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On Selection Algorithm

F. O. Isiogugu

Abstract. Strong convergence of an iteration scheme for approximating the common elements of the set of solutions $EP(F)$ of an equilibrium problem for a bifunction F and the set of fixed points $F(T)$ of a multi-valued (or single-valued) hemicontractive mapping T is established in a real Hilbert space H. This work contributes to the study on the applicability and computability of iteration schemes for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequence $\{K_n\}_{n=1}^{\infty}$ of closed convex subsets of H from an arbitrary $x_0 \in H$ and the sequence $\{x_n\}_{n=1}^{\infty}$ of the metric projections of x_0 into K_n. The results obtained extend and improve many results in this direction in the contemporary literature.

Keywords. Hilbert spaces, strong convergence, equilibrium problem, order inclusion transitive, resolvent, bifunction, hemicontractive mapping.

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1. Introduction

Let H be a real Hilbert space with an inner product $\langle .,. \rangle$ and a norm $\|.\|$, respectively and let K be a nonempty closed convex subset of H. Let $A : H \to H$ be an operator on H and $F : K \times K \to \mathbb{R}$ be a bifunction on K, where $\mathbb{R}$ is the set of real numbers. The variational inequality problem of A in K denoted by $VIP(A,K)$ is to find an $x^* \in K$ such that

$$\langle x - x^*, A(x^*) \rangle \geq 0, \;\; \forall x \in K, \tag{1}$$

while the equilibrium problem for F is to find $x^* \in K$ such that

$$F(x^*, x) \geq 0, \;\; \forall x \in K. \tag{2}$$

The set of solutions of (2) is denoted by $EP(F)$. Suppose $F(x,y) = \langle y - x, Ax \rangle$ for all $x, y \in K$, then $w \in EP(F)$ if and only if w is a solution of (1). Many problems in optimization, economics and physics reduce to finding a solution of (1), (see for examples, [1], [2] [4]) and the references therein. The following conditions are assumed for solving the equilibrium problems for a bifunction $F : K \times K \to \mathbb{R}$,

(A1) $F(x,x) = 0$ for all $x \in K$.

(A2) F is monotone, that is , $F(x,y) + F(y,x) \leq 0$, for all $x, y \in K$.

(A3) For each $x,y,z \in K$, $\lim\limits_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$.

(A4) For each $x \in K$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Let X be a nonempty set and let $T : X \to X$ be a map. A point $x \in X$ is called a fixed point of T if $x = Tx$. If $T : X \to 2^X$ is a multi-valued map from X into the family of nonempty subsets of X, then x is a fixed point of T if $x \in Tx$. If $Tx = \{x\}$, x is called a strict fixed point of T. The set $F(T) = \{x \in D(T) : x \in Tx\}$(respectively $F(T) = \{x \in D(T) : x = Tx\}$) is called the fixed point set of multi-valued(respectively single-valued) map T while the set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of T.

Let X be a normed space. A subset K of X is called proximinal if for each $x \in X$ there exists $k \in K$ such that

$$||x - k|| = \inf\{||x - y|| : y \in K\} = d(x,K). \tag{3}$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote the family of all nonempty closed and bounded

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subsets of X by $CB(X)$, the family of all nonempty subsets of X by 2^X, the family of all nonempty closed and convex subsets of X by $CC(X)$ and the family of all proximinal subsets of X by $P(X)$, for a nonempty set X.

Let H denote the Hausdorff metric induced by the metric d on X, that is, for every $A,B \in CB(X)$,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

Let X be a normed space. Let $T : D(T) \subseteq X \to 2^X$ be a multi-valued mapping on X. A multi-valued mapping $T : D(T) \subseteq X \to 2^X$ is called $L-Lipschitzian$ if there exists $L \geq 0$ such that for all $x,y \in D(T)$

$$H(Tx,Ty) \leq L||x-y||. \tag{4}$$

In (4), if $L \in [0,1)$ T is said to be a contraction while T is nonexpansive if $L = 1$.

Definitions 2.1 ([11])**.** T is said to be k-strictly pseudocontractive-type of Isiogugu [11] if there exists $k \in (0,1)$ such that given any pair $x,y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $||u-v|| \leq H(Tx,Ty)$ and

$$H^2(Tx,Ty) \leq ||x-y||^2 + k||x-u-(y-v)||^2. \tag{5}$$

If $k = 1$ in (5), T is called pseudocontractive-type.

T is said to be hemicontractive of Isiogugu and Osilike [14] if $F(T) \neq \emptyset$ and

$$H^2(Tx,Tp) \leq ||x-p||^2 + d^2(x,Tx). \tag{6}$$

for all $x \in D(T)$, $p \in F(T)$

Many authors have approximated the common elements of the set of fixed points $F(T)$ of a multi-valued (or single-valued) mapping T and the set of solutions $EP(F)$ of an equilibrium problem for a bifunction F (or the common elements of the sets of fixed points of a finite family of multi-valued (or single-valued) mappings and the sets of solutions of equilibrium problems for a finite family of bifunctons) (see for examples [5], [6], [7], [8], [9], [10] and references therein). In a real Hilbert space, several authors have studied the algorithms involving the construction of the sequences of sets $\{K_n\}_{n=1}^{\infty}$ and the metric projections $\{x_n\}_{n=1}^{\infty}$, from an arbitrary $x_0 \in H$, where $K_{n+1} = \{z \in K_n : \|z-u_n\|^2 \leq \|z-x_n\|^2\}$, $x_{n+1} = P_{K_{n+1}}x_0$, while P_{K_n} is the projection map and $\{u_n\}_{n=1}^{\infty}$ is the sequence of the resolvent of the bifunctions, (see for examples [3], [5], [6], [7], [9], [10] and references therein).

Two of the iteration schemes studied by authors are the modified Reich-Sabach-type Algorithm 1.1 and modified Mann-Reich-Sabach-type Algorithm 1.2 below defined for the approximation of (i) the solutions of an equilibrium problem for a bifunction; (ii) the common elements of the set of fixed points $F(T)$ of a multi-valued (or single-valued) $k-$ strictly Pseudocontractive-type mapping T and the set of solutions $EP(F)$ of an equilibrium problem for a bifunction F, respectively.

(i). Let H be a real Hilbert space, K a closed and convex subset of H. Let $F : K \times K \to \mathbb{R}$ be a bifunction and $r \in [a,\infty)$ for some $a > 0$. Then from an arbitrary $x_0 \in H$ the algorithm is generated as follows.

Algorithm 1.1.

$$\begin{cases} x_0 \in H, \\ y_n = x_n, \\ u_n \in K \text{ such that } F(u_n,y) \\ +\frac{1}{r}\langle y-u_n, u_n-y_n\rangle \geq 0, \ \forall y \in K, \\ K_{n+1} = \{z \in K_n : \|z-u_n\|^2 \leq \|z-x_n\|^2\} \\ x_{n+1} = P_{K_{n+1}}x_0. \end{cases}$$

(ii). Let H be a real Hilbert space, K a closed and convex subset of H, $F : K \times K \to \mathbb{R}$ a bifunction and $T : K \to P(K)$ multivalued $k-$strictly pseudocontractive-type mapping. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ and $r \in [a,\infty)$ for some $a > 0$. Then from an arbitrary $x_0 \in H$ the algorithm is generated as follows,

Algorithm 1.2.

$$\begin{cases} x_0 \in H, \\ y_n = \alpha_n x_n + (1-\alpha_n) v_n, \\ u_n \in K \text{ such that } F(u_n, y) \\ + \frac{1}{r}\langle y - u_n, u_n - y_n \rangle \geq 0, \ \forall y \in K, \\ K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\} \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases}$$

where $v_n \in Tx_n$ for multi-valued mapping T.

It has been noted by authors that, despite the fact that most of these algorithms yield strong convergence theoretically, the difficulty encountered by computer in the construction of the sequence of the metric projection $\{x_n\}_{n=1}^{\infty}$ and the sequence of sets $\{K_n\}_{n=1}^{\infty}$ has made such algorithms almost impossible for real life applications. Consequently, this non-computability and non-applicability of such algorithms has lead to the introduction of other iteration schemes which do not include the construction of these two sequences but require stronger conditions and many parameters.

Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let F be a bifunction and T an L-Lipschitzian pseudocontractive-type mapping such that $F : K \times K \to \mathbb{R}$ and $T : K \to CC(K)$ respectively. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$be sequences in [0,1] and $\{r_n\}_{n=1}^{\infty} \subset [a, \infty)$ for some $a > 0$, then from an arbitrary $x_0 \in H$, Isiogugu et al. [15] generated the sequence $\{x_n\}_{n=1}^{\infty}$ as follows.

Algorithm 1.3.

$$\begin{cases} x_0 \in H, \\ z_n = (1-\beta_n) x_n + \beta_n v_n, \\ y_n = (1-\alpha_n) x_n + \alpha_n w_n, \\ u_n \in K \text{ such that } F(u_n, y) \\ + \frac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0, \ \forall y \in K, \\ x_{n+1} = \frac{1}{2}(u_n + x_n), \end{cases}$$

where $w_n \in T(z_n) = T((1-\beta_n)x_n + \beta_n v_n)$ with $d((1-\beta_n)x_n + \beta_n v_n, T[(1-\beta_n)x_n + \beta_n v_n]) = ||(1-\beta_n)x_n + \beta_n v_n - w_n||$, $v_n \in Tx_n$ with $||x_n - v_n|| = d(x_n, Tx_n)$ and $||w_n - v_n|| \leq H(Tz_n, Tx_n)$.

They proved the following theorem.

Theorem 1.4. Let H, K, T, F, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be as in Algorithm 1.3. Suppose F satisfying (A1)-(A4), T satisfies condition 1 and $\mathbb{F} = F_s(T) \bigcap EP(F) \neq \emptyset$, then $\{x_n\}$ converges strongly to $p \in \mathbb{F}$ also, if H has order inclusion transitive property, $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}} x_0$ if for all $n \geq 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying (i) $0 \leq \alpha_n \leq \beta_n < 1$; (ii) $\liminf_{n\to\infty} \alpha_n = \alpha > 0$; (iii) $\sup_{n \geq 1} \beta_n \leq \beta \leq \frac{1}{\sqrt{1+(L)^2}+1}$.

The aim of this research is to extend the results of Isiogugu et al. [15] above to the class of hemicontractive mappings. The results obtained are contributions towards the resolution of the controversy over the applicability and computability of iteration schemes for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequences $\{K_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ as in algorithms 1 and 2 above. They also generalize, extend, complement and improve many corresponding results in the contemporary literature.

2. Preliminaries

Lemma 2.2: Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let P_K be the convex projection onto K. Then, convex projection is characterized by the following relations;
(i) $x^* = P_K(x) \Leftrightarrow \langle x - x^*, y - x^* \rangle \leq 0$, for all $y \in K$.
(ii) $||x - P_K x||^2 \leq ||x - y||^2 - ||y - P_K x||^2$.
(iii)$||x - P_K y||^2 \leq ||x - y||^2 - ||P_K y - y||^2$.

Lemma 2.3 ([1]). Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \times K \to \mathbb{R}$

a bifunction satisfying (A1)-(A4). Let $r>0$ and $x \in H$. Then, there exists $z \in K$ such that

$$F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in K.$$

Lemma 2.4 **([2]).** Let K be a nonempty closed convex subset of a real Hilbert space H. Assume that $F: K\times K \to \mathbb{R}$ that satisfies (A1)-(A4). Let $r>0$ and $x \in H$, define $T_r : H \to 2^K$ by

$$T_r(x)=\{z \in K : F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0\}, \quad \forall y \in K.$$

Then the following conditions hold:
(1) T_r is single valued.
(2) T_r is firmly nonexpansive, that is for any $x,y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle$.
(3) $F(T_r)=EP(F)$.
(4) $EP(F)$ is closed and convex.

Lemma 2.5 **([3]).** Let K be a nonempty closed convex subset of a real Hilbert space H and $F: K\times K \to \mathbb{R}$ a bifunction satisfying (A1)-(A4). Let $r>0$ and $x\in H$. Then for all $x \in H$ and $p \in F(T_r)$

$$\|p-T_r x\|^2+\|T_r x - x\|^2 \leq \|p-x\|^2.$$

Definition 2.6 ([12]). Let $\{K_n\}_{n=1}^{\infty}$ be sequence of sets, a sequence $\{z_n\}_{n=1}^{\infty}$ is called a selection of $\{K_n\}_{n=1}^{\infty}$ if $z_n \in K_n$ for each n.

Definition 2.7 **([12]).** A norm $\|.\|$ on a Hilbert space H is order inclusion transitive on $CC(H)$ if given any $A,B \in CC(H)$ with $A \subseteq B$ and arbitrary $x \in H$, then $d(x,B)=\inf_{\overline{b}\in B}\|\overline{b}-x\| = ||b-x||$ and $d(b,A)=\inf_{\overline{a}\in A}\|\overline{a}-b\| = ||a-b||$ imply that $d(x,A)=\inf_{\overline{a}\in A}\|\overline{a}-x\|=||a-x||$

Definition 2.8 **([12]).** A Hilbert H is said to have order inclusion transitive property on $CC(H)$ if its norm is order inclusion transitive on $CC(H)$. It is easy to see that the set of real numbers with the usual norm has order inclusion transitive property.

Lemma 2.9 **([12]).** Let H be a real Hilbert space and $K=K_0$ be a closed and convex subset of H. Let $x_0 \in H$ be arbitrary and $\{u_n\}_{n=1}^{\infty}$ a sequence in K. Define $K_{n+1} := \{z \in K_n : \|z-u_n\|^2 \leq \|z-x_n\|^2\}$, if we define $x_{n+1}=\frac{1}{2}(u_n+x_n)$, then the following conditions are true.
(C_1). $\{x_n\}_{n=1}^{\infty}$ is a selection of $\{K_n\}_{n=1}^{\infty}$.
(C_2). $x_{n+1}=P_{K_{n+1}}x_n$.
$C_3)$. If H has order inclusion transitive property on $CC(H)$ then, $x_{n+1}=P_{K_{n+1}}x_0$.

Definition 2.10 ([13]). A multi-valued mapping $T: K \to P(K)$ is said to satisfy condition 1 if there exists a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$d(x,Tx) \geq f(d(x,F(T))), \ \forall x \in K.$$

3. Main results

Proposition 3.1. Let H be a real Hilbert space and $T: D(T)\subseteq H \to P(H)$ be a multi-valued $L-$ Lipschtizian hemicontractive mapping, then, fixed point set of T is closed.

Proof. let $\{g_n\}_{n=1}^{\infty} \subseteq F(T)$ such $g_n \to x^*$. Then,

$$\begin{aligned} d^2(x^*,Tx^*) &\leq d(x^*,g_n)+d(g_n,Tg_n)+H(Tg_n,Tx^*)\\ &= \|x^*-g_n\|+H(Tg_n,Tx^*)\\ &\leq (1+L)\|g_n-x^*\| \to 0 \ as \ n\to\infty. \end{aligned}$$

Therefore, $d(x^*,Tx^*)=0$. Since T is proximinal, there exist $v\in Tx^*$ such that $\|x^*-v\|=d(x^*,Tx^*)=0$. Consequently, $x^*\in Tx^*$. $\square$

We now consider the following algorithm.

Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let F be a bifunction and T an L-Lipschitzian hemicontractive mapping such that $F: K\times K\to\mathbb{R}$ and $T: K \to CC(K)$ respectively. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$be sequences in [0,1] and $\{r_n\}_{n=1}^{\infty}\subset[a,\infty)$ for some $a>0$, then from an arbitrary $x_0\in H$ we generate the sequence $\{g_n\}_{n=1}^{\infty}$ as follows.

Algorithm 3.4.

$$\begin{cases} x_0 \in H, \\ z_n = (1-\beta_n)g_n + \beta_n v_n, \\ y_n = (1-\alpha_n)g_n + \alpha_n w_n, \\ u_n \in K \text{ such that } F(u_n, y) \\ +\frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \ \forall y \in K, \\ g_{n+1} = \frac{1}{2}(u_n + g_n), \end{cases}$$

where $w_n \in T(z_n) = T((1-\beta_n)g_n + \beta_n v_n)$ with $d((1-\beta_n)g_n + \beta_n v_n, T[(1-\beta_n)g_n + \beta_n v_n]) = ||(1-\beta_n)g_n + \beta_n v_n - w_n||$, $v_n \in Tg_n$ with $||g_n - v_n|| = d(g_n, Tg_n)$ and $||w_n - v_n|| \leq H(Tz_n, Tg_n)$.

Theorem 3.5. Let H, K, T, F, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be as in Algorithm 3.4. Suppose F satisfying (A1)-(A4), T satisfies condition 1 and $\mathbb{F} = F_s(T)\bigcap EP(F) \neq \emptyset$, then $\{g_n\}$ converges strongly to $p \in \mathbb{F}$ also, if H has order inclusion transitive property, $\{g_n\}$ converges strongly to $p \in P_{\mathbb{F}}x_0$ if for all $n \geq 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying (i) $0 \leq \alpha_n \leq \beta_n < 1$; (ii) $\liminf\limits_{n\to\infty} \alpha_n = \alpha > 0$; (iii) $\sup\limits_{n\geq 1} \beta_n \leq \beta \leq \frac{1}{\sqrt{1+(L)^2}+1}$.

Proof. Using Lemma 2.4, for all $p \in \mathbb{F}$ we have

$$\begin{aligned}
||g_{n+1} - p||^2 &= ||\frac{1}{2}(g_n - u_n) - p||^2 \\
&= \frac{1}{2}||g_n - p||^2 + \frac{1}{2}||u_n - p||^2 \\
&\quad -\frac{1}{4}||g_n - u_n||^2 \\
&\leq \frac{1}{2}||g_n - p||^2 - \frac{1}{4}||g_n - u_n||^2 \\
&\quad +\frac{1}{2}||p - T_{r_n}y_n||^2 \\
&\leq \frac{1}{2}||g_n - p||^2 - \frac{1}{4}||g_n - u_n||^2 \\
&\quad +\frac{1}{2}||p - y_n||^2 \\
&= \frac{1}{2}||g_n - p||^2 - \frac{1}{4}||g_n - u_n||^2 \\
&\quad +\frac{1}{2}||(1-\alpha_n)g_n + \alpha_n w_n - p||^2 \\
&= \frac{1}{2}||g_n - p||^2 - \frac{1}{4}||g_n - u_n||^2 \\
&\quad +\frac{1}{2}||(1-\alpha_n)(g_n - p) + \alpha_n(w_n - p)||^2 \\
&= \frac{1}{2}||g_n - p||^2 - \frac{1}{4}||g_n - u_n||^2 \\
&\quad +\frac{1}{2}\Big[(1-\alpha_n)||g_n - p||^2 \\
&\quad +\alpha_n||w_n - p||^2 \\
&\quad -\alpha_n(1-\alpha_n)||g_n - w_n||^2\Big] \\
&\leq \frac{1}{2}||g_n - p||^2 - \frac{1}{4}||g_n - u_n||^2 \\
&\quad +\frac{1}{2}\Big[(1-\alpha_n)||g_n - p||^2 \\
&\quad +\alpha_n H^2(Tz_n, Tp) \\
&\quad -\alpha_n(1-\alpha_n)||g_n - w_n||^2\Big] \\
&\leq \frac{1}{2}||g_n - p||^2 - \frac{1}{4}||g_n - u_n||^2 \\
&\quad +\frac{1}{2}\Big[(1-\alpha_n)||g_n - p||^2 \\
&\quad +\alpha_n\Big[||z_n - p||^2 + ||z_n - w_n||^2\Big] \\
&\quad -\alpha_n(1-\alpha_n)||g_n - w_n||^2\Big] \\
&= \frac{1}{2}||g_n - p||^2 - \frac{1}{4}||g_n - u_n||^2 \\
&\quad +\frac{1}{2}\Big[(1-\alpha_n)||g_n - p||^2 + \alpha_n||z_n - p||^2 \\
&\quad +\alpha_n d^2(z_n, Tz_n) \\
&\quad -\alpha_n(1-\alpha_n)||g_n - w_n||^2\Big]. \ (7)
\end{aligned}$$

$$\begin{aligned}
||z_n - w_n||^2 &= ||(1-\beta_n)g_n + \beta_n v_n - w_n||^2 \\
&= ||(1-\beta_n)(g_n - w_n) + \beta_n(v_n - w_n)||^2 \\
&= (1-\beta_n)||g_n - w_n||^2 + \beta_n||v_n - w_n||^2 \\
&\quad -\beta_n(1-\beta_n)||g_n - v_n||^2. \qquad (8)
\end{aligned}$$

(7) and (8) imply that

$$\begin{aligned}
||p - y_n||^2 &= (1-\alpha_n)||g_n - p||^2 \\
&\quad +\alpha_n||w_n - p||^2
\end{aligned}$$

$$
\begin{aligned}
& -\alpha_n(1-\alpha_n)||g_n-w_n||^2 \\
\leq\ & (1-\alpha_n)||g_n-p||^2+\alpha_n H^2(Tz_n,Tp) \\
& -\alpha_n(1-\alpha_n)||g_n-w_n||^2 \\
\leq\ & (1-\alpha_n)||g_n-p||^2+\alpha_n||z_n-p||^2 \\
& +\alpha_n\Big[(1-\beta_n)||g_n-w_n||^2+\beta_n||v_n-w_n||^2 \\
& -\beta_n(1-\beta_n)||g_n-v_n||^2\Big] \\
& -\alpha_n(1-\alpha_n)||g_n-w_n||^2. \qquad (9)
\end{aligned}
$$

Also,

$$
\begin{aligned}
||z_n-p||^2 &= ||(1-\beta_n)g_n+\beta_n v_n-p||^2 \\
&= ||(1-\beta_n)(g_n-p)+\beta_n(v_n-p)||^2 \\
&= (1-\beta_n)||g_n-p||^2+\beta_n||v_n-p||^2 \\
&\quad -\beta_n(1-\beta_n)||g_n-v_n||^2 \\
&\leq (1-\beta_n)||g_n-p||^2+\beta_n H^2(Tg_n,Tp) \\
&\quad -\beta_n(1-\beta_n)||g_n-v_n||^2 \\
&\leq (1-\beta_n)||g_n-p||^2+\beta_n\Big[||g_n-p||^2 \\
&\quad +||g_n-v_n||^2\Big]-\beta_n(1-\beta_n)||g_n-v_n||^2 \\
&= ||g_n-p||^2+\beta_n{}^2||g_n-v_n||^2. \qquad (10)
\end{aligned}
$$

(9) and (10) imply that

$$
\begin{aligned}
\|p-y_n\|^2 &\leq (1-\alpha_n)||g_n-p||^2 \\
&\quad +\alpha_n\Big[||g_n-p||^2+\beta_n{}^2||g_n-v_n||^2\Big] \\
&\quad +\alpha_n\Big[(1-\beta_n)||g_n-w_n||^2 \\
&\quad +\beta_n||v_n-w_n||^2 \\
&\quad -\beta_n(1-\beta_n)||g_n-v_n||^2\Big] \\
&\quad -\alpha_n(1-\alpha_n)||g_n-w_n||^2 \\
&= (1-\alpha_n)||g_n-p||^2+\alpha_n||g_n-p||^2 \\
&\quad +\alpha_n\beta_n{}^2||g_n-v_n||^2 \\
&\quad +\alpha_n(1-\beta_n)||g_n-w_n||^2 \\
&\quad +\alpha_n\beta_n||v_n-w_n||^2 \\
&\quad -\alpha_n\beta_n(1-\beta_n)||g_n-v_n||^2 \\
&\quad -\alpha_n(1-\alpha_n)||g_n-w_n||^2 \\
&\leq ||g_n-p||^2+\alpha_n\beta_n{}^2||g_n-v_n||^2 \\
&\quad +\alpha_n\beta_n H^2(Tg_n,Tz_n) \\
&\quad -\alpha_n(\beta_n-\alpha_n)||g_n-w_n||^2 \\
&\quad -\alpha_n\beta_n(1-\beta_n)||g_n-v_n||^2 \\
&\leq ||g_n-p||^2+\alpha_n\beta_n{}^2||g_n-v_n||^2 \\
&\quad +\alpha_n\beta_n{}^3L^2\|g_n-v_n\|^2 \\
&\quad -\alpha_n\beta_n(1-\beta_n)||g_n-v_n||^2 \\
&\quad -\alpha_n(\beta_n-\alpha_n)||g_n-w_n||^2 \\
&= ||g_n-p||^2-\alpha_n\beta_n[1-2\beta_n \\
&\quad -L^2\beta_n{}^2]||g_n-v_n||^2 \\
&\quad -\alpha_n(\beta_n-\alpha_n)||g_n-w_n||^2 \\
&\leq ||g_n-p||^2-\alpha_n\beta_n[1-2\beta_n \\
&\quad -L^2\beta_n{}^2]||g_n-v_n||^2 \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
\|g_{n+1}-p\|^2 &\leq \frac{1}{2}\|g_n-p\|^2-\frac{1}{4}\|g_n-u_n\|^2 \\
&\quad +\frac{1}{2}\Big[||g_n-p||^2-\alpha_n\beta_n[1-2\beta_n \\
&\quad -L^2\beta_n{}^2]||g_n-v_n||^2\Big] \\
&\leq \|g_n-p\|^2-\frac{1}{4}\|g_n-u_n\|^2 \\
&\quad -\frac{1}{2}\alpha_n\beta_n[1-2\beta_n-L^2\beta_n{}^2]||g_n-v_n||^2
\end{aligned}
$$

It then follows that $\lim_{n\to\infty}\|g_n-p\|$ exists hence $\{g_n\}$ is bounded. Also, from (11), we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty}\alpha^2[1-2\beta-L^2\beta^2]||g_n-v_n||^2 &\leq \sum_{n=0}^{\infty}\alpha_n\beta_n[1-2\beta_n \\
&\quad -L^2\beta_n{}^2]||g_n-v_n||^2
\end{aligned}
$$

$$\begin{aligned} &\leq \sum_{n=0}^{\infty}[||g_n-p||^2 \\ &\quad -||g_{n+1}-p||^2] \\ &\leq ||x_0-p||^2 \\ &\quad +D<\infty. \end{aligned}$$

It then follows that

$$\lim_{n\to\infty}||g_n-v_n||=0. \tag{12}$$

Since $d(g_n,Tg_n) = ||g_n - v_n||$, we have that $d(g_n,Tg_n)\to 0$ as $n\to\infty$. Furtheremore,

$$\lim_{n\to\infty}||g_n-u_n||=0. \tag{13}$$

Consequently,

$$\lim_{n\to\infty}||g_{n+1}-g_n||=\lim_{n\to\infty}\|\frac{1}{2}(g_n-u_n)\|=0 \tag{14}$$

which implies that $\{g_n\}$ is a Cauchy sequence in K. Also, since K is closed and convex, $\{g_n\}$ converges strongly to some $p^*\in K$. Since T satisfies condition (1), $\lim_{n\to\infty}d(g_n,F(T))=0$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{g_n\}$ such that $\|x_{n_k}-p_k\|\leq\frac{1}{2^k}$ for some $\{p_k\}_{k=1}^{\infty}\subseteq F(T)$. We now show that $\{p_k\}_{n=1}^{\infty}$ is a Cauchy sequence in $F(T)$. Observe that from (14), $\lim_{n\to\infty}\|x_{n_{k+1}}-x_{n_k}\|=0$ for all subsequences $\{x_{n_k}\}$ of $\{g_n\}$. It then follows that,

$$\begin{aligned} \|p_{k+1}-p_k\| &\leq \|p_{k+1}-x_{n_{k+1}}\|+\|x_{n_{k+1}}-x_{n_k}\|+\|x_{n_k}-p_k\| \\ &\leq \frac{1}{2^{k+1}}+\frac{1}{2^k}+\|x_{n_{k+1}}-x_{n_k}\| \\ &\leq \frac{1}{2^{k-1}}+\|x_{n_{k+1}}-x_{n_k}\|. \end{aligned}$$

Therefore $\{p_k\}$ is a Cauchy sequence and converges to some $q\in F(T)$ because $F(T)$ is closed. Now,

$$\|x_{n_k}-q\|\leq\|x_{n_k}-p_k\|+\|p_k-q\|.$$

Hence $x_{n_k}\to q$ as $k\to\infty$.

$$\begin{aligned} d(q,Tq) &\leq \|q-p_k\|+\|p_k-x_{n_k}\|+d(x_{n_k},Tx_{n_k}) \\ &\quad +H(Tx_{n_k},Tq) \\ &\leq \|q-p_k\|+\|p_k-x_{n_k}\|+d(x_{n_k},Tx_{n_k}) \\ &\quad +L\|x_{n_k}-q\|. \end{aligned}$$

Hence, $q\in Tq$ and $\{x_{n_k}\}$ converges strongly to q. Since g_n converges strongly to p^*, uniqueness of limit of a convergent sequence guarantees that $p^*=q$. Hence $p^*\in F(T)$.

It remains to show that p^* is in $EP(F)$. Using (13) and (14),

$$\lim_{n\to\infty}\|g_{n+1}-u_n\|=0. \tag{15}$$

Hence from $\lim_{n\to\infty}\|g_n-p^*\|=0$ and (13) we have that

$$\lim_{n\to\infty}\|u_n-p^*\|=0. \tag{16}$$

Also, from (11),

$$\begin{aligned} \|y_n-p^*\|^2 &\leq ||g_n-p^*||^2-\alpha_n\beta_n[1-2\beta_n \\ &\quad -L^2{\beta_n}^2]||g_n-v_n||^2 \end{aligned} \tag{17}$$

Observe that

$$\begin{aligned} \|p^*-g_n\|^2-\|p^*-u_n\|^2 &= \|g_n\|^2-\|u_n\|^2 \\ &\quad -2\langle p^*,g_n-u_n\rangle \\ &\leq \|g_n-u_n\|(\|g_n\|+\|u_n\|) \\ &\quad +2\|p^*\|\|g_n-u_n\|. \end{aligned}$$

It follows from (13)and (16) that

$$\lim_{n\to\infty}\|p^*-g_n\|-\|p^*-u_n\|=0. \tag{18}$$

Now from (17)

$$\|p^*-y_n\|\leq\|p^*-g_n\|. \tag{19}$$

Also, using $u_n=T_{r_n}y_n$, Lemma 2.3 and (19) we have

$$\begin{aligned} \|u_n-y_n\|^2 &= \|T_{r_n}y_n-y_n\|^2 \\ &\leq \|p^*-y_n\|^2-\|p^*-T_{r_n}y_n\|^2 \\ &\leq \|p^*-g_n\|^2-\|p^*-T_{r_n}y_n\|^2 \\ &= \|p^*-g_n\|^2-\|p^*-u_n\|^2. \end{aligned} \tag{20}$$

Therefore, from (18) and (20)

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{21}$$

Consequently, from (16) and (21)

$$\lim_{n\to\infty}\|y_n-p^*\|=0. \tag{22}$$

From the assumption that $r_n \geq a > 0$,

$$\lim_{n\to\infty} \frac{\|u_n - y_n\|}{r_n} = 0. \tag{23}$$

Since $u_n = T_{r_n} y_n$ implies

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0,$$

we deduce from (A2) that

$$\begin{aligned} \frac{\|u_n - y_n\|^2}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \\ &\geq -F(u_n, y) \geq F(y, u_n). \ \forall y \in K \end{aligned}$$

By taking limit as $n \to \infty$ of the above inequality and from (A4), (16) and (22), $F(y, p^*) \leq 0$, for all $y \in K$. Let $t \in (0,1)$ and for all $y \in K$, since $p^* \in K$, $y_t = ty + (1-t)p^* \in K$. Hence $F(y_t, p^*) \leq 0$. Therefore, from (A1),

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, p^*) \leq tF(y_t, y),$$

that is, $F(y_t, y) \geq 0$. Letting $t \downarrow 0$, from (A3) we obtain $F(p^*, y) \geq 0$ for all $y \in K$ so that $p^* \in EP(F)$. Finally, if H has order inclusion transitive property, $g_n = P_{K_n} x_0$ consequently, from Lemma 2.2(i)

$$\langle g_n - y, x_0 - g_n\rangle \geq 0, \ \ \forall\, y \in K_n. \tag{24}$$

Since $EP(F) \subseteq K_n$ for all $n \geq 1$, we have that

$$\langle g_n - q, x_0 - g_n\rangle \geq 0, \ \ \forall\, q \in EP(F). \tag{25}$$

Taking the limits as $n \to \infty$ in (25) we obtain

$$\langle p^* - q, x_0 - p^*\rangle \geq 0, \ \ \forall q \in EP(F).$$

Thus, from Lemma 2.2(i) $p^* = P_{EP(F)} x_0$. This completes the proof. $\square$

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