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Thermodynamic properties and mass spectra of a quarkonium system with Ultra Generalized Exponential–Hyperbolic Potential

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Abstract: *We solved the N-dimensional Klein-Gordon equation analytically using the Nikiforov-Uvarov method to obtain the energy eigenvalues and corresponding wave function in terms of Laguerre polynomials with the ultra-generalized exponential –hyperbolic potential. The results were applied for calculating the mass spectra of heavy mesons including charmonium* ($c\bar{c}$) *and bottomonium* ($b\bar{b}$) *for different quantum states. Also, the thermodynamic properties such as free energy, mean energy, entropy, and specific heat were obtained. The data obtained in the study was in excellent agreement with experimental results and with results obtained from others with a maximum error of* $0.0059\ GeV$.

Keywords: *Ultra generalized exponential–hyperbolic potential; Thermodynamic properties; Klein-Gordon equation; Heavy mesons; Nikiforov-Uvarov method*

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1.0 Introduction

The study of thermodynamic properties is important in various areas of physical and chemical sciences. This is made possible through the solutions of the quantum mechanical problems, which contain all the essential data to portray the quantum system under study (Florkowski, 2010). The thermodynamic properties of systems are significance in the analyses of quark-gluon plasma and can provide useful information towards that can unfold some composition of the strange quark matter. For example, Modarres, and Mohamadnejad (2013) calculated some thermodynamic properties of the quark-gluon plasma (QGP), as a function of baryon density(chemical potential) and temperature using the framework of a one gluon exchange model. Their results reveal the dependence of the investigated function on bag pressure and the quantum chromo dynamic (QCD) coupling constant. The statistical thermodynamics function needed for the determination of the thermodynamic properties of any physical system is the partition function (Ikot *et al.*,2016, 2018). Also, the solutions the Schrodinger and Klein-Gordon are relevant in probing the mass spectra of heavy quarkonia such as bottomonium and charmonium (Anisiu, 2015). The potential commonly utilized in simulating the interaction for this system is the confining-type potential such as Cornell potential or Killingbeck potential that has two important variables that account for the Coulomb interaction and the confinement of the quarks, respectively (Mocsy, 2009). In recent times the solutions of the Schrödinger equation (SE) and Klein-Gordon equation (KGE) under the quarkonium interaction potential model such as the Cornell or the Killingbeck potential have attracted much interest from researchers (Abu-Shady & Ikot, 2019; Al-Jamel, 2019; Ciftci & Kisoglu, 2018; Abu-Shady, *et al.*,2018; Mansour, & Gamal, 2018; Abu-Shady, 2016; Vega & Flores, 2016; Al-Oun, *et al.*, 2015; Al-Jamel and Widyan 2012). The KGE for some potential can be solved exactly for angular momentum quantum number $(l=0)$, but it is complicated for other systems. Therefore, approximations techniques are necessary in other to obtain quantum mechanical solutions for systems with $l \neq 0$. Some of the approximation methods include, asymptotic iteration method (AIM) (Khokha, *et al.*, 2016). Laplace transformation method, Abu-Shady, (2015), super symmetric quantum mechanics method (SUSQM) (Omugbe, *et al.*, 2020; Abu-Shady, *et al.*, 2019; Mutuk, 2018), Nikiforov-Uvarov(NU) method (Inyang, *et al.*, 2021; Ekpo, *et al.*, 2020; Edet, *et al.*, 2020; Ntibi, *et al.*, 2020; William, *et al.*,2020; Okoi, *et al.*,2020; Edet, *et al.*, 2020; Ikot, *et al.*, 2020; Inyang, *et al.*,2020; Edet, *et al.*, 2020; William, *et al.*, 2020; Edet, *et al.*, 2019;Abu-Shady, 2016) ,series expansion method (SEM) (Inyang, *et al.*,2021; Ibekwe, *et al.*, 2020;Abu-Shady & Fath-Allah, 2019) , analytical exact iterative method(AEIM) Ikot *et al.*, (2021), and others (Inyang *et al.*, 2020). Numerous researchers have solved exact solutions; like harmonic oscillator and approximate solutions of SE and KGE utilizing diverse insightful methods with potentials to obtained thermodynamic properties of some physical systems (Edet & Ikot, 2021; Abu-Shady, *et al.*, 2019; Ikot, *et al.*, 2019; Lutfuoglu, 2018; Okorie, *et al.*, 2018; Song, *et al.*, 2017; Jia, *et al.*, 2017; Oyewumi, *et al.*, 2014).

Recently, studies on mass spectra with exponential-type potentials have been carried out by most researchers. For instance, Inyang *et al*. (2021) obtained the mass spectra with Yukawa potential using the NU method for relativistic and non relativistic regimes for heavy quarkonium system (HQS). Akpan, *et al.* (2021) obtained the mass spectra of heavy mesons in the non relativistic model with Huthern-Hellmann potential using the NU method for HQS. While Inyang *et al.* (2021) also adopted analytical solution to find the solution to the SE via the NU method to obtained the energy eigenvalues and corresponding wave function in terms of Laguerre polynomials with the Varshni potential. They applied their results for the calculation of the mass spectra of heavy mesons for different quantum states including charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$). In their study, Ibekwe, *et al*. (2021) solved the SE for HQS with screened Kratzer potential using the series expansion method. The results were used to obtain the mass spectra of heavy mesons.

The trigonometric hyperbolic potential plays a vital role in atomic and molecular physics. Since it can be used to model inter-atomic and inter-molecular forces (Rampho, *et al.*,2020; Suparmi, *et al.*,2014;). The ultra-generalized exponential –hyperbolic potential (UGEHP) takes the form (Parmar, 2020).

$$V(r)=\frac{ae^{-4\alpha r}+be^{-2\alpha r}}{r^2}+\frac{ce^{-2\alpha r}-dCosh(\eta\alpha r)e^{-\alpha r}+g\,\mathrm{Cosech}\;\alpha re^{\alpha r}}{r}+f, \tag{1}$$

where $a,b,c,d,\eta,g\ and\ f$ are potential strengths and α is the screening parameter. When $\eta=1$, then

$$\left.\begin{aligned} Cosh\alpha r &= \frac{e^{\alpha r}+e^{-\alpha r}}{2} \\ \text{Cosech}\,\alpha r &= \frac{2}{e^{\alpha r}-e^{-\alpha r}} \end{aligned}\right\} \quad (2)$$

The expansion of the exponential terms in equation 1 and 2 (up to order three, in order to model the potential to interact in the quark-antiquark system) and substitution of the results obtained into equation 1 yields equation 3,

$$V(r)=\frac{\beta_0}{r^2}-\frac{\beta_1}{r}+\beta_2 r-\beta_3 r^2+\beta_4, \quad (3)$$

The terms in equations 1 to 3 are defined as follows:

$$\left.\begin{aligned} &\beta_0=a+b, \beta_1=4a\alpha+2b\alpha+d-g, \beta_2=2c\alpha^2+\alpha d \\ &\beta_3=\alpha(d-g), \beta_4=8a\alpha^2+2b\alpha^2-2c\alpha-\alpha d+g\alpha+f \end{aligned}\right\} \quad (4)$$

The third term of equation 3 is a linear variable for the confinement feature while the second term is the Coulomb potential that describes the short distance between quarks. Literature is scanty on application of generalized exponential-hyperbolic potential for calculating some thermodynamic properties of complex systems.

Therefore, in this present work, we aim at studying the KGE with the ultra-generalized exponential –hyperbolic potential (UGEHP) using the NU method to calculate the thermodynamic properties and mass spectra of heavy mesons such as charmonium $(c\bar{c})$ and bottomonium $(b\bar{b})$. The study is designed for a threefold approach which started with modeling the potential to interact in the quark-antiquark system. This was succeeded by solving the, model potential with KGE using the NU method and finally the calculations of the thermodynamic properties and mass spectra of quark that were considered as spineless particles.

2.0 Bound state solution of the Klein-Gordon equation with the ultra-generalized exponential –hyperbolic potential (UGEHP)

The Klein-Gordon equation for a spinless particle for $\hbar=c=1$ in N-dimensions is given as (Louis, *et al*., 2018)

$$\left[-\nabla^2+\left(M+S(r)\right)^2+\frac{(N+2l-1)(N+2l-3)}{4r^2}\right]\psi(r,\theta,\varphi)=\left[E_{nl}-V(r)\right]^2\psi(r,\theta,\varphi) \quad (5)$$

where ∇^2 is the Laplacian, M is the reduced mass, E_{nl} is the energy spectrum, n and l are the radial and orbital angular momentum quantum numbers respectively. A wave function that satisfies equation 5 can be represented according to equation 6 as follows:

$$\psi(r,\theta,\varphi)=\frac{R_{nl}}{r}Y_{lm}(\theta,\varphi) \quad (6)$$

The separation of the angular component of the wave function gave expression for the radial parts as follows:

$$\frac{d^2R(r)}{dr^2}+\left[\left(E_{nl}{}^2-M^2\right)+V^2(r)-S^2(r)-2\left(E_{nl}V(r)+MS(r)\right)-\frac{(N+2l-1)(N+2l-3)}{4r^2}\right]R(r)=0 \quad (7)$$

Thus, for equal vector and scalar potentials $V(r)=S(r)=2V(r)$, then equation 7 becomes

$$\frac{d^2R(r)}{dr^2}+\left[\left(E_{nl}{}^2-M^2\right)-2V(r)\left(E_{nl}+M\right)-\frac{(N+2l-1)(N+2l-3)}{4r^2}\right]R(r)=0 \quad (8)$$

(8)

Upon substituting equation 3 into equation 8, we obtain

$$\frac{d^2R(r)}{dr^2}+\left[\begin{array}{l}\left(E_{nl}{}^2-M^2\right)+\left(-\frac{2\beta_0}{r^2}+\frac{2\beta_1}{r}-2\beta_2 r+2\beta_3 r^2-2\beta_4\right)\left(E_{nl}+M\right)\\ -\frac{(N+2l-1)(N+2l-3)}{4r^2}\end{array}\right]R(r)=0 \qquad (9)$$

Transformation of the r (in equation 9) to s coordinates yields equation 10

$$s=\frac{1}{r} \qquad (10)$$

Therefore, the second derivatives in equation 10 can be expressed according to equation 11,

$$\frac{d^2R(\mathrm{r})}{dr^2}=2s^3\frac{dR(\mathrm{s})}{ds}+s^4\frac{d^2R(\mathrm{s})}{ds^2} \qquad (11)$$

The substitution of equations 10 and 11 into equation 9 gave equation 12 as follows:

$$\frac{d^2R(s)}{ds^2}+\frac{2}{s}\frac{dR}{ds}+\frac{1}{s^4}\left[\begin{array}{l}\left(E_{nl}{}^2-M^2\right)+\left(-2\beta_0 s^2+2\beta_1 s-\frac{2\beta_2}{s}+\frac{2\beta_3}{s^2}-2\beta_4\right)\left(E_{nl}+M\right)\\ -\frac{(N+2l-1)(N+2l-3)s^2}{4}\end{array}\right]R(\mathrm{s})=0$$

(12)

We were able to propose the following approximation scheme on the term $\frac{\beta_2}{s}$ and $\frac{\beta_3}{s^2}$ through the assumption that there is a characteristic radius r_0 of the meson. The scheme was achieved by the expansion of $\frac{\beta_2}{s}$ and $\frac{\beta_3}{s^2}$ in a power series around r_0; i.e., around $\delta\equiv\frac{1}{r_0}$, in the x-space up to the second order. This is similar to Pekeris-type approximation, which helps to deform the centrifugal term such that the potential can be solved by the NU method (Akpan *et al*., 2021)

Setting $y=s-\delta$ and around $y=0$, it can be expanded into a series of powers as;

$$\frac{\beta_2}{s}=\frac{\beta_2}{y+\delta}=\frac{\beta_2}{\delta\left(1+\frac{y}{\delta}\right)}=\frac{\beta_2}{\delta}\left(1+\frac{y}{\delta}\right)^{-1} \qquad (13)$$

which yields

$$\frac{\beta_2}{s}=\beta_2\left(\frac{3}{\delta}-\frac{3s}{\delta^2}+\frac{s^2}{\delta^3}\right) \qquad (14)$$

Similarly,

$$\frac{\beta_3}{s^2}=\beta_3\left(\frac{6}{\delta^2}-\frac{8s}{\delta^3}+\frac{3s^2}{\delta^4}\right) \qquad (15)$$

Also, the substitution of equations 14 and 15 into equation 12 led to equation 16 as follows:

$$\frac{d^2R(\mathrm{s})}{ds^2}+\frac{2s}{s^2}\frac{dR(\mathrm{s})}{ds}+\frac{1}{s^4}\left[-\varepsilon+\beta s-\gamma s^2\right]R(\mathrm{s})=0 \qquad (16)$$

where

$$\left.\begin{array}{l}-\varepsilon=\left(\left(E_{nl}^2-M^2\right)-\frac{6\beta_2}{\delta}\left(E_{nl}+M\right)+\frac{12\beta_3}{\delta^2}\left(E_{nl}+M\right)-2\beta_4\left(E_{nl}+M\right)\right)\\ \beta=\left(2\beta_1\left(E_{nl}+M\right)+\frac{6\beta_2}{\delta^2}\left(E_{nl}+M\right)-\frac{16\beta_3}{\delta^3}\left(E_{nl}+M\right)\right)\\ \gamma=\left(2\beta_0\left(E_{nl}+M\right)+\frac{2\beta_2}{\delta^3}\left(E_{nl}+M\right)-\frac{6\beta_3}{\delta^4}\left(E_{nl}+M\right)+\frac{(N+2l-1)(N+2l-3)}{4}\right)\end{array}\right\} \qquad (17)$$

The comparison of equation 16 with equation A1 resulted in the following functions (equation 18)

$$\left.\begin{array}{l}\tilde{\tau}(\mathrm{s})=2s,\ \sigma(\mathrm{s})=s^2\\ \tilde{\sigma}(\mathrm{s})=-\varepsilon+\beta s-\gamma s^2\\ \sigma'(\mathrm{s})=2s,\ \sigma''(\mathrm{s})=2\end{array}\right\} \qquad (18)$$

(18)

We substitute equation 18 into equation A9 and obtain

$$\pi(s) = \pm\sqrt{\varepsilon - \beta s + (\gamma + k)s^2} \tag{19}$$

In the process of determination of the k, in equation 19, the discriminant of the function (equation 20) and equation 21 was obtained.

$$k = \frac{\beta^2 - 4\gamma\varepsilon}{4\varepsilon} \tag{20}$$

$$\pi(s) = \pm\left(\frac{\beta s}{2\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}}\right) \tag{21}$$

Differentiation of the negative part of equation 21, required for bound state problems yielded a physical acceptable solution given by equation 22

$$\pi'_{-}(s) = -\frac{\beta}{2\sqrt{\varepsilon}} \tag{22}$$

The substitution of equations 18 and 22 into equation A7 yielded equation 23 and upon differentiation, equation 24 was obtained.

$$\tau(s) = 2s - \frac{\beta s}{\sqrt{\varepsilon}} + \frac{2\varepsilon}{\sqrt{\varepsilon}} \tag{23}$$

$$\tau'(s) = 2 - \frac{\beta}{\sqrt{\varepsilon}} \tag{24}$$

The simplification of equation 25 using equations A10 and A11 yielded equations 25 and 26 respectively.

$$\lambda = \frac{\beta^2 - 4\gamma\varepsilon}{4\varepsilon} - \frac{\beta}{2\sqrt{\varepsilon}} \tag{25}$$

$$\lambda_n = \frac{n\beta}{\sqrt{\varepsilon}} - n^2 - n \tag{26}$$

The comparison of equations 25 with equation 26 followed by substitution of equations 4 and 17 to the respective terms yielded the energy eigenvalue equation of the UGEHP in the relativistic limit (equation 27)

$$\begin{aligned} M^2 - E_{nl}^2 &= \frac{6(2c\alpha^2 + \alpha d)}{\delta}(E_{nl} + M) - \frac{12\alpha(d-g)}{\delta^2}(E_{nl} + M) \\ &+ 2(8a\alpha^2 + 2b\alpha^2 - 2c\alpha - \alpha d + g\alpha + f)(E_{nl} + M) \\ &+ \frac{1}{4}\left[\frac{(4a\alpha + 2b\alpha + d - g)(E_{nl} + M) + \frac{6(2c\alpha^2 + \alpha d)(E_{nl} + M)}{\delta^2} - \frac{16\alpha(d-g)(E_{nl} + M)}{\delta^3}}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - 2(a+b)(E_{nl} + M) + \frac{2(2c\alpha^2 + \alpha d)}{\delta^3}(E_{nl} + M) - \frac{6\alpha(d-g)}{\delta^4}(E_{nl} + M) + \frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2 \end{aligned} \tag{27}$$

2.1 Non relativistic limit

In order to analyze the non-relativistic limit arising from the analytical processes employed in this study, the form of transformations considered were $M + E_{nl} \to \frac{2\mu}{\hbar^2}$ and $M - E_{nl} \to -E_{nl}$, where μ is the reduced mass , and substituting it into equation 27, we have the non-relativistic energy eigenvalue equation as

$$\begin{aligned} E_{nl} &= \frac{12\alpha(d-g)}{\delta^2} - \frac{6(2c\alpha^2 - \alpha d)}{\delta} - 2(8a\alpha + 2b\alpha^2 - 2c\alpha - \alpha d + \alpha g + f) \\ &- \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu}{\hbar^2}(4a\alpha + 2b\alpha + d - g) + \frac{12\mu}{\delta^2\hbar^2}(2c\alpha^2 + \alpha d) - \frac{32\mu\alpha}{\delta^3\hbar^2}(d-g)}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu}{\hbar^2}(a+b) + \frac{4\mu}{\delta^3\hbar^2}(2c\alpha^2 + \alpha d) - \frac{12\mu\alpha}{\delta^4\hbar^2}(d-g) + \frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2 \end{aligned} \tag{28}$$

However, the following special case was also considered

1. Setting $a=b=c=g=f=\alpha=0$,we obtain the energy equation for Coulomb potential

$$E_{nl}=-\frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu d}{\hbar^2}}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2 \tag{29}$$

The result expressed by equation 29 is consistent with the one expressed by equation 36(obtained by Edet *et al*., 2020) especially when $N=3$.

The unnormalized wave function in terms of Laguerre polynomials is given as

$$\psi(s)=B_{nl}s^{-\frac{\alpha_1}{2\sqrt{\varepsilon}}}e^{-\frac{\varepsilon}{s\sqrt{\varepsilon}}}L_n^{\frac{\alpha_1}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{s\sqrt{\varepsilon}}\right), \tag{30}$$

where L_n is the associated Laguerre polynomials and B_{nl} is normalization constant, which can be obtain from

$$\int_0^\infty |\mathrm{B}_{nl}(r)|^2 dr=1 \tag{31}$$

3.0 Thermodynamic properties of the KGE with the ultra-generalized exponential –hyperbolic potential (UGEHP)

Thermodynamic properties of UGEHP can be obtain from the partition function by the simplification of equation 28 to the form expressed by equation 32,

$$E_{nl}=P_1-\frac{\hbar^2}{8\mu}\left[\frac{P_2}{(n+\sigma)}\right]^2 \tag{32}$$

where,

$$\sigma=\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{4\mu}{\hbar^2}(a+b)+\frac{4\mu}{\delta^3\hbar^2}\left(2c\alpha^2+\alpha d\right)-\frac{12\mu\alpha}{\delta^4\hbar^2}(d-g)+\frac{(N+2l-1)(N+2l-3)}{4}} \tag{33}$$

$$P_1=\frac{12\alpha(d-g)}{\delta^2}-\frac{6\left(2c\alpha^2-\alpha d\right)}{\delta}-2\left(8a\alpha+2b\alpha^2-2c\alpha-\alpha d+\alpha g+f\right) \tag{34}$$

$$P_2=\frac{4\mu}{\hbar^2}(4a\alpha+2b\alpha+d-g)+\frac{12\mu}{\delta^2\hbar^2}\left(2c\alpha^2+\alpha d\right)-\frac{32\mu\alpha}{\delta^3\hbar^2}(d-g) \tag{35}$$

3.1 Partition function $Z(\beta)$

The partition function according to Abu-Shady *et al.* (2019) is as equation 36

$$Z(\beta)=\sum_{n=0}^{\lambda}e^{-\beta E_{nl}} \tag{36}$$

where, β is the reciprocal of the product of Boltzmann constant (K) and the absolute temperature (equation 37)

$$\beta=\frac{1}{KT} \tag{37}$$

n is the principal quantum number, $n=0,1,2,3...$ and λ is the maximum or upper bound quantum number.

The substitution of equation 32 into equation 36 yielded equation 38

$$Z(\beta)=\sum_{n=0}^{\lambda}e^{-\beta\left(P_1-\frac{\hbar^2}{8\mu}\left[\frac{P_2}{(n+\sigma)}\right]^2\right)} \tag{38}$$

In the classical limit, at high temperature T, the summation can be replaced by an integral,

$$Z(\beta)=\int_0^{\lambda}e^{M_1\beta+\frac{N\beta}{\rho^2}}d\rho \tag{39}$$

The parameters in the above equations are defined as follows:

$$n+\sigma=\rho \tag{40}$$

$$M_1=-P_1 \tag{41}$$

$$N_1=\frac{\hbar^2 P_2^2}{8\mu} \tag{42}$$

The integration of equation 39 yielded the expression for the partition function shown in equation 43,

$$Z(\beta)=\frac{1}{2}e^{M_1\beta}\sqrt{N_1\beta}\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}-2\sqrt{N_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)}{\sqrt{N_1\beta}}-2\sqrt{\pi}\right) \tag{43}$$

However, the imaginary error function $erfi(\mathrm{y})$ is defined according to equation 44 as follows (Okorie, *et al.*, 2018)

$$erfi(\mathrm{y})=\frac{erf(iy)}{i}=\frac{2}{\sqrt{\pi}}\int_0^y e^{t^2}dt. \tag{44}$$

3.2 Mean energy $U(\beta)$

According to Abu-Shady *et al*.(2019), the mean energy of HQS takes the following form

$$U(\beta)=-\frac{\partial}{\partial\beta}InZ(\beta), \tag{45}$$

The substitution of equation 43 into equation 45 gave equation 46

$$U(\beta)=-\frac{\left[M_1e^{M_1\beta}\sqrt{N_1\beta}\Delta_1+\frac{\frac{1}{4}e^{M_1\beta}\Delta_1N_1}{\sqrt{N_1\beta}}+\frac{1}{2}e^{M_1\beta}\sqrt{N_1\beta}\Delta_2\right]}{e^{M_1\beta}\sqrt{N_1\beta}\Delta_1} \tag{46}$$

where,

$$\Delta_1=\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}-2\sqrt{N_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)}{\sqrt{N_1\beta}}-2\sqrt{\pi} \tag{47}$$

$$\Delta_2=-\frac{\lambda e^{\frac{N\beta}{\lambda^2}}N_1}{(N_1\beta)^{\frac{3}{2}}}-\frac{\sqrt{N_1}\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)}{\sqrt{N_1\beta^2}}+\frac{N_1^{\frac{3}{2}}\sqrt{\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)}{(N_1\beta)^{\frac{3}{2}}} \tag{48}$$

3.3 Free energy $\mathrm{F}(\beta)$

The free energy of the HQS can be expressed as follows (Abu-Shady *et al.*,2019):

$$F(\beta)=-KT\ln Z(\beta) \tag{49}$$

The substitution of equations 37 and 43 into equation 49 gave equation 50, which is exponential form of the free energy

$$F(\beta)=-\frac{1}{\beta}\ln\left[\frac{1}{2}e^{M_1\beta}\sqrt{N_1\beta}\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}-2\sqrt{N_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)}{\sqrt{N_1\beta}}-2\sqrt{\pi}\right)\right] \tag{50}$$

3.4 Entropy $S(\beta)$

The entropy of the HQS can be expressed as follows (Abu-Shady *et al.*,2019):

$$S(\beta)=K\ln Z(\beta)-K\beta\frac{\partial}{\partial\beta}\ln Z(\beta) \tag{51}$$

The substitution of equations 43 and 46 into equation 51 gave equation 52

$$S(\beta)=K\ln\left[\frac{1}{2}e^{M_1\beta}\sqrt{N_1\beta}\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}-2\sqrt{N_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)}{\sqrt{N_1\beta}}-2\sqrt{\pi}\right)\right] \tag{52}$$

$$-K\beta\frac{\left(M_1e^{M_1\beta}\sqrt{N_1\beta}\Delta_1+\frac{\frac{1}{4}e^{M_1\beta}\Delta_1N_1}{\sqrt{N_1\beta}}+\frac{1}{2}e^{M_1\beta}\sqrt{N_1\beta}\Delta_2\right)}{e^{M_1\beta}\sqrt{N_1\beta}\Delta_1}$$

3.5 Specific heat $C(\beta)$

The specific heat of the HQS can be expressed as follows (Abu-Shady *et al.*,2019):

$$C(\beta)=\frac{\partial U}{\partial T}=-K\beta^2\frac{\partial U}{\partial\beta} \tag{53}$$

The substitution of equations 47 and 49 into equation 46, yielded equation 54

$$C(\beta)=-K\beta^2\left\{\begin{array}{l}\frac{1}{\gamma_1}\left[\begin{array}{l}\left(M_1^2\right)\gamma_1+\frac{M_1e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\sqrt{N_1\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)-2\sqrt{\pi}\right)N_1}{\sqrt{N_1\beta}}-\frac{2M_1e^{M_1\beta}\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\beta}\\-\frac{1}{4}\frac{e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\sqrt{N_1\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)-2\sqrt{\pi}\right)N_1^{\ 2}}{\left(N_1\beta\right)^{\frac{3}{2}}}+\frac{1}{2}\frac{e^{M_1\beta}\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\beta^2}-\frac{e^{M_1\beta}N_1e^{\frac{N_1\beta}{\lambda^2}}}{3\lambda}\end{array}\right]\\-\frac{1}{\gamma_1}\left[M_1\gamma_1+\frac{\gamma_2}{2}-\frac{e^{M_1\beta}\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\beta}\right]\\-\frac{1}{\gamma_3}\left[\begin{array}{l}\left[\frac{1}{2}\left(M_1e^{M_1\beta}\right)\sqrt{N_1\beta}\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\sqrt{N_1\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right)-2\sqrt{\pi}\right)+\frac{\gamma_2}{4}-\frac{1}{2}\frac{e^{M_1\beta}}{\beta}\right]N_1\\+\left[M_1\gamma_1+\frac{\gamma_2}{2}-\frac{e^{M_1\beta}+\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\beta}\right]\lambda e^{\frac{N_1\beta}{\lambda^2}}\end{array}\right]\end{array}\right\} \tag{54}$$

where

$$\gamma_1 = e^{M_1\beta}\sqrt{N_1\beta}\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\sqrt{N_1\beta}} - 2\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right) - 2\sqrt{\pi}\right) \quad (55)$$

$$\gamma_2 = \frac{e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\sqrt{N_1\beta}} - 2\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right) - 2\sqrt{\pi}\right)N_1}{\sqrt{N_1\beta}} \quad (56)$$

$$\gamma_3 = e^{M_1\beta}N_1\beta^2\left(\frac{2\lambda e^{\frac{N_1\beta}{\lambda^2}}}{\sqrt{N_1\beta}} - 2\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\lambda}\right) - 2\sqrt{\pi}\right)^2 \quad (57)$$

4.0 Results and Discussion

4.1 Results

We calculate mass spectra of the heavy quarkonium system such as charmonium and bottomonium in 3-dimensional space ($N = 3$) that have the quark and antiquark flavor, using the following relation (Inyang, *et al.*,2021; Inyang, *et al.*, 2020).

$$M = 2m + E_{nl}^{N=3} \quad (58)$$

where m is quarkonium bare mass, and $E_{nl}^{N=3}$ is energy eigenvalues. By substituting equation 28 into equation 58 we obtain the mass spectra for UGEHP as:

$$M = 2m + \frac{12\alpha(d-g)}{\delta^2} - \frac{6(2c\alpha^2 - \alpha d)}{\delta} - 2(8a\alpha + 2b\alpha^2 - 2c\alpha - \alpha d + \alpha g + f)$$
$$-\frac{\hbar^2}{8\mu}\left[\frac{\frac{4\mu}{\hbar^2}(4a\alpha + 2b\alpha + d - g) + \frac{12\mu}{\delta^2\hbar^2}(2c\alpha^2 + \alpha d) - \frac{32\mu\alpha}{\delta^3\hbar^2}(d-g)}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu}{\hbar^2}(a+b) + \frac{4\mu}{\delta^3\hbar^2}(2c\alpha^2 + \alpha d) - \frac{12\mu\alpha}{\delta^4\hbar^2}(d-g) + \frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2 \quad (59)$$

Table 1: **Mass spectra of charmonium in** (GeV)

$\left(a = -22.17885\ GeV, b = 13.73217\ GeV, c = 10.73524\ GeV^2, d = 3.010241\ GeV^{-1}, g = 10.64213\ , f = 0.05\ GeV^3, \alpha = 0.01, \delta = 1.00252\ GeV, m_c = 1.209\ GeV, N = 3, \hbar = 1, \mu = 0.6045\ GeV\right)$

State	Present work	Abu-Shady *et al*, 2019	Abu-Shady 2016	Experiment(Tanabashi, *et al.*2018)
1S	3.096	3.096	3.096	3.096
2S	3.686	3.686	3.672	3.686
1P	3.526	3.255	3.521	3.525
2P	3.767	3.779	3.951	3.773
3S	4.040	4.040	4.085	4.040
4S	4.262	4.269	4.433	4.263
1D	3.768	3.504	3.800	3.770
2D	4.034	-	-	4.159
1F	4.162	-	-	-

Table 2: **Mass spectra of bottomonium in** (GeV)

$$\left(\begin{array}{l} a=-20.99857\ GeV, b=13.6254385\ GeV, c=13.73524\ GeV^2, d=4.110240\ GeV^{-1}, g=11.542130, \\ f=0.05\ GeV^3, \alpha=0.01, \delta=1.00252\ GeV, m_c=4.823\ GeV, N=3, \hbar=1, \mu=2.4115\ GeV \end{array}\right)$$

State	Present work	Abu-Shady *et al*, 2019	Abu-Shady 2016	Experiment(Tanabashi, *et al*.2018)
1S	9.460	9.460	9.4620	9.460
2S	10.023	10.023	10.027	10.023
1P	9.761	9.619	9.9630	9.899
2P	10.261	10.114	10.299	10.260
3S	10.355	10.355	10.361	10.355
4S	10.579	10.567	10.624	10.580
1D	9.998	9.864	10.209	10.164
2D	10.206	-	-	-
IF	10.109	-	-	-

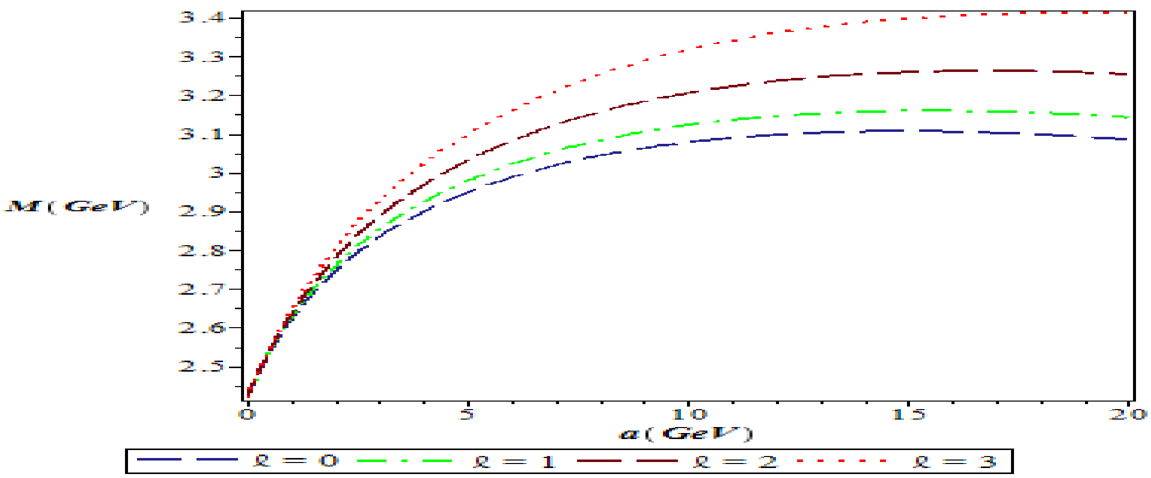

Fig. 1: Variation of mass spectra with potential strength (a) **for different quantum numbers**

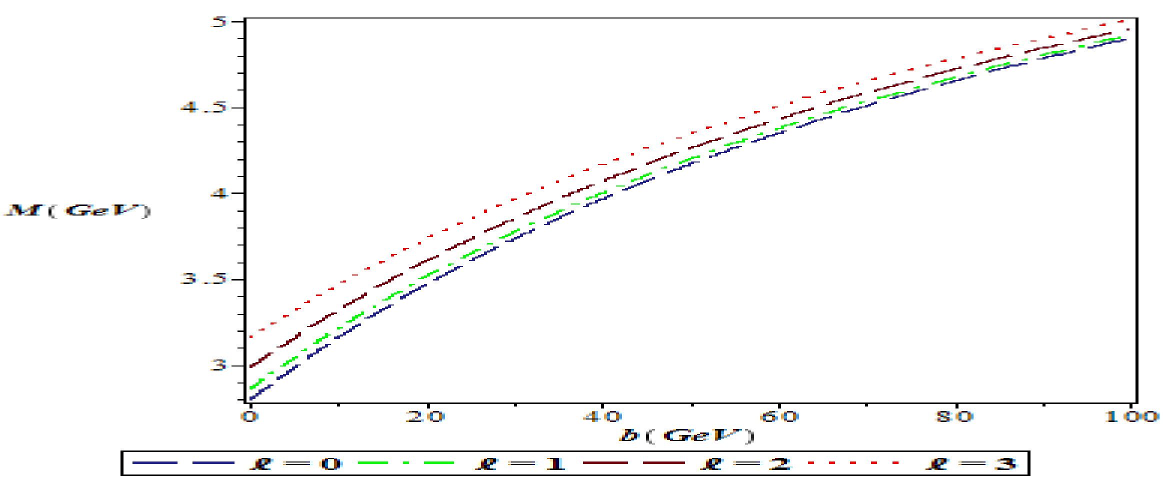

Fig. 2: Variation of mass spectra with potential strength (b) **for different quantum numbers**

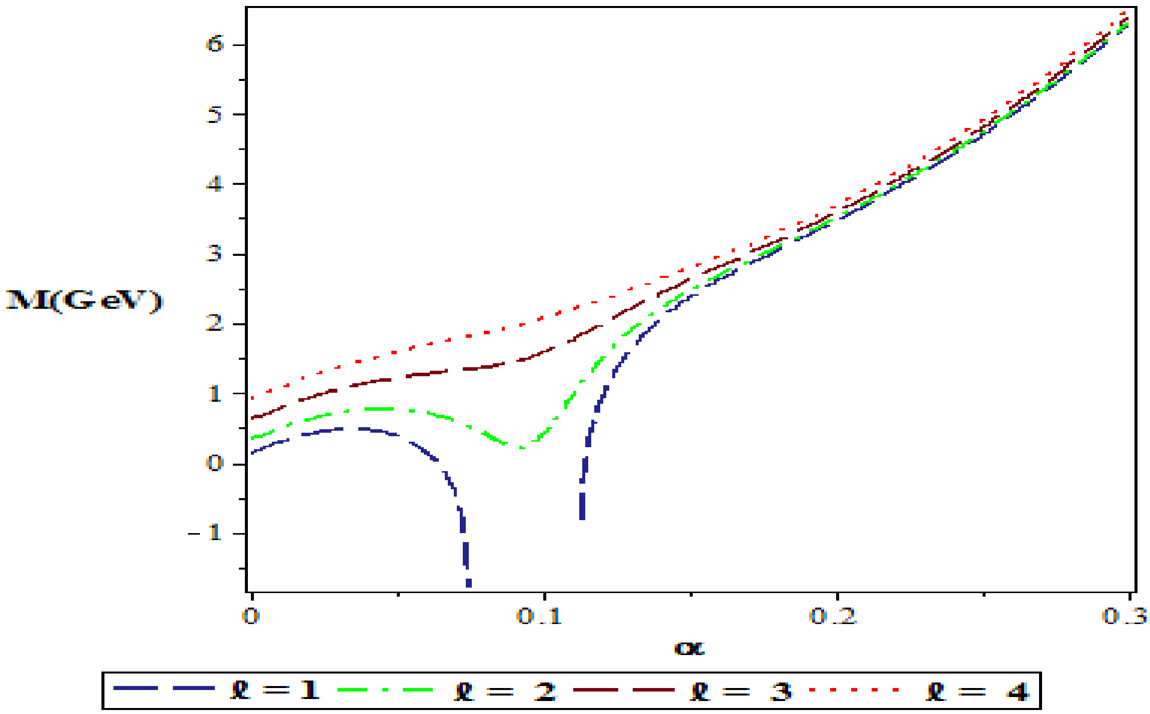

Fig. 3: Variation of mass spectra with screening parameter (α) for different quantum numbe

4.2 *Thermodynamic properties plots*

In this subsection, we present the plots of thermodynamic properties.

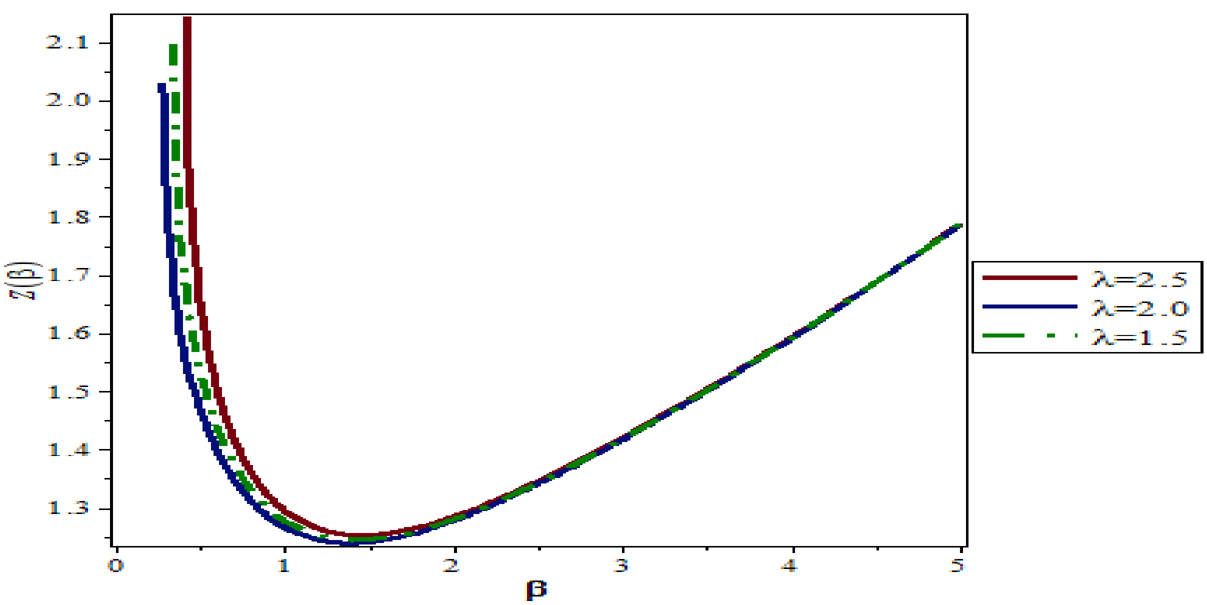

Fig. 4: Variation of the partition function $Z(\beta)$ versus temperature (β) for different values of maximum quantum number (λ)

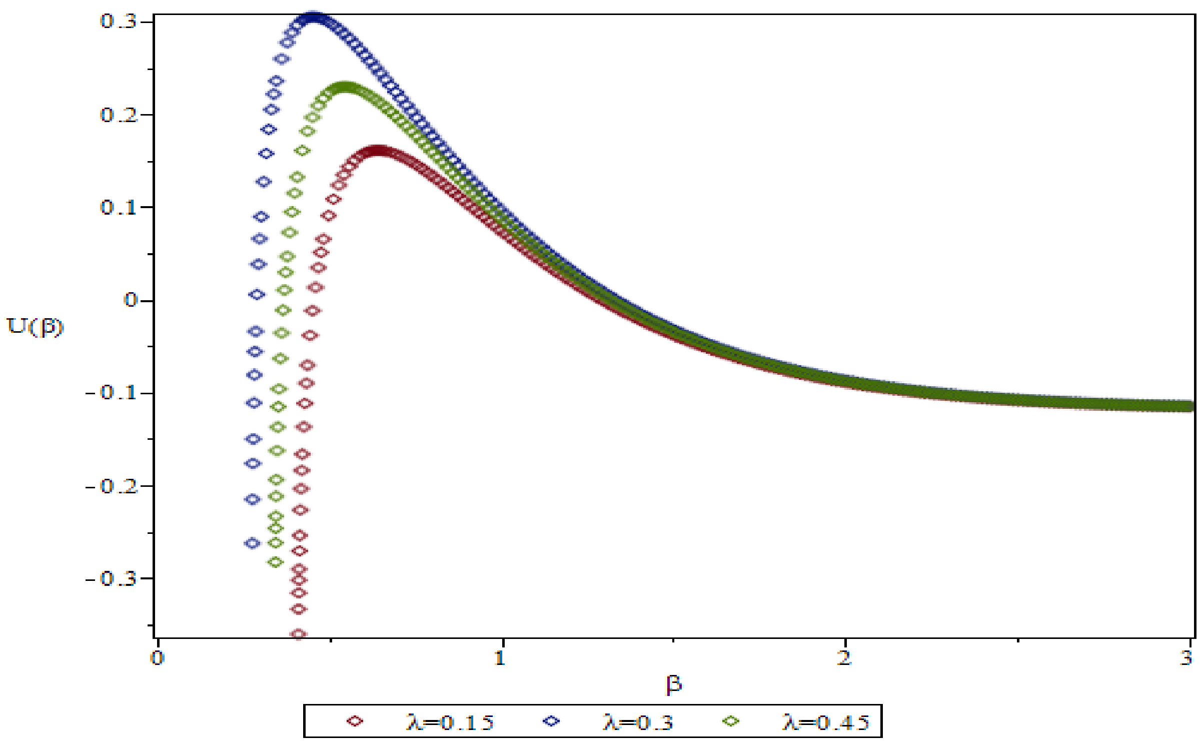

Fig. 5: Variation of the mean energy $U(\beta)$ versus temperature (β) for different values of maximum quantum number (λ)

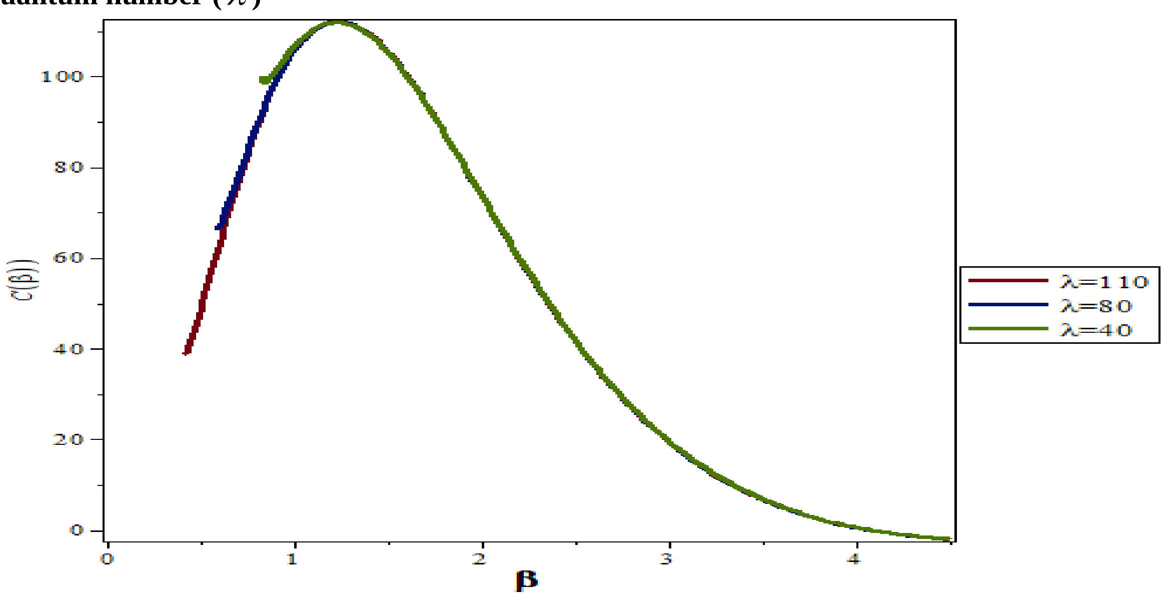

Fig 6: Variation of the specific heat $C(\beta)$ versus temperature (β) for different values of maximum quantum number (λ)

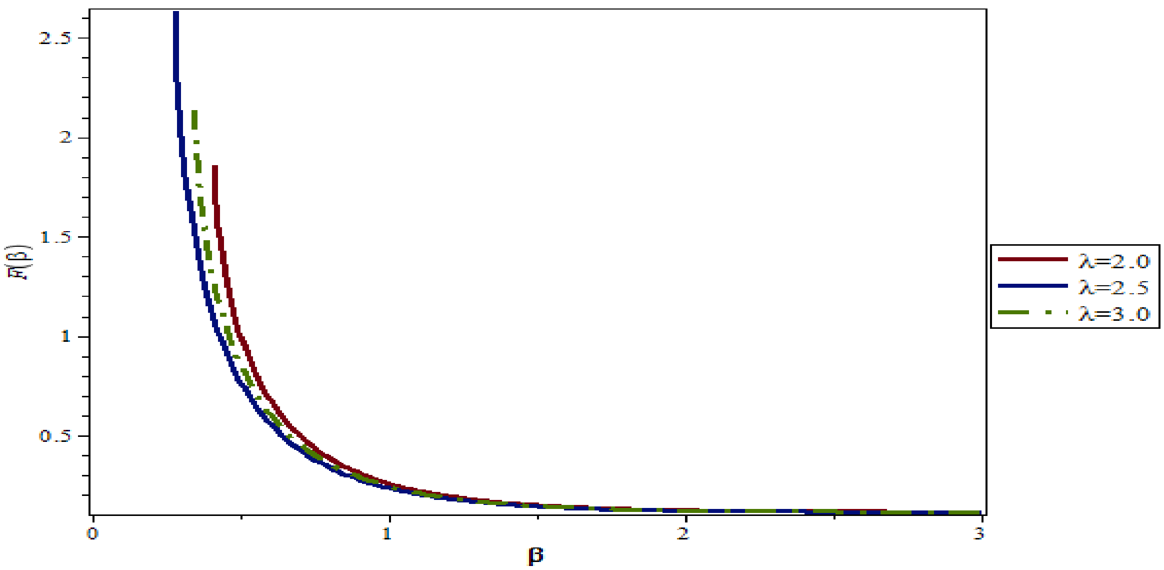

Fig. 7: Variation of the free energy $F(\beta)$ versus temperature (β) for different values of maximum quantum number (λ)

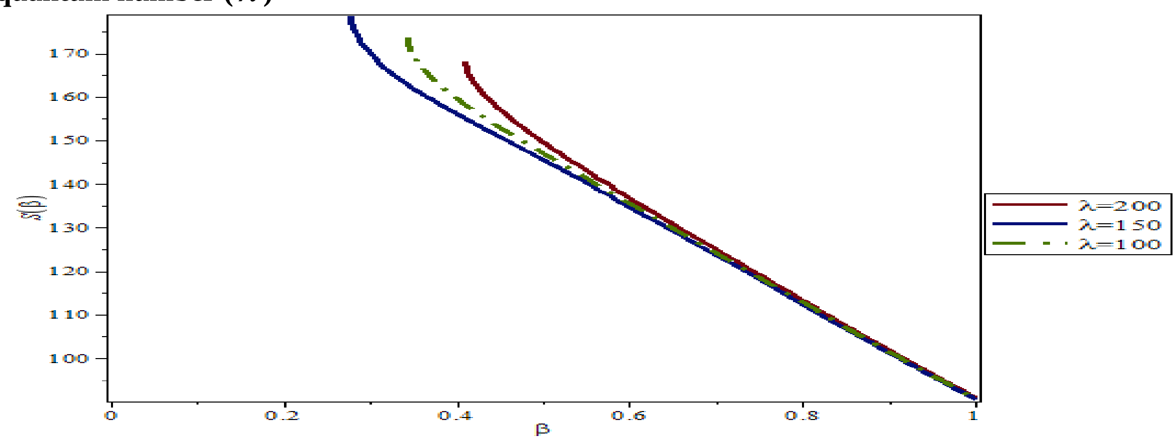

Fig.8: Variation of the entropy $S(\beta)$ versus temperature (β) for different values of maximum quantum number

4.3 Discussion of results

Equation 49 was used to calculate mass spectra of charmonium and bottomonium for different quantum states and the free parameters that were obtained from equation 59 were consequences of the two algebraic equations.

For bottomonium $b\bar{b}$ and charmonium $c\bar{c}$ systems the numerical values of these masses as m_b = 4.823 GeV and m_c = 1.209 GeV were adopted (Barnett, *et al.*, 2012). The corresponding reduced mass are μ_b = 2.4115 GeV and μ_c = 0.6045 GeV, respectively. The experimental data were taken from the results reported Tanabashi *et al.* (2018). We observed that the results obtained from the calculations of mass spectra of charmonium and bottomonium are in good agreement with experimental data with those reported by other researchers Abu-Shady *et al.* (2019) and Abu-Shady (2016) as presented in Tables 1 and 2. In order to test for the accuracy of the predicted results, we used a Chi squared function to determine the error between the experimental data and theoretical predicted values. The maximum error in comparison with the experimental data was 0.0059 GeV. We plotted the variation of mass spectra energy with respect to potential strengths, and screening parameter (α) respectively. In Figs. 1 and 2, the mass spectra energy increases as the potential strength increases for the different quantum numbers. Also from Fig. 3, the mass spectra energy was also to increase with increasing screening parameter.

The thermodynamic properties were obtained by first obtaining the partition function. Fig. 4 reveals that the partition function $Z(\beta)$ decreases exponentially with temperature β. The plots of mean energy $U(\beta)$ with different values of β and λ are shown in Fig.5, which clearly reveal a monotonic increase with increase values of β and λ before the decreasing trend was observed. Fig. 6 show the plot of specific heat $C(\beta)$ with temperature for different values of maximum quantum number (λ).The Figure reveals that the heat capacity tend to increase monotonically as β increases and then decreases as β and λ increases with each plot converging. The free energy $F(\beta)$ is plotted as a function of temperature shown in Fig.7. The free energy is seen to decrease exponentially as β and λ increases and converges at a point close to zero. The plot of entropy $S(\beta)$ as a function of temperature β and maximum quantum number λ is shown in Fig. 8. We observed that the entropy decreases with increasing β. This finding is in agreement with Okorie *et al.* (2018) in which the entropy increases with increasing temperature for the system.

5.0 Conclusion

In this study, we modelled the adopted ultra-generalized exponential–hyperbolic potential to interact in quark-antiquark system. We obtained the approximate solutions of the KGE for energy eigenvalues and unnormalized wave function using the NU method. We applied the present results to compute heavy-meson masses of charmonium and bottomonium for different quantum states. The result agreed with experimental data (with a maximum error of 0.0059 GeV) and with the results obtained by other researchers. Mass spectra variation with potential strengths and screening parameter (α) were plotted and discussed. We also obtained thermodynamic properties such as free energy, mean energy, entropy, and specific heat and their plots were in acceptable concurrence with the work of Abu-Shady *et al.* (2019) and Okorie *et al.* (2018). Therefore, ultra-generalized exponential–hyperbolic potential provides satisfied results for thermodynamic properties and mass spectra of heavy quarkonium system.

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Conflict of Interest

The authors declared no conflict of interest

APPENDIX A: Review of Nikiforov-Uvarov (NU) method

The NU method according to Nikiforov & Uvarov (1988) is used to solve the second-order differential equation which takes the following form:

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \quad \text{(A1)}$$

where $\tilde{\sigma}(s)$ and $\sigma(s)$ are polynomials of maximum second degree and $\tilde{\tau}(s)$ is a polynomial of maximum first degree. The exact solution of equation (A1) takes the form

$$\psi(s)=\phi(s)\chi(s) \quad \text{(A2)}$$

Substituting equation (A2) into equation (A1), we obtain

$$\sigma(s)\chi''(s)+\tau(s)\chi'(s)+\lambda\chi(s)=0 \quad \text{(A3)}$$

Where the function $\phi(s)$ satisfies the following relation

$$\frac{\phi'(s)}{\phi(s)}=\frac{\pi(s)}{\sigma(s)} \quad \text{(A4)}$$

And $\chi(s)$ is a hypergeometric-type function, whose polynomial solutions are obtained from the Rodrigues relation

$$\chi_n(s)=\frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \quad \text{(A5)}$$

where B_n is the normalization constant and $\rho(s)$ the weight function which satisfies the condition below;

$$\frac{d}{ds}\left(\sigma(s)\rho(s)\right)=\tau(s)\rho(s) \quad \text{(A6)}$$

where also

$$\tau(s)=\tilde{\tau}(s)+2\pi(s) \quad \text{(A7)}$$

For bound solutions, it is required that

$$\frac{d\tau(s)}{ds}<0 \quad \text{(A8)}$$

We can then obtain the eigenfunction and eigenvalues using the definition of the following function $\pi(s)$ and parameter λ, given as:

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)} \quad \text{(A9)}$$

and

$$\lambda=k+\pi'(s) \quad \text{(A10)}$$

The value of k can be calculated if the function under the square root in equation (A10) is the square of a polynomial. This is possible if its discriminate is equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda_n+n\tau'(s)+\frac{n(n-1)}{2}\sigma''(s)=0 \quad \text{(A11)}$$