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A Fifth-Order Five-Stage Trigonometrically-Fitted Improved Runge-Kutta Method for Oscillatory Initial Value Problems

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Abstract: *A fifth-order five-stage trigonometrically-fitted Improved Runge-Kutta (TFIRK5-5) method for solving first order initial value problems (IVPs) has been derived and analysed in this work. The method is shown to integrate exactly the initial value problem whose solution is a linear combination of the set functions sin(ωx) and cos(ωx) for trigonometrically fitted, and $e^{i\omega x}$ and $e^{-i\omega x}$ for exponentially fitted, where $\omega > 0 \in R$, being the main frequency of the problem, is used to increase the accuracy of the method. The numerical results revealed the effectiveness of the new method in comparison with other existing methods in the literature.*

Key Words: *Improved Runge-Kutta (IRK) method, Initial value Problem, Oscillating solution, Trigonometric fitting.*

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1.0 Introduction

The Runge-Kutta method is a unique method for numerical solution of the first order IVP$y'(x) = f(x, y(x)),\ y(x_0) = y_0\ \ x\,\epsilon[x_0, X]$ (1)

The general $s-$stage Runge-Kutta method is governed by the the relation

$$\left.\begin{aligned} y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i k_i \\ k_i &= f\left(x_n + c_i h, y_n + h\sum_{j=1}^{s} a_{i,j}k_j\right), i = 1, \dots, s \end{aligned}\right\}$$

where the $k_i's$ represent various estimates of the slope $f(x, y)$ from x_n to x_{n+1}, s is the number of stages of the method and represents the number of evaluations of $f(x, y)$ in each step. Except for $s > 4$, the number of stages is typically the order of the method. One technique for improving the order, and in effect the accuracy, of Runge-Kutta (RK) methods is to increase the number of function evaluations, which consequentially reduces the efficiency of the method. Several authors have contributed a great deal toward improving the effectiveness of RK methods without jeopardizing the method's efficiency, that is, by reducing the number of function evaluations per step. Some of the improvement are those reported by Goeken and Johnson (2000), Xinyuan (2003), Phohomsiri and Udwadia (2004) and Udwadia and Farahani (2008). The Improved Runge-Kutta (IRK) methods is a class of two-step methods that require lower number of function evaluations, i.e., stages, compared to the classical Runge-Kutta method; therefore, they are computationally more efficient at achieving the same order of accuracy. More so, the IRK method introduces new terms of k_{-i} which are calculated from k_i, $(i > 2)$ in the previous step. These methods can also be used for autonomous as well as non-autonomous systems. The IRK method with $\mathrm{s}-$stages for solving (1) has the form:

$$y_{n+1} = y_n + h\left(b_1 k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i(k_i - k_{-i})\right) \quad (3)$$

for $1 \le n \le N-1$, where

$$\left.\begin{aligned} k_1 = f(x_n, y_n), \qquad k_{-1} = f(x_{n-1}, y_{n-1}) \\ k_i = f\left(x_n + c_i h, y_n + h\sum_{i=1}^{s} a_{i,j}k_j\right), 2 \le i \le s \\ k_{-i} = f\left(x_{n-1} + c_i h, y_{n-1} + h\sum_{i=1}^{s} a_{i,j}k_{-j}\right), 2 \le i \le s \end{aligned}\right\} \quad (4)$$

for $c_2 \dots c_s \epsilon [0, 1]$ and f depend on both x and y while k_i and k_{-i} depend on the value of k_j and k_{-j} for $j = 1, \dots\dots, i-1$. where s is the number of function evaluations performed at each integration step and increases with the order of local accuracy of the IRK method. A one step method must provide the approximate solution of y_1 at the first step since the method is not self-starting and must ensure that the difference, $y_1 - y(x_1)$, is of order p or higher (Rabiei and Ismail, 2012).

Trigonometric fitting is a method that approximate a function f by series of trigonometric functions. The approximation g of f can be written as

$$g(x) = \sum_{i=1}^{m} \rho_i \cos\,(\omega_i x + \varphi_i)$$

where m is the number of terms required for approximating f, ρ_i is the amplitude, ω_i is the frequency and φ_i is the amplitude of the ith cosine function. Among the early works on this technique include, Gautschi (1961) and Lyche (1972). However, Vigo-Aguiar and Simos (2001) presented a simple procedure for adapting Cowell methods in any algebraic, trigonometric and exponential order. Fang *et al.* (2014) is two derivatives trigonometrically fitted RK method for solving oscillatory differential equations. Fawzi *et al.* (2016b) worked on an explicit Runge Kutta method with trigonometrical fitting for solving first order ODEs. Fawzi *et al.* (2016a) derived a fourth algebraic order explicit trigonometrically fitted modified Runge-Kutta method for the numerical solution of periodic IVPs. Ehigie *et al*. (2017) study was concentrated on a continuous Runge-Kutta Nystrom collocation method with trigonometric coefficients for periodic initial value problems. Neta and Changbum (2020) also presented a new trigonometrically fitted method for second order initial value problems. In this research, we examine the construction of a fifth-order five-stage trigonometrically fitted (TFIRK5-5) method based on two-step explicit fifth-order five-stage Improved Runge-Kutta (IRK5-5) method proposed by Rabiei *et al*. (2013). Ismail *et al.* (2018) developed algebraic order conditions for two-point block hybrid method up to order five using the approach of B-series. A fifth order two-point block explicit hybrid method for solving special second order ordinary differential equations (ODEs) was derived based on the order conditions. The existing explicit hybrid method of order five is employed at the first point. Eventually, the method is trigonometrically fitted in order to be suitable for solving highly oscillatory problems arising from special second order ODEs. The trigonometrically-fitted block method is then validated with a set of oscillatory problems over a very large in

2.0 Materials and Methods

A more compact form of the general IRK method (3) and (4) is

$$\begin{aligned} y_{n+1} = y_n + hb_1 f(x_n, y_n) - hb_{-1} f(x_{n-1}, y_{n-1}) \\ + h\sum_{i=2}^{s} b_i\big(f(x_n + c_i h, Y_i) - f(x_{n-1} + c_i h, Y_{-i})\big) \end{aligned} \quad (5)$$

where, $Y_i = y_n + h\sum_{j=1}^{i-1} a_{i,j}\, f\big(x_n + c_j h, Y_j\big)$ (6)

$$Y_{-i} = y_{n-1} + h\sum_{j=1}^{i-1} a_{i,j}\, f\big(x_{n-1} + c_j h, Y_{-j}\big) \quad (7)$$

with y_{n+1} and y_n being an approximation to $y(x_{n+1})$ and $y(x_n)$ respectively.

Trigonometrically- fitted techniques are derived to explicitly approximate the initial value problems whose solution are linear combination of the functions $\{e^{\omega x}, e^{-\omega x}\}$, where ω can be real or complex number (Berghe *et al*., 2000).

If a function $y(x)$ is integrated exactly by TFIRK method for all problems whose solution is $y(x)$ then,

$$y_n = y(x_n) = e^{i\omega x_n} \quad (8)$$

$$y_{n-1} = y(x_{n-1}) = y(x_n - h) = e^{i\omega(x_n - h)} \quad (9)$$

$$y_n' = i\omega e^{i\omega x_n} = f(x_n, y_n) \quad (10)$$

$$y_{n-1}' = i\omega e^{i\omega(x_n - h)} = f(x_{n-1}, y_{n-1}) \quad (11)$$

$$Y_i = e^{i\omega(x_n + c_i h)} \quad (12)$$

$$Y_{-i} = e^{i\omega(x_{n-1} + c_i h)} \quad (13)$$

Therefore, we obtain the recursive relations

$$cos(c_i z) = 1 - z\sum_{j=1}^{i-1} a_{ij}\sin(c_j z), \quad i = 2,3,\dots,s \quad (14)$$

$$sin(c_i z) = z\sum_{j=1}^{i-1} a_{ij}\, cos(c_j z), \qquad i = 2,3,\dots,s \quad (15)$$

$$cos(z) = 1 - zb_{-1}\, sin(z) - z\sum_{i=2}^{s} b_i\, sin(c_i z) + z\sum_{i=2}^{s} b_i\, sin\big(z(c_i - 1)\big) \quad (16)$$

$$sin(z) = zb_1 - zb_{-1}\, cos(z) + z\sum_{i=2}^{s} b_i\, cos(c_i z) - z\sum_{i=2}^{s} b_i\, cos\big(z(c_i - 1)\big) \quad (17)$$

The relations (14), (15), (16) and (17) are relations of order conditions of the trigonometrically-fitted method. These relations replace the equations of order conditions of two-step Improved Runge-Kutta method, which can be solved to give the coefficients of a particular method based on existing coefficients.

To derive a trigonometrically-fitted method with order $p = 5$ and stage $s = 5$, consider the order conditions up to order five from IRK methods (Rabiei *et al*., 2013).

$$\left.\begin{aligned}
&\text{First order}: \; b_1 - b_{-1} = 1\\
&\text{Second order}: \; b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2}\\
&\text{Third order}: \; \sum_{i=2}^{s} b_i c_i = \frac{5}{12}\\
&\sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3}\\
&\text{Fourth order}: \; \sum_{i=3,j=2}^{s} b_i a_{i,j} c_j = \frac{1}{6}\\
&\sum_{i=3}^{s} b_i c_i^3 = \frac{31}{120}\\
&\text{Fifth order}: \; \sum_{i=3,j=2}^{s} b_i c_i a_{ij} c_j = \frac{31}{240}\\
&\sum_{i=3,j=2}^{s} b_i a_{ij} c_j^2 = \frac{31}{360}\\
&\sum_{i=4,j=3,k=2}^{s} b_i a_{ij} a_{jk} c_k = \frac{31}{720}
\end{aligned}\right\} \quad (18)$$

And the classical fifth order five stage (IRK5-5) has the butcher tableau as

Table 1. Coefficients of IRK5 − 5 methods

0					
$\frac{1}{4}$	$\frac{1}{4}$				
$\frac{1}{4}$	$\frac{-41}{5000}$	$\frac{1293}{5000}$			
$\frac{1}{2}$	$\frac{193}{500}$	$\frac{332}{625}$	$\frac{1611}{2500}$		
$\frac{3}{4}$	$\frac{103}{500}$	$\frac{4501}{5000}$	$\frac{4459}{5000}$	$\frac{5517}{10000}$	
$\frac{1}{45}$	$\frac{46}{45}$	$\frac{-559}{3879}$	$\frac{1502}{19395}$	$\frac{-1}{10}$	$\frac{29}{45}$

To derive the fifth order five stage Trigonometrically-fitted IRK method, we substitute $s = 5, c_1 = 0$ in the recursive relations $(14) - (15)$

for $i = 2$

$$\cos(c_2 z) - 1 = 0 \quad (19)$$

$$\sin(c_2 z) - z a_{2,1} = 0 \quad (20)$$

for $i = 3$

$$\cos(c_3 z) - 1 + z a_{3,2} \sin(c_2 z) = 0 \quad (21)$$

$$\sin(c_3 z) - z\left[a_{3,1} + a_{3,2}\cos(c_2 z)\right] = 0 \quad (22)$$

for $i = 4$

$$\cos(c_4 z) - 1 + z\left[a_{4,2}\sin(c_2 z) + a_{4,3}\sin(c_3 z)\right] = 0 \quad (23)$$

$$\sin(c_4 z) - z\left[a_{4,1} + a_{4,2}\cos(c_2 z) + a_{4,3}\cos(c_3 z)\right] = 0 \quad (24)$$

for $i = 5$

$$\cos(c_5 z) - 1 + z\left[a_{5,2}\sin(c_2 z) + a_{5,3}\sin(c_3 z) + a_{5,4}\sin(c_4 z)\right] = 0 \quad (25)$$

$$\cos(c_5 z) - z\left[a_{5,1} + a_{5,2}\sin(c_2 z) + a_{5,3}\sin(c_3 z) + a_{5,4}\sin(c_4 z)\right] = 0 \quad (26)$$

Now, substituting $s = 5$, $c_1 = 0$ in equations $(16) - (17)$

$$\cos(z) - 1 + z b_{-1}\sin(z) + z[b_2\sin(c_2 z) + b_3\sin(c_3 z) + b_4\sin(c_4 z) + b_5\sin(c_5 z)] - z\left[b_2\sin\left((c_2 - 1)z\right) + b_3\sin\left((c_3 - 1)z\right) + b_4\sin\left((c_4 - 1)z\right) + b_5\sin\left((c_5 - 1)z\right)\right] = 0 \quad (27)$$

$$\sin(z) - z b_1 + z b_{-1}\cos(z) - z[b_2\cos(c_2 z) + b_3\cos(c_3 z) + b_4\cos(c_4 z) + b_5\cos(c_5 z)] + z\left[b_2\cos\left((c_2 - 1)z\right) + b_3\cos\left((c_3 - 1)z\right) + b_4\cos\left((c_4 - 1)z\right) + b_5\cos\left((c_5 - 1)z\right)\right] = 0 \quad (28)$$

Equations $(25) - (28)$ are now the equations of order conditions for fifth-order five-stage trigonometrically-fitted method that replaces order conditions (18) of the original method. In order to obtain the coefficients of the new method, equations 27 and 28 are solved simultaneously with additional equations from the order condition (18) namely,

$$b_1 - b_{-1} = 1 \quad (29)$$

$$b_{-1} + b_2 + b_3 + b_4 + b_5 = \frac{1}{2} \quad (30)$$

$$b_2 c_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 = \frac{5}{12} \quad (31)$$

$$b_2 a_{2,1} c_1 + b_3 a_{3,2} c_2 + b_4 a_{4,3} c_3 + b_5 a_{5,4} c_4 = \frac{1}{6} \quad (32)$$

These sum up to six equations in ten unknowns $(b_{-1}, b_1, b_2, b_3, b_4, b_5, c_2, c_3, c_4$ and $c_5)$. The equations are solved in terms of four free parameters $(c_2 = \frac{1}{4}, c_3 = \frac{1}{4}, c_4 = \frac{1}{2}, c_5 = \frac{3}{4})$ whose values are obtained from Table 1. Equations$(29), (30), (31)$ and (32) are chosen to augment the updated (27) and (28) so that $(b_{-1}, b_1, b_2, b_3, b_4 and\ b_5)$ are not taken as free parameters. Solving equations $(27), (28), (29), (30), (31)$ and (32) the following values are obtained for b_{-1}, b_1, b_2, b_3, $b_4$and b_5

$$\left.\begin{aligned} b_{-1} &= -\frac{1}{3}\frac{M_1}{M_2} \\ b_1 &= \frac{1}{3}\frac{\mathrm{M}_3}{\mathrm{M}_4} \\ b_2 &= -\frac{1}{3879}\frac{\mathrm{M}_5}{\mathrm{M}_6} \\ b_3 &= -\frac{1}{7758}\frac{M_7}{M_8} \\ b_4 &= \frac{1}{6}\frac{M_9}{M_{10}} \\ b_5 &= -\frac{1}{6}\frac{M_{11}}{M_{12}} \end{aligned}\right\} \tag{33}$$

where, $M_1 = -2(z)\cos\left(\frac{3}{4}z\right) + 2(z)\cos\left(\frac{1}{4}z\right) + 3\sin(z) - 3(z)$

$$M_2 = z\left(-1 + 2\cos\left(\frac{1}{4}z\right) - 2\cos\left(\frac{3}{4}z\right) + \cos(z)\right)$$

$$M_3 = 3(z)\cos(z) - 4(z)\cos\left(\frac{3}{4}z\right) + 4(z)\cos\left(\frac{1}{4}z\right) - 3\sin(z)$$

$$M_4 = z\left(-1 + 2\cos\left(\frac{1}{4}z\right) - 2\cos\left(\frac{3}{4}z\right) + \cos(z)\right)$$

$$\begin{aligned} M_5 = 549 - 1098\cos(z) & \\ & - 1098\cos\left(\frac{1}{4}z\right) \\ & + 1098\cos\left(\frac{3}{4}z\right) \\ & + 549(z)\sin(z) \\ & - 3949(z)\sin\left(\frac{1}{4}z\right) \\ & - 3949(z)\sin\left(\frac{3}{4}z\right) \\ & + 6251(z)\sin\left(\frac{1}{2}z\right) \\ & + 549\cos(z^2) \\ & + 366\sin(z)z\cos\left(\frac{3}{4}z\right) - 366\sin(z)z\cos\left(\frac{1}{4}z\right) + 943\sin\left(\frac{1}{4}z\right)z\cos(z) \\ & + 943\sin\left(\frac{3}{4}z\right)z\cos(z)\,z - 1337\sin\left(\frac{1}{2}z\right)z\cos(z) + 5950\sin\left(\frac{1}{2}z\right)z\cos\left(\frac{3}{4}z\right) \\ & - 5950\sin\left(\frac{1}{2}z\right)z\cos\left(\frac{1}{4}z\right) - 549\sin(z^2) + 3006\sin\left(\frac{1}{4}z\right)\sin(z) + 3006\sin\left(\frac{3}{4}z\right)\sin(z) \\ & - 4914\sin\left(\frac{1}{2}z\right)\sin(z) \\ & + 1098\cos(z)\cos\left(\frac{1}{4}z\right) - 1098\cos(z)\cos\left(\frac{3}{4}z\right) - 3890\sin\left(\frac{1}{4}z\right)z\cos\left(\frac{3}{4}z\right) \\ & + 3890\sin\left(\frac{1}{4}z\right)z\cos\left(\frac{1}{4}z\right) - 3890\sin\left(\frac{3}{4}z\right)z\cos\left(\frac{3}{4}z\right) + 3890\sin\left(\frac{3}{4}z\right)z\cos\left(\frac{1}{4}z\right) \end{aligned}$$

$$M_6 = z\left(\sin\left(\frac{1}{4}z\right) + \sin\left(\frac{3}{4}z\right) - 2\sin\left(\frac{1}{2}z\right)\right)\left(-1 + 2\cos\left(\frac{1}{4}z\right) - 2\cos\left(\frac{3}{4}z\right) + \cos(z)\right)$$

$$M_7 = 2781 - 5562\cos(z) - 5562\cos\left(\frac{1}{4}z\right) + 5562\cos\left(\frac{3}{4}z\right) + 2781(z)\sin(z) + 13070(z)\sin\left(\frac{1}{4}z\right) + 13070(z)\sin\left(\frac{3}{4}z\right) - 34483(z)\sin\left(\frac{1}{2}z\right) + 2781\cos(z^2) + 1854\sin(z)(z)\cos\left(\frac{3}{4}z\right) - 1854\sin(z)(z)\cos\left(\frac{1}{4}z\right) + 700\sin\left(\frac{1}{4}z\right)z\cos(z) + 700\sin\left(\frac{3}{4}z\right)z\cos(z) + 1381\sin\left(\frac{1}{2}z\right)z\cos(z) - 24830\sin\left(\frac{1}{2}z\right)z\cos\left(\frac{3}{4}z\right) + 24830\sin\left(\frac{1}{2}z\right)z\cos\left(\frac{1}{4}z\right) - 2781\sin(z^2) - 13770\sin\left(\frac{1}{4}z\right)\sin(z) - 13770\sin\left(\frac{3}{4}z\right)\sin(z) + 33102\sin\left(\frac{1}{2}z\right)\sin(z) + 5562\cos(z)\cos\left(\frac{1}{4}z\right) - 5562\cos(z)\cos\left(\frac{3}{4}z\right) + 7780\sin\left(\frac{1}{4}z\right)z\cos - 7780\sin\left(\frac{1}{4}z\right)z\cos\left(\frac{1}{4}z\right) + 7780\sin\left(\frac{3}{4}z\right)z\cos\left(\frac{3}{4}z\right) - 7780\sin\left(\frac{3}{4}z\right)z\cos\left(\frac{1}{4}z\right)$$

$$M_8 = z\left(\sin\left(\frac{1}{4}z\right) + \sin\left(\frac{3}{4}z\right) - 2\sin\left(\frac{1}{2}z\right)\right)\left(-1 + 2\cos\left(\frac{1}{4}z\right) - 2\cos\left(\frac{3}{4}z\right) + \cos(z)\right)$$

$$M_9 = 6 + 6(z)\sin(z) - 9(z)\sin\left(\frac{1}{4}z\right) - 9(z)\sin\left(\frac{3}{4}z\right) - 12\cos\left(\frac{1}{4}z\right) + 12\cos\left(\frac{3}{4}z\right) - 12\cos(z) + 4\sin(z)(z)\cos\left(\frac{3}{4}z\right) - 4\sin(z)(z)\cos\left(\frac{1}{4}z\right) + 3\sin\left(\frac{1}{4}z\right)(z)\cos(z) - 10\sin\left(\frac{1}{4}z\right)(z)\cos\left(\frac{3}{4}z\right) + 10\sin\left(\frac{1}{4}z\right)(z)\cos\left(\frac{1}{4}z\right) + 3\sin\left(\frac{3}{4}z\right)(z)\cos(z) - 10\sin\left(\frac{3}{4}z\right)(z)\cos\left(\frac{3}{4}z\right) + 10\sin\left(\frac{3}{4}z\right)(z)\cos\left(\frac{1}{4}z\right) + 6\cos(z^2) - 6\sin(z^2) + 6\sin\left(\frac{1}{4}z\right)\sin(z) + 6\sin\left(\frac{3}{4}z\right)\sin(z) + 12\cos(z)\cos\left(\frac{1}{4}z\right) - 12\cos(z)\cos\left(\frac{3}{4}z\right)$$

$$M_{10} = z\left(\sin\left(\frac{1}{4}z\right) + \sin\left(\frac{3}{4}z\right) - 2\sin\left(\frac{1}{2}z\right)\right)\left(-1 + 2\cos\left(\frac{1}{4}z\right) - 2\cos\left(\frac{3}{4}z\right) + \cos(z)\right)$$

$$M_{11} = 3 + 3(z)\sin(z) - 4(z)\sin\left(\frac{1}{4}z\right) - z\sin\left(\frac{1}{2}z\right) - 4(z)\sin\left(\frac{3}{4}z\right) - 6\cos\left(\frac{1}{4}z\right) + 6\cos\left(\frac{1}{4}z\right) - 6\cos(z) + 2\sin(z)(z)\cos\left(\frac{3}{4}z\right) - 2\sin(z)(z)\cos\left(\frac{1}{4}z\right) - 2\sin\left(\frac{1}{4}z\right)(z)\cos(z) - 2\sin\left(\frac{3}{4}z\right)(z)\cos(z) + 7\sin\left(\frac{1}{2}z\right)z\cos(z) - 10\sin\left(\frac{1}{2}z\right)(z)\cos\left(\frac{3}{4}z\right) + 10\sin\left(\frac{1}{2}z\right)(z)\cos\left(\frac{1}{4}z\right) + 3\cos(z^2) - 3\sin(z^2) + 6\sin\left(\frac{1}{4}z\right)\sin(z) + 6\sin\left(\frac{3}{4}z\right)\sin(z) - 6\sin\left(\frac{1}{2}z\right)\sin(z) + 6\cos(z)\cos\left(\frac{1}{4}z\right) - 6\cos(z)\cos\left(\frac{3}{4}z\right)$$

$$M_{12} = z\left(\sin\left(\frac{1}{4}z\right) + \sin\left(\frac{3}{4}z\right) - 2\sin\left(\frac{1}{2}z\right)\right)\left(-1 + 2\cos\left(\frac{1}{4}z\right) - 2\cos\left(\frac{3}{4}z\right) + \cos(z)\right)$$

In order to recover the original method $IRK5 - 5$ as z approaches zero, the Taylor expansions of the coefficients $b_{-1}, b_1, b_2, b_3, b_4$ and b_5 in (33) are obtained as follow

$$\left.\begin{aligned} b_{-1} &= \frac{1}{45} + \frac{13}{3780}z^2 + \frac{17}{172800}z^4 + o(z^5) \\ b_1 &= \frac{46}{45} + \frac{13}{3780}z^2 + \frac{17}{172800}z^2 + o(z^5) \\ b_2 &= -\frac{559}{3879} + o(z^2) \\ b_3 &= \frac{1502}{19395} + o(z^2) \\ b_4 &= -\frac{1}{10} + o(z^2) \\ b_5 &= \frac{29}{45} + o(z^2) \end{aligned}\right\} \qquad (34)$$

It is clear from (34) that, as z approache zero, the classical method $IRK5-5$ presented in Table 1 is recovered exactly.

To verify the order of the method as earlier claimed, we substitute the coefficients of the method into order conditions (18) up to order five and take the Taylor expansion of each to obtain

$$\left.\begin{aligned} &\text{Order 1: } b_1 - b_{-1} = 1 + o(z^2) \\ &\text{Order 2: } b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} + o(z^2) \\ &\text{Order 3: } \sum_{i=2}^{s} b_i c_i = \frac{5}{12} + o(z^2) \\ &\text{Order 4: } \sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3} + o(z^2) \\ &\qquad \sum_{i=2,j=1}^{s} b_i a_{i,j} c_j = \frac{1}{6} + o(z^2) \\ &\text{Order 5: } \sum_{i=3,j=2}^{s} b_i c_i a_{ij} c_j = \frac{31}{240} + o(z^2) \\ &\qquad \sum_{i=3,j=2}^{s} b_i a_{ij} c_j^2 = \frac{31}{360} + o(z^2) \\ &\qquad \sum_{i=4,j=3,k=2}^{s} b_i a_{ij} a_{jk} c_k = \frac{31}{720} + o(z^2) \end{aligned}\right\} \qquad (35)$$

From (35), as z tends to zero, the order conditions of the Improved Runge-Kutta method up to order five are recovered; this signifies that the coefficients of the Trigonometrically-fitted fifth order five stage method satisfies the IRK order five conditions.

3.0 Results and Discussion

In order to evaluate the effectiveness of the derived method $TFIRK5-5$, we seek the approximate solutions on the partition $[x_0, x_N]$ where errors are calculated at the endpoints $|y_n - y(x_n)|$ with $y_n,\ y(x_n)$ as approximate and exact solutions at the point x_n and y_n respectively. The $TFIRK5-5$ method is implemented in fixed step. The solutions to the test problems are obtained using Maple 2019 software package and the results presented in Tables $2-6$.

Problem 1:

$y'(x) = -10\sin(10x) + 10\cos(10x) + \cos(x) \quad y(0) = 1\ ,\ \omega = 10$

Exact solution: $y(x) = \cos(10x) + \sin(10x) + \sin(x)$

Problem 2

$y'(x) = 2\cos(2x), y(0) = 1, \omega = 2$

Exact solution: $y(x) = \sin(2x) - 3$

Problem 3

$y'(x) = \cos(x) + \sin(x), y(0) = 1, \omega = 1$

Exact solution: $y(x) = \sin(x) - \cos(x)$

94Table 2 Results of Problem 1 on $[0, 1]$, $h = 0.05$, $\omega = 10$

x	Exact	TFIRK5 − 5	Error	IRK5 − 5	Error
0.05	1.4069872700	1.7990160250	0.392028755+0000	1.7990160250	0.392028755+0000
0.10	1.4816067070	1.4816067073	1.2431882246E-11	1.4816091455	0.0000024382+000
0.15	1.2176703210	1.2176703207	3.3078753008E-11	1.2176749471	0.0000046264+000
0.20	0.6918199210	0.6918199210	6.1889005863E-11	0.6918259499	0.0000060288+000
0.25	0.0447324870	0.0447324877	9.8790630183E-11	0.0447387388	0.0000063021+000
0.30	-0.5533522810	-0.5533522820	1.4369139112E-11	-0.5533469026	0.0000053793+000
0.35	-0.9443421070	-0.9443421077	1.9647906017E-10	-0.9443386211	0.0000034865+000
0.40	-1.0210277740	-1.0210277741	2.5702169563E-10	-1.0210266869	0.0000010869+000
0.45	-0.7533603830	-0.7533603833	3.2516797246E-10	-0.7533616148	0.0000012318+000
0.50	-0.1958365500	-0.1958365510	4.0074756045E-10	-0.1958394526	0.0000029020+000
0.55	0.5258166770	0.5258166772	4.8357154999E-10	0.5258131629	0.0000035147+000
0.60	1.2453972620	1.2453972613	5.7343292425E-10	1.2453943418	0.0000029200+000
0.65	1.7968940200	1.7968940189	6.7010707658E-10	1.7968927561	0.0000012635+000
0.70	2.0551065400	2.0551065395	7.7335237195E-10	2.0551075897	0.0000010494+000
0.75	1.9662740550	-3.5257387533	8.8291075089E-10	1.9662775069	0.0000034522+000
0.80	1.5612143040	1.5612143027	9.9850837452E-10	1.5612196605	0.0000053568+000
0.85	0.9477556150	0.9477556140	1.1198563090E-09	0.9477619118	0.0000062967+000
0.90	0.2843151330	0.2843151317	1.2466512476E-09	0.2843211750	0.0000060420+000
0.95	-0.2589077710	-0.2589077732	1.3785762691E-09	-0.2589031170	0.0000046549+000
1.00	-0.5416216550	-0.5416216567	1.5153016296E-09	-0.5416191801	0.0000024750+000

Table 3 Results of Problem 2 on $[0, 1]$, $h = 0.05$, $\omega = 2$

x	Exact	TFIRK5 − 5	Error	IRK5 − 5	Error
0.05	-2.9001665830	1.1995004170	4.099667+0000000	1.1995004170	4.099667+0000000
0.10	-2.8013306690	-2.8013306692	1.1682541000E-22	-2.8013306692	1.6172533540E-11
0.15	-2.7044797930	-2.7044797933	4.1057953000E-22	-2.7044797933	4.2983874049E-11
0.20	-2.6105816580	-2.6105816577	8.7832725000E-22	-2.6105816576	8.0166131475E-11
0.25	-2.5205744610	-2.5205744614	1.5153950100E-21	-2.5205744613	1.2734779299E-10
0.30	-2.4353575270	-2.4353575266	2.3154174400E-21	-2.4353575264	1.8405743503E-10
0.35	-2.3557823130	-2.3557823128	3.2704009700E-21	-2.3557823125	2.4972843361E-10
0.40	-2.2826439090	-2.2826439091	4.3708037200E-21	-2.2826439088	3.2370462579E-10
0.45	-2.2166730900	-2.2166730904	5.6056308300E-21	-2.2166730900	4.0524686594E-10
0.50	-2.1585290150	-2.1585290152	6.9625443100E-21	-2.1585290147	4.9354041094E-10
0.55	-2.1087926400	-2.1087926399	8.4279863500E-21	-2.1087926394	5.8770306087E-10
0.60	-2.0679609140	-2.0679609140	9.9873147100E-21	-2.0679609133	6.8679397367E-10
0.65	-2.0364418150	-2.0364418146	1.1624949120E-20	-2.0364418138	7.8982306568E-10
0.70	-2.0145502700	-2.0145502700	1.3324526870E-20	-2.0145502691	8.9576090428E-10
0.75	-2.0025050130	-2.0025050134	1.5069066340E-20	-2.0025050124	1.0035489936E-09
0.80	-2.0004263970	-2.0084073216	1.6841136660E-20	-2.0004263958	1.1121103507E-09
0.85	-2.0083351800	-2.0083351895	1.8623031910E-20	-2.0083351883	1.2203602663E-09
0.90	-2.0261523690	-2.0261523691	2.0396947970E-20	-2.0261523678	1.3272171432E-09
0.95	-2.0536999120	-2.0536999123	2.2145160440E-20	-2.0536999109	1.4316133026E-09
1.00	-2.0907025730	-2.0907025732	2.3850201790E-20	-2.0907025164	1.5325056528E-09

Table 4 Results of Problem 1 on $[0, 100]$, $\omega = 10$

h	TFIRK5 − 5	IRK5 − 5	NFEs
$\frac{1}{20}$	0.0000047390+000	0.0000063488+000	10000
$\frac{1}{40}$	2.0894095404E-10	2.2814408005E-07	20000
$\frac{1}{80}$	6.5497409108E-12	7.5716810189E-09	40000
$\frac{1}{160}$	2.0483705160E-13	2.4330250628E-10	80000
$\frac{1}{320}$	6.5892568647E-15	7.7057923430E-12	160000
$\frac{1}{640}$	3.7331342510E-13	2.4239343778E-13	320000

Table 5 Results of Problem 2 on $[0, 100]$, $\omega = 2$

h	$TFIRK5-5$	$IRK5-5$	NFEs
$\frac{1}{20}$	1.9946645755E-19	2.1672396262E-09	10000
$\frac{1}{40}$	7.3192400529E-17	6.7793987299E-11	20000
$\frac{1}{80}$	2.1057113817E-15	2.1191000515E-12	40000
$\frac{1}{160}$	8.2491095960E-13	6.6226006091E-14	80000
$\frac{1}{320}$	2.1322953740E-10	2.0695949661E-15	160000
$\frac{1}{640}$	5.0321906060E-09	6.4675094733E-17	320000

Table 6: Results of Problem 3 on [0,100]. $\omega = 1$

h	$TFIRK5-5$	$IRK5-5$	NFEs
$\frac{1}{20}$	6.8940692385E-11	8.3529664416E-11	10000
$\frac{1}{40}$	2.9571439914E-10	2.5844114437E-12	20000
$\frac{1}{80}$	5.9451500460E-08	8.0354336439E-14	40000
$\frac{1}{160}$	1.6549221638E-08	2.5046629353E-15	80000
$\frac{1}{320}$	8.2694118862E-09	7.8170351418E-17	160000
$\frac{1}{640}$	7.0192065120E-07	2.4412536408E-18	320000

Table 2 shows the results of variation in performance between $TFIRK5-5$ and $IRK5-5$ over the interval $[0,1]$ when applied to Problem 1. It reveals that *TFIRK5-5* performs better than the classical $IRK5-5$ by displaying lesser errors. In addition, Table 3 reveals the results of the comparison between $TFIRK5-5$ and $IRK5-5$ when applied to Problem 2, which shows $TFIRK5-5$ display more accurate performance than the non-fitted $IRK5-5$ method. The Table 4 results contain maximum errors obtained from the interval $[0,100]$ when $TFIRK5-5$ and $IRK5-5$ are compared in application to Problem 1. In Table 5, it is observed that the performance of $TFIRK5-5$ over $IRK5-5$ reduces as z approaches zero. Lastly, in Table 6, the performance of $TFIRK5-5$ over $IRK5-5$ method diminishes as step length tends to zero when applied to problem 3, this is because the method approaches the original method.

4.0 Conclusion

We have proposed a fifth-order five-stage trigonometrically-fitted Improved Runge-Kutta methods for the solution of general first order initial value problems of the form $y'(x) = f(x, y(x))$ with oscillatory solutions. Implementation of the method with numerical examples showed that the method is superior to the classical method having the same number of function evaluations per step. The $TFIRK5-5$ method requires only five function evaluations at each integration step and in general requires $\left(5 \cdot \left(\frac{T}{h}\right)\right)$ NFEs on the entire interval of integration. The effectiveness (in terms of accuracy) and efficiency (in terms of execution time) of the method as displayed by the numerical results, indicates the superiority of the method over other existing methods in literature.

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6.0 References

Berghe, G. V., De Meyer, H., Van Daele, M. & Van Hecke, T. (2000). Exponentially fitted Runge-Kutta methods. *Journal of Computational and Applied Mathematics,* 125, 1-2, pp. 107-115.

Ehigie, J. O., Jator, S. N & Okunuga, S. A. (2017). A continuous Runge-Kutta Nystrom collocation method with trigonometric coefficients for periodic initial value problems. *Journal of* *the Nigerian Mathematical Society*, 35, pp. 139-162.

Fang, Y., You, X., Ming, Q. (2014). Trigonometrically fitted two-derivative Runge-Kutta methods for solving oscillatory differential equations. *Numerical Algorithms, 65,* 3, pp. 651-667.

Fawzi, F. A., Senu, N., Ismail, F. & Majid, Z. A. (2016a). A fourth algebraic order explicit trigonometrically-fitted modified Runge-Kutta method for the numerical solution of periodic IVPs. *Indian Journal of Science and Technology*, 9, 48, pp. 1-7. doi:10.17485/ijst/2016/v9i47/97774.

Fawzi, F. A., Senu, N., Ismail, F. & Majid, Z. A. (2016b). Explicit Runge-Kutta method with trigonometrically fitted for solving first order ODEs. *AIP Conference proceedings 1739, 020044,* https://doi.org/10.1063/1.4952524.

Gautschi, W. (1961). Numerical integration of ordinary differential equation based on trigonometric polynomials. *Numerical Mathematics,* 3, pp. 381-97.

Goeken, D. & Johnson, O. (2000). Runge-Kutta with higher order derivative approximations. *Applied Numerical Mathematics*, 34, pp. 207-218.

Ismail, F., Ahmad, S. Z., Jikantoro, Y. D. and Senu, N. (2018). Block hybrid method with trigonometric-fitting for solving oscillatory problems. *Sains Malaysiana,* 47, 9, pp. 2223–2230.

Ismail, F. & Suleiman, M. (2013). Improved Runge-Kutta method for solving ordinary differential equations. *Sains Malaysiana*, 42, 11, pp. 1679-1687.

Lyche, T. (1972). Chebyshevian multistep methods for ordinary differential equation. *Numerical Mathematics,* 19, pp. 65-75.

Neta, B. & Changbum C. (2020). A new trigonometrically-fitted method for second order initial value problems. Naval Postgraduate School Monterey CA United States

Phohomsiri, P. & Udwadia, F. E. (2004). Acceleration of Runge-Kutta integration schemes. *Discrete Dynamics in Nature and Society,* 2, pp.307-314.

Rabiei, F. & Ismail, F. (2012). Fifth-order improved Runge-Kutta method with reduced number of function evaluations. *Australian Journal of Basic and Applied Sciences,* 6, 3, pp. 97-105.

Rabiei, F., Ismail, F. & Suleiman, M. (2013). Improved Runge-Kutta method for solving ordinary differential equations. *Sains Malaysiana*, 42, 11, pp. 1679-1687.

Simos, T. E. (2005). A family of fifth algebraic order trigonometrically fitted Runge-Kutta methods for the numerical solution of the Schrodinger equation. *Computational Materials Science,* 34, 4, pp. 342-354.

Udwadia, F. E. & Farahani, A. (2008). Accelerated Runge-Kutta methods. *Discrete Dynamics in Nature and Society,* 790619, pp. 1-39. doi:10.1155/2008/790619, 2008.

Vanden Berghe, G., De Meyer, H. Van Daele, M. & Van Hecke, T. (2000). Exponentially-fitted Runge-Kutta methods. *J. Comput. Appl. Math*., 125, pp. 107–115.

Vigor-Aguiar, J. & Simos, T. E. (2001). A modified phase-fitted Runge-Kutta method for the numerical solution of the Schrodinger equation. *Journal of Mathematical Chemistry*, 30, pp. 121-31.

Xinyuan, W. (2003). A class of Runge-Kutta formulae of order three and four with reduced evaluations of function. *Applied Mathematics and Computation*, 146, pp. 417-432.

Conflict of interest

The authors declared no conflict of interest