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha - 1\right]^\varphi \log\left[\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha - 1\right]\right\} \quad (36)$$

$$\frac{\partial(l(\Theta))}{\partial\alpha} = U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha + (\varphi-1)\sum_{i=1}^{n}\frac{\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha \log\left(1 + \frac{x_i^\beta}{\lambda}\right)}{\left[\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha - 1\right]} - \varphi\rho\sum_{i=1}^{n} \left[\left[\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha \log\left(1 + \frac{x_i^\beta}{\lambda}\right)\right]\left[\left(1 + \frac{x_i^\beta}{\lambda}\right)^\alpha - 1\right]^{\varphi-1}\right] \quad (37)$$

$$\frac{\partial(l(\Theta))}{\partial\beta} = U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{\log(x_i)x_i^\beta}{\lambda} \left\{ \frac{\lambda}{x_i^\beta} + \left(\frac{(\alpha-1)}{\left(1+\frac{x_i^\beta}{\lambda}\right)} + \frac{(\varphi-1)}{\left[\left(1+\frac{x^\beta}{\lambda}\right)^\lambda - 1\right]} \right) \right\} - \rho\varphi\alpha$$
$$\times \sum_{i=1}^{n} \frac{\log(x_i)x_i^\beta}{\lambda} \left(\left[\left(1+\frac{x^\beta}{\lambda}\right)^\alpha - 1 \right]^{\varphi-1} \left(1+\frac{x^\beta}{\lambda}\right)^{\alpha-1} \right) \quad (38)$$

$$\frac{\partial(l(\Theta))}{\partial\lambda} = U_\lambda = -\frac{n}{\lambda} - \sum_{i=1}^{n} \frac{x_i^\beta}{\lambda^2} \frac{(\alpha-1)}{\left(1+\frac{x_i^\beta}{\lambda}\right)} - \sum_{i=1}^{n} \frac{x_i^\beta}{\lambda^2} \left(1+\frac{x_i^\beta}{\lambda}\right)^{\alpha-1}$$
$$\times \left[\frac{\alpha(\varphi-1)}{\left[\left(1+\frac{x_i^\beta}{\lambda}\right)^\alpha - 1\right]} - \alpha\rho\varphi \left[\left(1+\frac{x_i^\beta}{\lambda}\right)^\alpha - 1 \right]^{\varphi-1} \right] \quad (39)$$

Solving the equations (35), (36), (6.1), (38), and (39) algebraically may be intractable. To avoid this problem, one can obtain the MLEs numerically by applying any of the following methods: NewtonRaphson algorithm.

7 Data Analysis

Here, an application of the Weibull-Power Lomax Distribution, is provided by comparing the fit of this model with that of some Weibull-G families. The glass fibres data set analyzed by Smith and Naylor (1987) was used for this comparison. The data set originate from 63 observations of strengths of 1.5cm glass fibres, primitively obtained by workers at the UK National Physical Laboratory as reported by Bourguignon *et al.*(2014). The data set is presented below:

Table 1 MLEs for 63 observations of strengths of glass fibres

Distribution	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
WPL	0.1404	0.2314	2.1273	13.5221	16.0408
WL	0.1700	3.5281	5.9560	6.0375	-
EW	0.7309	3.2677	2.3134	-	-
WIL	0.6583	0.8122	0.7737	0.0371	-
WBXII	0.9866	0.4816	0.2873	0.6236	1.7957
PL	-	-	4.5151	6.1960	77.3271
WLL	19.9972	0.3236	17.8183	2.7373	-

Table 2 The Statistics $\ell(.)$, AIC, CAIC, BIC, and HQIC, for strengths of glass fibres

Distribution model	$\ell(.)$	AIC	BIC	CAIC	HQIC
WPL	12.5516	35.1032	36.1559	45.0446	39.3177
WL	14.4533	36.9065	37.5962	45.6791	40.2781
WLL	15.2515	385030	39.1927	47.0756	41.8746
PL	16.3075	38.6151	39.0218	45.8189	45.1438
EW	19.0089	44.0274	44.4342	50.4568	46.5561
WIL	96.8718	185.7433	185.0530	177.1768	182.3717
WBXII	172.6723	355.3446	356.3973	366.0603	359.5592

The method of maximum likelihood is employed to fit the proposed Weibull-Power Lomax distribution, Weibull distribution, Exponentiated Weibull (EW)" distribution, Weibull-Lomax, the baseline Power-Lomax (PL) distribution, Weibull-Inverse Lomax (WIL), Weibull-Burr Twelve (WBXII) and Weibull Log-Logistic (WLL) as provided in the work of Bourguignon *et al.*(2014), to these data. Criteria such as Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan Quinn information criterion (HQIC), and Consistent Akaike Information Criterion (CAIC) are used to compare the distribution models. The distribution model with the smallest AIC, BIC, HQIC, and CAIC values is considered to be the best distribution. Table 1 and Table 2 shows MLEs of the parameters for each of the fitted distributions and the statistics: AIC, BIC, HQIC, and CAIC respectively. The results from the Weibull-G family of distributions showed that, the proposed Weibull-Power Lomax distribution, has the least AIC, BIC, HQIC and CAIC values. Hence, this is an indication that Weibull-Power Lomax distribution, is a very strong competitor to other distributions used here for fitting the data set.

8 CONCLUSION

In this paper, we developed a new distribution called Weibull-Power Lomax distribution, which generalizes the Power Lomax distribution. The pdf, cdf, and hazard function were derived.Moreover, some of the mathematical and statistical properties like moments, moments generating function, order statistics and entropy were alsoderived. The model parameters were estimated by using the maximum likelihood estimation procedure. Finally, we fit the proposed model to a data and compared it with estimates from other Weibull-G family distributions. The new distribution was found to provide a better fit than its competitors.

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