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Inverse Cube Root Transformation: Theory and Application to Time Series Data

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Abstract: *Constant variance assumption is one of the necessary assumptions of several time series models. The violation of this assumption often necessitates data transformation. This paper investigated the properties of the inverse cube root transformation of a time series based on the multiplicative model. The probability density function of the inverse cube root of the left truncated normally distributed error components of the model was derived; the relationship between the variances of the errors in both the transformed and untransformed multiplicative models was examined through simulations. The error variance for the untransformed model was empirically found to be nine (9) times the error variance for the transformed model. Other results obtained in this paper revealed that an inverse cube roots transformation of a time series that results in normally distributed errors is possible when* $0 < \sigma < 0.38$*. In other words, the variance of the inverse cube root transformed error component was greater than that of the untransformed component for all* σ*.*

Keywords*: Bartlett's transformation, Left truncated normal distribution, Anderson – Darling test*

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1.0 Introduction

Transformation techniques play a crucial role in time series analysis, particularly in addressing issues of non-normality and non-constant variance. Among the various transformation methods, two widely recognized approaches are the Box-Cox transformation (Box & Cox, 1964) and Bartlett's transformation (Bartlett, 1947). While these transformations are frequently used, some studies have pointed out that they do not always guarantee improved forecasting accuracy (Iwueze, 2007). Bartlett's transformation, first applied to time series by Iwueze, has been noted for its simplicity and ease of interpretation, making it a preferred choice in certain contexts.

Despite the extensive research on transformation techniques, no theoretical framework exists for the inverse cube root transformation, which may be suitable for specific types of time series data. Ruppert (1999) emphasized that any transformation's impact on the error structure must be carefully examined. Given the existence of datasets where the inverse cube root transformation may provide significant advantages, this study aims to bridge the gap by investigating its statistical properties and establishing the conditions under which it is an effective method for stabilizing variance in time series models.

Several studies have explored transformation methods to improve time series modeling. Iwueze (2007) investigated the effect of logarithmic transformation on error components and established conditions for its validity. Akpanta and Iwueze (2009) applied Bartlett's transformation to time series data and analyzed its effect on variance stabilization. However, while these transformations provide theoretical and empirical justifications, they do not encompass all possible data structures, necessitating alternative transformation approaches such as the inverse cube root transformation.

A critical gap in the literature is the lack of research on the inverse cube root transformation and its practical applicability to time series models. While traditional transformations have been extensively studied, their limitations in certain scenarios call for further exploration of alternative methods. This study addresses this gap by providing a theoretical foundation for the inverse cube root transformation and evaluating its efficacy in stabilizing variance. The inverse cube root transformation can be defined as

$$Y_t = \frac{1}{\sqrt[3]{X_t}} \qquad (1)$$

where represents the original time series data, and is the transformed variable. Applying this transformation to a time series stabilizes variance and alters the distributional properties of the data. According to Bartlett (1947), the relationship between the periodic standard deviations (σ_i) and periodic means (μ_i) can be expressed as:

$$log\sigma_i = \alpha + \beta log_e\mu_i \qquad (2)$$

For the inverse cube root transformation, the slope should be approximately. This ensures that the transformation effectively stabilizes variance without distorting the essential statistical properties of the data.

2.0 Materials and Methods

2.1 Derivation of the Probability Density Function (pdf) of the Inverse Cube Root Transformed Error Component, $e_t^* = \frac{1}{\sqrt[3]{X}} = Y, g(Y)$

To proceed with this probability density function derivation, it is of great importance to consider the transformed error term to be $e_t^* = \frac{1}{\sqrt[3]{e_t}} = \frac{1}{\sqrt[3]{X}} = Y$, where $e_t = X$.

Proposition 2.1:

Suppose that X follows a left truncated normal distribution with a unit mean (1) and a constant variance (σ^2)**.** The probability density function (pdf) of $Y = \frac{1}{\sqrt[3]{X}} = X^{-\frac{1}{3}}$ is

$$g(Y) = \begin{cases} \dfrac{3e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\dfrac{1}{\sigma}\right)\right]y^4}, & 0 < y < \infty \\ 0 \qquad otherwise & -\infty < y < 0 \end{cases}$$

Proof: Given $f_{LTN}(X) = f^*(X) = \dfrac{e^{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2}}{\sigma\sqrt{\pi}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)}$ (3)

From $y = \frac{1}{\sqrt[3]{X}} \Longrightarrow \frac{1}{\sqrt[3]{x}} = \frac{1}{y} \Longrightarrow \left(\sqrt[3]{x}\right)^3 = \left(\frac{1}{y}\right)^3 \Longrightarrow x = \frac{1}{y^3} = y^{-3}$

The derivative of x with respect to y

$\frac{dx}{dy} = \frac{d(y^{-3})}{dy} = -y^{-3-1} = -3y^{-4} = -3 \times \frac{1}{y^4} = \frac{-3}{y^4}$. Therefore $\frac{dx}{dy} = \frac{-3}{y^4}$

Having gotten $\frac{dx}{dy}$ we now derived the probability density function (pdf) of the inverse cube-root transformed variable (y) using transformation of variable technique given in Equation (2) below.

$$g(Y) = f^*(X).\left|\frac{dx}{dy}\right| \qquad (4)$$

$(Freund\ and\ Walpole, 1986)$

Where $\left|\frac{dx}{dy}\right|$ is the Jacobian of the transformation. For this inverse cube root transformation;

$\left|\frac{dx}{dy}\right| = \left|\frac{-3}{y^4}\right| = \frac{3}{y^4}$, now we substitute $f^*(x)$ as shown in Equation (3) into Equation (4) and as well replace $x = \frac{1}{y^3}$

$$g(Y) = \frac{e^{-\frac{1}{2}\left(\frac{\frac{1}{y^3}-1}{\sigma}\right)^2}}{\sigma\sqrt{\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} \cdot \frac{3}{y^4}$$

$$g(Y) = \begin{cases} \frac{3e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]y^4}, & 0 < y < \infty \\ 0 \quad otherwise & -\infty < y < 0 \end{cases} \tag{5}$$

Now that the probability density function of the inverse cube root transformed variable, $g(y)$ have been derived, we shall now show that $g(y)$ is a proper probability density function (pdf). Equation (3) shall be a proper probability density function of the inverse cube root transformed variable (y) if and only if $\int_0^\infty g(y)dy = 1$

We shall now carry out the integration of $g(y)$ to show that Equation (5) is a proper pdf.

$$\int_0^\infty g\,(y)\,dy \;= \int_0^\infty \frac{3e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]y^4} . dy$$

$$= \frac{3}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} \int_0^\infty \frac{1}{y^4} e^{-\frac{1}{2\sigma^2}\left(\frac{y^{-3}-1}{\sigma}\right)^2} . dy$$

Let $u = \frac{y^{-3}-1}{\sigma}$; and $-\frac{1}{\sigma} < u < \infty$. Consequently,

$$\frac{du}{dy} = \frac{d}{dy}\left(\frac{y^{-3}-1}{\sigma}\right) = \frac{1}{\sigma} . \frac{d}{dy}\left(y^{-3}-1\right)$$

$dy = \frac{-\sigma y^4 du}{3}$. Therefore,

$$\int_0^\infty g\,(y)\,dy = \frac{3}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} \int_{-\frac{1}{\sigma}}^{\infty} \frac{1}{y^4} . e^{-\frac{1}{2}.u^2} . \frac{\sigma y^4}{3} du$$

$$= \frac{1}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} \int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}} du$$

Note: $\int_{-\frac{1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} . du = \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}} . du$

$= 1 - \Phi\left(-\frac{1}{\sigma}\right)$

Using the above, the pdf derivation becomes,

$$\frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} \int_{-\frac{1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} . e^{-\frac{u^2}{2}} du = \frac{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} = 1$$

Hence, $\int_0^\infty g\;(y)\,dy = 1$

As such, it can be undoubtedly stated that $g(y)$ is a proper probability density function of y.

Plots of $f(x)$ and g(y) for $\sigma = 0.1000$, $\sigma = 0.2000$, $\sigma = 0.3700$ and $\sigma = 0.5000$ are shown in Fig. 1 to Fig. 4 sequentially. For all values of σ, $g(y)$ is unimodal. $g(y)$ also has a larger peak than $f(x)$.

Proposition 2.2:

Let y_{max} denote the mode of $g(y)$. Then $y_{max} = \sqrt[3]{\frac{-3+\sqrt{9+48\sigma^2}}{8\sigma^2}}$

Proof:

In order to determine the mode of $g(y)$, we obtain the first derivative $g^{/}(y)$ of $g(y)$ with respect to y. $g^{/}(y) = 0$ is used to obtain the maximum point for a given value of σ.

Given $g(y) = \frac{3}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} \cdot y^{-4} \cdot e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}$ (6)

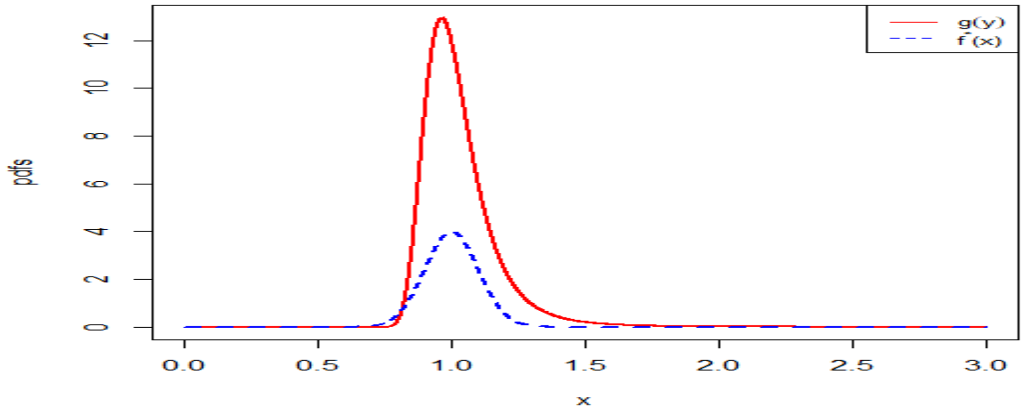

Fig. 1: Curve Shapes for $\sigma = 0.100$

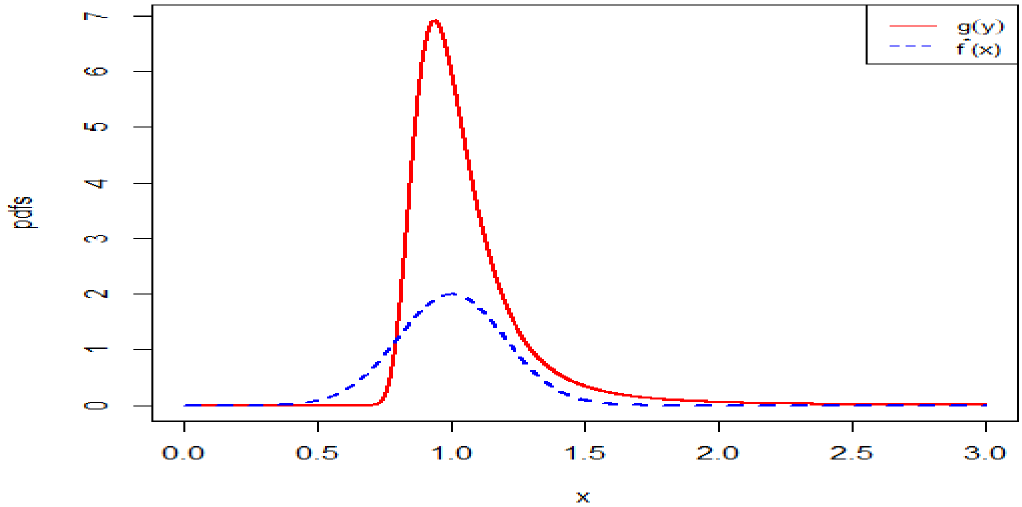

Fig. 2: Curve Shapes for $\sigma = 0.200$

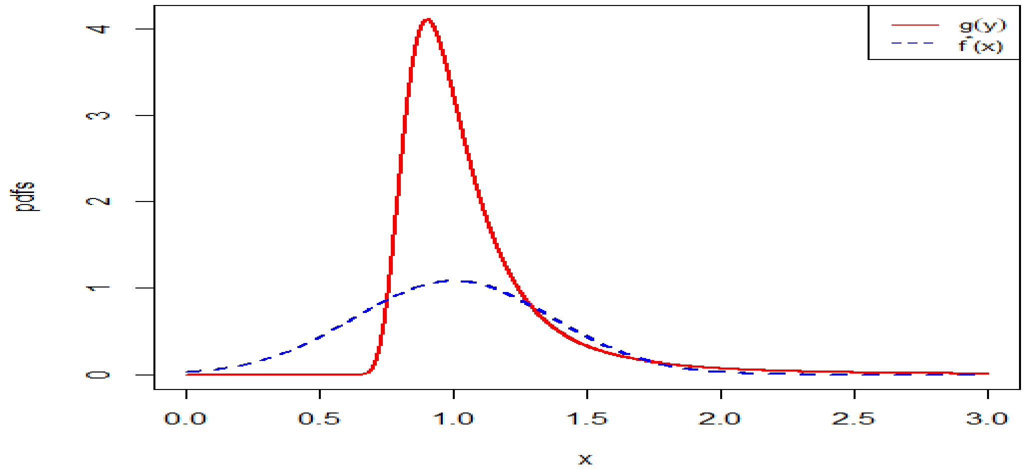

Fig. 3: Curve Shapes for $\sigma = 0.370$

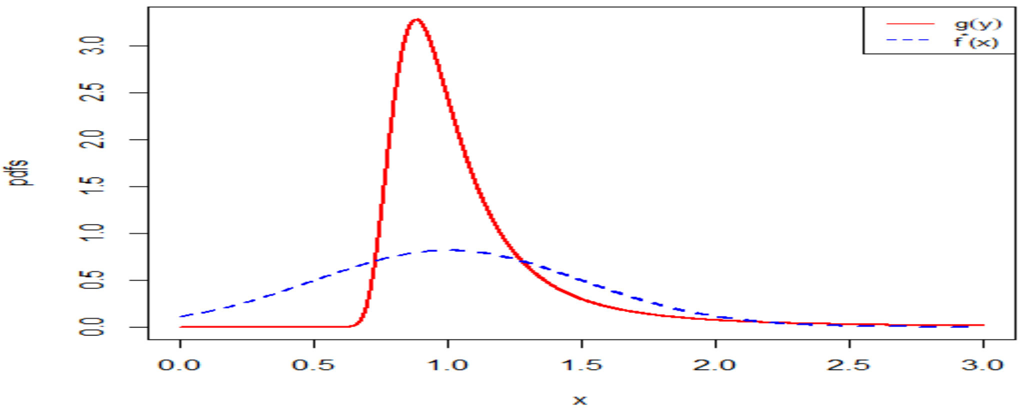

Fig. 4: Curve Shapes for $\sigma =$ 0.500

Recall that, given $f(x) = u(x).v(x)$, then $f^{/}(x) = u^{/}(x).v(x) + v^{/}(x).u(x)$. This concept was used to differentiate $g(y)$. To reduce complexity during differentiation let $m = \frac{3}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}$, in Equation (6), for the purpose of differentiation;

Let $u = y^{-4}$, $u^{/} = -4y^{-5} = \frac{-4}{y^5}$, $v = e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}$,

$$v^{/} = \left(-\frac{1}{2\sigma^2}\right)\left(\frac{1}{y^3}-1\right)\left(-\frac{6}{y^4}\right)e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}$$

$$g^{/}(y) = m\left[\left[\left(-\frac{4}{y^5}\right).e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}\right] + \left[\left(\frac{-1}{2\sigma^2}\right)\left(\frac{1}{y^3}-1\right)\left(\frac{-6}{y^4}\right)e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}.y^{-4}\right]\right]$$

$$g^{/}(y) = m\left[\frac{-4}{y^5}e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2} + \left\{\left(\frac{6}{2\sigma^2 y^4}\right)\left(\frac{1}{y^4}\right)\left(\frac{1}{y^3}-\frac{y^3}{y^3}\right)e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}\right\}\right]$$

$$g^{/}(y) = m\left[\frac{-4}{y^5}.e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2} + \frac{6}{2\sigma^2 y^8}\left(\frac{1}{y^3}-\frac{y^3}{y^3}\right).e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}\right]$$

$$g^{/}(y) = m\left[\left(\frac{-4}{y^5} + \frac{6}{2\sigma^2 y^8}\left(\frac{1-y^3}{y^3}\right)\right).e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}\right]$$

$$g^{/}(y) = \frac{3}{\sigma\sqrt{2\pi}\left[1-\varphi\left(-\frac{1}{\sigma}\right)\right]}\left[\left(\frac{-4}{y^5} + \frac{6}{2\sigma^2 y^8}\left(\frac{1-y^3}{y^3}\right)\right).e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}\right] \quad (7)$$

In Equation (7) equating $g^{/}(y) = 0$, we have

$$\left(\frac{-4}{y^5} + \frac{6}{2\sigma^2 y^8}\left(\frac{1-y^3}{y^3}\right)\right) = 0$$

$$\frac{6(1-y^3)}{2\sigma^2 y^{11}} - \frac{4}{y^5} = 0 => \frac{3(1-y^3)}{\sigma^2 y^{11}} - \frac{4}{y^5} = 0 => \frac{3-3y^3}{\sigma^2 y^{11}} = \frac{4}{y^5}$$

$$y^5(3-3y^3) = 4\sigma^2 y^{11} => 3y^5 - 3y^8 = 4\sigma^2 y^{11}$$

$4\sigma^2 y^{11} = 3y^5 - 3y^8 => 4\sigma^2 y^{11} + 3y^8 - 3y^5 = 0$ *divide through by* y^5

$$4\sigma^2 y^6 + 3y^3 - 3 = 0 \quad (8)$$

$4\sigma^2(y^3)^2 + 3y^3 - 3 = 0$. Putting $d = y^3$in Equation (8), it becomes

$$4\sigma^2 d^2 + 3d - 3 = 0 \quad (9)$$

Solving Equation (9) using quadratic general formula $d = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Where $a = 4\sigma^2$, $b = 3$ and $c = -3$

$$d = \frac{-3 \pm \sqrt{3^2 - 4\,(4\sigma^2)\,(-3)}}{2(4\sigma^2)}$$

$$d = \frac{-3 \pm \sqrt{9 + 48\sigma^2}}{8\sigma^2}$$

Recall $d = y^3 \Rightarrow y^3 = d \Rightarrow y = \sqrt[3]{d} \;\; => y = \sqrt[3]{\frac{-3 \pm \sqrt{9+48\sigma^2}}{8\sigma^2}}$ Since y_{max} is positive then,

$$y_{max} = \sqrt[3]{\frac{-3 + \sqrt{9+48\sigma^2}}{8\sigma^2}} \qquad (10)$$

The probability density function (pdf) of $g(y)$ attains its maximum at y_{max}.
The bell-shaped condition would implied $y_{max} \approx 1$. The computation of y_{max} for various values of the standard deviation, σ is shown in Table (2).

Table 2: Computation of y_{max} for $\sigma \epsilon$ [0.010, 0.395]

σ	y_{max}	$1 - y_{max}$	σ	y_{max}	$1 - y_{max}$
0.010	0.999956	0.0000444	0.205	0.982870	0.0171302
0.015	0.999900	0.0001000	0.210	0.982095	0.0179051
0.020	0.999822	0.0001776	0.215	0.981307	0.0186925
0.025	0.999723	0.0002774	0.220	0.980508	0.0194922
0.030	0.999601	0.0003992	0.225	0.979696	0.0203039
0.035	0.999457	0.0005430	0.230	0.978873	0.0211270
0.040	0.999291	0.0007086	0.235	0.978039	0.0219615
0.045	0.999104	0.0008960	0.240	0.977193	0.0228068
0.050	0.998895	0.0011050	0.245	0.976337	0.0236628
0.055	0.998665	0.0013355	0.250	0.975471	0.0245290
0.060	0.998413	0.0015873	0.255	0.974595	0.0254052
0.065	0.998140	0.0018604	0.260	0.973709	0.0262910
0.070	0.997846	0.0021544	0.265	0.972814	0.0271862
0.075	0.997531	0.0024693	0.270	0.971910	0.0280903
0.080	0.997195	0.0028048	0.275	0.970997	0.0290032
0.085	0.996839	0.0031607	0.280	0.970075	0.0299245
0.090	0.996463	0.0035368	0.285	0.969146	0.0308540
0.095	0.996067	0.0039328	0.290	0.968209	0.0317912
0.100	0.995651	0.0043486	0.295	0.967264	0.0327360
0.105	0.995216	0.0047839	0.300	0.966312	0.0336881
0.110	0.994762	0.0052383	0.305	0.965353	0.0346472
0.115	0.994288	0.0057118	0.310	0.964387	0.0356129
0.120	0.993796	0.0062038	0.315	0.963415	0.0365851
0.125	0.993286	0.0067143	0.320	0.962437	0.0375635
0.130	0.992757	0.0072429	0.325	0.961452	0.0385478
0.135	0.992211	0.0077893	0.330	0.960462	0.0395377
0.140	0.991647	0.0083531	0.335	0.959467	0.0405331
0.145	0.991066	0.0089342	0.340	0.958466	0.0415336
0.150	0.990468	0.0095321	0.345	0.957461	0.0425390
0.155	0.989853	0.0101466	0.350	0.956451	0.0435492
0.160	0.989223	0.0107773	0.355	0.955436	0.0445638
0.165	0.988576	0.0114240	0.360	0.954417	0.0455826
0.170	0.987914	0.0120862	0.365	0.953395	0.0466055

0.175	0.987236	0.0127637	0.370	0.952368	0.0476321
0.180	0.986544	0.0134561	0.375	0.951338	0.0486624
0.185	0.985837	0.0141631	0.380	0.950304	0.0496960
0.190	0.985116	0.0148843	0.385	0.949267	0.0507328
0.195	0.984381	0.0156195	0.390	0.948227	0.0517726
0.200	0.983632	0.0163682	0.395	0.947185	0.0528152

Tables 3: Conditions for median ≈ $\boldsymbol{mode} \approx \mathbf{1} \approx \boldsymbol{mean}$, Where $y_{max} \approx \mathbf{1}$

Decimal places	Mean ≈ mode ≈ 1 ≈ median
4	$0 < \sigma \leq 0.10$ $(0 < \sigma < 0.15)$
3	$0 < \sigma \leq 0.030$ $(0 < \sigma < 0.035)$
2	$0 < \sigma \leq 0.105$ $(0 < \sigma < 0.110)$
1	$0 < \sigma \leq 0.380$ $(0 < \sigma < 0.385)$

The summary of the values of $y_{max} \approx 1$ for different values of σ, shows that $g(y)$ is approximately symmetrical about one (1) with mean ≈ mode ≈ median ≈ 1 when $0 < \sigma \leq 0.10$ $(0 < \sigma < 0.15)$ correct to four decimal places, $0 < \sigma \leq 0.030$ $(0 < \sigma < 0.035)$ correct to three decimal places, $0 < \sigma \leq 0.105$ $(0 < \sigma < 0.110)$ correct to two decimal places and correct to one decimal place when $0 < \sigma \leq 0.380$ $(0 < \sigma < 0.385)$. With $0 < \sigma \leq 0.380$ $(0 < \sigma < 0.385)$ as the chosen interval.

Testing Normality Using the Anderson-Darling Test

Due to the fact that values of Skewness, which is the degree of asymmetry (departure from symmetry) of a distribution, positively or negatively and Kurtosis, which is the degree of peakedness of a distribution (leptokurtic-high peak, platykurtic-flat-top, and mesokurtic- not too flat, the Anderson-Darling (AD) test is used to test if the sample of the data used came from a population with normal distribution. AD test allows a more sensitive test since it makes use of a specific distribution in calculating critical values. It is one of the most powerful statistical tools for detecting departures from normality. AD is widely used in practice. The Anderson-Darling (AD) Test statistic is,

The Anderson-Darling (AD) Test statistic is,

$$AD = \sum_{i=1}^{n} \frac{1-2i}{n}\{\ln(F_0[Z_{(i)}]) + \ln(1 - F_0[Z_{(n+1-i)}])\} - n \tag{11}$$

Where "F_0" is the assumed (normal) distribution with the assumed or sample estimated parameters (μ, σ); "$Z_{(i)}$" is the ith sorted ordered data, standardized, samples value;

"n" is the sample size, "ln" in the natural logarithm (base) and subscript "i" runs from 1 to n.

In the AD test, we test data that are normally distributed against the alternative hypothesis (H_1). The test is a one tailed test, and the true distribution (H_0), F_0 is rejected (at significance level $\propto = 0.05$, for sample size n) if the AD test statistic is greater than the critical value (CV) or if the p-value is less than the significance level ($\propto$) used.

Simulation

The region where the bell-shaped conditions are satisfied (mean ≈ median ≈ 1). To find this satisfactory region, we obtained it by using artificial data (simulated data) generated from a truncated normal distribution for e_t, and subsequently transformed to obtain $e_t^* = \frac{1}{\sqrt[3]{e_t}}$ for $0.05 \leq \sigma \leq 0.200$

To conFig. this simulation, interval of 0.01 is used, that is, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.11, up to 0.25 due to space consumption a selected few shall be shown here. Values of the required statistics such as medians, means, standard deviations and variances of the untransformed e_t and transformed e_t^* will be obtained to examine the relationship between the two distributions.

Many replications were performed for values of σ in steps of 0.01, as highlighted above. For want of space, the results of the first 15 replications are shown for the selected configurations, (n = 100, σ = 0.05), (n = 100, σ = 0.07), (n = 100, σ = 0.10), (n = 100, σ = 0.12), (n = 100, σ = 0.14), (n = 100, σ = 0.19), (n = 100, σ = 0.22) and (n = 100, σ = 0.25).

For the fact that the values of Skewness and Kurtosis are not sufficient to confirm normality, the used of Anderson Darling (AD) test statistic for normality of the untransformed distribution and that of the transformed distribution shall be employed.

2.2 Functional Expressions for the Mean and Variance of Transformed Error Term $e_t^* = Y$

By definition, the mean of Y, $E(Y)$ is given by $E(Y) = \int_0^\infty y\, g(y)\, dy$ (12)

Thus, $E(Y) = \int_0^\infty y \frac{3e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}}{\sigma\sqrt{2\pi}\,[1-\Phi\left(-\frac{1}{\sigma}\right)]}.\frac{1}{y^4}dy$

$= \frac{3}{\sigma\sqrt{2\pi}\;[1-\Phi\left(\frac{-1}{\sigma}\right)]}\int_0^\infty y^{-3} e^{-\frac{1}{2}\cdot\left(\frac{\frac{1}{y^3}-1}{\sigma}\right)^2} dy($ (13) $Let\ u = \frac{\frac{1}{y^3}-1}{\sigma} \Rightarrow \sigma u = \frac{1}{y^3} - 1 \Rightarrow \frac{1}{y^3} = \sigma u + 1 \Rightarrow y = (\sigma u + 1)^{-\frac{1}{3}}$

$\frac{dy}{du} = \frac{-\sigma\,(\sigma u+1)^{-\frac{4}{3}}}{3}$

$dy = \frac{-\sigma(\sigma u+1)^{-\frac{4}{3}}}{3}.du$

Hence, $E(Y) = \frac{3}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\int_{-\frac{1}{\sigma}}^{\infty}\left[(\sigma u+1)^{-\frac{1}{3}}\right]^{-3} e^{-\frac{1}{2}\cdot u^2}.\frac{\sigma(\sigma u+1)^{-\frac{4}{3}}}{3}.du$

$= \frac{3}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\cdot\frac{\sigma}{3}\int_{-\frac{1}{\sigma}}^{\infty}(\sigma u+1)^{\frac{3}{3}}.\,e^{-\frac{1}{2}\cdot u^2}.(\sigma u+1)^{-\frac{4}{3}}.du$

$$= \frac{1}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\int_{-\frac{1}{\sigma}}^{\infty}(\sigma u+1)^{-\frac{1}{3}}\, e^{-\frac{u^2}{2}}\, du \quad (14)$$

For simplicity, $V = \frac{1}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}$. Using the Binomial expansion (Smith and Minton, 2008),

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots \quad (15)$$

Hence, $(\sigma u+1)^{-\frac{1}{3}}$ in Equation (14) is rewritten as $(1+\sigma u)^{-\frac{1}{3}}$ $(1+\sigma u)^{-\frac{1}{3}} = 1 + \frac{\left(-\frac{1}{3}\right)(\sigma u)}{1!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)(\sigma u)^2}{2!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)(\sigma u)^3}{3!} + \cdots$

$\therefore (1+\sigma u)^{-\frac{1}{3}} = 1 - \frac{\sigma u}{3} + \frac{4(\sigma u)^2}{18} - \frac{28(\sigma u)^3}{162} + \cdots$

It follows that

$$E(Y) = V\int_{-\frac{1}{\sigma}}^{\infty}\left[1 - \frac{\sigma u}{3} + \frac{4(\sigma u)^2}{18} - \frac{28\,(\sigma u)^3}{162} + \cdots\right] e^{\frac{-u^2}{2}}\, du \quad (16)$$

Table 4: Simulation Results When σ =0.05

$X = e_t \sim N(1, \sigma^2), \sigma = 0.05$ $Y = e_t^* = \frac{1}{\sqrt[3]{e_t}}, e_t \sim N\ (1, \sigma^2), \sigma = 0.05$

Mean(μ)	S.D(σ)	$Var(\sigma^2)$	Median	AD	P-value	Mean(μ)	S.D(σ)	$Var^*(\sigma^2)$	Median	AD	P-value	$\frac{Var(e_t)}{Var(e_t^*)}$	≈
1	0.05	0.0025	1.00059	0.424	0.317	1.00006	0.01640	0.000270	0.99980	0.557	0.605	9.25926	9
1	0.05	0.0025	0.99645	0.967	0.015	1.00006	0.01670	0.000279	1.00120	0.597	0.118	8.96057	9
1	0.05	0.0025	1.00310	0.380	0.404	1.00007	0.01719	0.000295	0.99898	0.983	0.114	8.47458	9
1	0.05	0.0025	1.00350	0.643	0.093	1.00006	0.01640	0.000270	0.99880	0.707	0.105	9.25926	9
1	0.05	0.0025	0.99572	0.689	0.070	1.00006	0.01700	0.000290	1.00140	0.545	0.158	8.62069	9
1	0.05	0.0025	0.99706	0.217	0.838	1.00006	0.01660	0.000280	1.00100	0.277	0.648	8.92857	9
1	0.05	0.0025	1.00440	0.323	0.522	1.00006	0.01610	0.000269	0.99850	0.507	0.197	9.29368	9
1	0.05	0.0025	0.99327	0.219	0.834	1.00006	0.01680	0.000280	1.00230	0.291	0.604	8.92857	9
1	0.05	0.0025	1.00204	0.528	0.174	1.00006	0.01700	0.000290	0.99930	0.784	0.041	8.62069	9
1	0.05	0.0025	0.98703	0.379	0.399	1.00006	0.01740	0.000289	1.00440	0.273	0.671	8.65052	9
1	0.05	0.0025	0.99903	0.416	0.332	1.00006	0.01730	0.000301	1.00030	0.567	0.142	9.3633	9
1	0.05	0.0025	1.00120	0.335	0.507	1.00006	0.01660	0.000277	1.00020	0.788	0.041	9.0253	9
1	0.05	0.0025	0.99994	0.288	0.619	1.00007	0.01700	0.000289	1.00000	0.379	0.406	8.6505	9
1	0.05	0.0025	1.00101	1.126	0.006	1.00006	0.01717	0.000295	0.99970	0.352	0.074	8.4803	9
1	0.05	0.0025	1.00420	0.441	0.290	1.00006	0.01671	0.000279	0.99861	0.423	0.105	8.9606	9

$NB: (i)\ \frac{2\sigma^2}{9} = \frac{2(0.05)^2}{9} = \frac{2(0.0025)}{9} = \frac{0.005}{9} = 0.0006 \quad (ii) E(e_t^*) \approx 1 + \frac{2\sigma^2}{9} \approx 1$

Table 5:Simulation Results when σ=0.07

$X = e_t \sim N(1, \sigma^2), \sigma = 0.07$ $Y = e_t^* = \frac{1}{\sqrt[3]{e_t}}, e_t \sim N\ (1, \sigma^2), \sigma = 0.07$

Mean(μ)	S.D(σ)	$Var(\sigma^2)$	Median	AD	P-value	Mean(μ)	S.D(σ)	$Var^*(\sigma^2)$	Median	AD	P-value	$\frac{Var(e_t)}{Var(e_t^*)}$	≈
1	0.07	0.0049	1.00590	0.230	0.802	1.00012	0.02300	0.00053	0.99810	0.200	0.881	9.24528	9
1	0.07	0.0049	0.99363	0.252	0.731	1.00013	0.02390	0.00057	1.00210	0.351	0.464	8.59649	9
1	0.07	0.0049	1.01080	0.613	0.108	1.00012	0.02319	0.00054	0.99644	0.688	0.070	9.07407	9
1	0.07	0.0049	0.99830	0.695	0.068	1.00012	0.02270	0.00052	1.00060	0.503	0.200	9.42308	9
1	0.07	0.0049	1.00940	0.396	0.365	1.00012	0.02350	0.00055	0.99690	0.673	0.077	8.90909	9

1	0.07	0.0049	1.00590	0.230	0.802	1.00012	0.02300	0.00053	0.99810	0.200	0.881	9.24528	9
1	0.07	0.0049	1.01950	0.655	0.085	1.00013	0.02392	0.00057	0.99359	0.852	0.028	8.59649	9
1	0.07	0.0049	1.00600	0.234	0.791	1.00012	0.02298	0.00053	0.99801	0.437	0.291	9.24528	9
1	0.07	0.0049	0.99953	0.473	0.238	1.00012	0.02330	0.00054	1.00020	0.787	0.040	9.07407	9
1	0.07	0.0049	0.99479	0.376	0.406	1.00012	0.02310	0.00053	1.00170	0.294	0.594	9.24528	9
1	0.07	0.0049	0.99870	0.253	0.727	1.00012	0.02310	0.00053	1.00040	0.157	0.952	9.24528	9
1	0.07	0.0049	0.99448	0.427	0.307	1.00012	0.02340	0.00055	1.00180	0.582	0.127	8.90909	9
1	0.07	0.0049	1.00344	0.962	0.015	1.00012	0.02350	0.00055	0.99890	1.103	0.010	8.90909	9
1	0.07	0.0049	1.00580	0.292	0.600	1.00012	0.02348	0.00055	0.99808	0.459	0.257	8.90909	9
1	0.07	0.0049	0.99740	0.550	0.153	1.00012	0.02335	0.00054	1.00087	0.323	0.521	9.07407	9

$(i)\ \frac{2\sigma^2}{9} = \frac{2(0.07)^2}{9} = \frac{2(0.0049)}{9} = \frac{0.0098}{9} = 0.00109 \qquad (ii) E(e_t^*) \approx 1 + \frac{2\sigma^2}{9} \approx 1.0011$

Table 6:Simulation Results when σ=0.10

$\boldsymbol{X = e_t \sim N(1, \sigma^2), \sigma = 0.10}$ $\boldsymbol{Y = e_t^* = \frac{1}{\sqrt[3]{e_t}}, e_t \sim N\,(1, \sigma^2), \sigma = 0.10}$

Mean(μ)	**S.D(σ)**	$\boldsymbol{Var(\sigma^2)}$	**Median**	**AD**	**P-value**	**Mean(μ)**	**S.D(σ)**	$\boldsymbol{Var^*(\sigma^2)}$	**Median**	**AD**	**P-value**	$\boldsymbol{\frac{Var(e_t)}{Var(e_t^*)}}$	$\approx$
1	0.1	0.01	1.00861	0.231	0.800	1.00024	0.03300	0.00109	0.99710	0.392	0.372	9.17431	9
1	0.1	0.01	1.00750	0.180	0.913	1.00025	0.03335	0.00111	0.99752	0.479	0.230	9.00901	9
1	0.1	0.01	1.00342	0.332	0.508	1.00024	0.03290	0.00108	0.99890	0.856	0.027	9.25926	9
1	0.1	0.01	1.00590	0.237	0.781	1.00024	0.03250	0.00106	0.99800	0.473	0.238	9.43396	9
1	0.1	0.01	0.99030	0.267	0.680	1.00024	0.03270	0.00107	1.00330	0.251	0.737	9.34579	9
1	0.1	0.01	0.99144	0.377	0.404	1.00025	0.03360	0.00113	1.00290	0.363	0.434	8.84956	9
1	0.1	0.01	0.98460	0.495	0.210	1.00025	0.03380	0.00114	1.00520	0.256	0.720	8.77193	9
1	0.1	0.01	0.97720	0.564	0.141	1.00026	0.03400	0.00116	1.00770	0.363	0.434	8.62069	9
1	0.1	0.01	1.01140	0.547	0.155	1.00025	0.03329	0.00111	0.99622	0.558	0.146	9.00901	9
1	0.1	0.01	1.00620	0.181	0.912	1.00024	0.03310	0.00110	0.99793	0.434	0.296	9.09091	9
1	0.1	0.01	0.99284	0.306	0.559	1.00026	0.03420	0.00117	1.00240	0.775	0.043	8.54701	9
1	0.1	0.01	1.00880	0.282	0.631	1.00026	0.03410	0.00116	0.99710	0.517	0.185	8.62069	9
1	0.1	0.01	1.01370	0.385	0.386	1.00024	0.03272	0.00107	0.99548	0.647	0.089	9.34579	9
1	0.1	0.01	0.98638	0.357	0.449	1.00025	0.03330	0.00111	1.00460	0.276	0.651	9.00901	9
1	0.1	0.01	1.01760	0.569	0.137	1.00024	0.03320	0.00110	0.99420	0.658	0.084	9.09091	9

$(i)\ \frac{2\sigma^2}{9} = \frac{2(0.10)^2}{9} = \frac{2(0.01)}{9} = \frac{0.02}{9} = 0.00222$ $\qquad (ii) E(e_t^*) \approx 1 + \frac{2\sigma^2}{9} \approx 1.00222$

Table 7:Simulation Results when σ = 0.14

$X = e_t \sim N(1, \sigma^2), \sigma = 0.14$ $\qquad Y = e_t^* = \frac{1}{\sqrt[3]{e_t}}, e_t \sim N\ (1, \sigma^2), \sigma = 0.14$

Mean(μ)	**S.D(σ)**	$\boldsymbol{Var(\sigma^2)}$	**Median**	**AD**	**P-value**	**Mean(μ)**	**S.D(σ)**	$\boldsymbol{Var^*(\sigma^2)}$	**Median**	**AD**	**P-value**	$\boldsymbol{\frac{Var(e_t)}{Var(e_t^*)}}$	**≈**
1	0.14	0.0196	0.9888	0.412	0.334	1.00048	0.04670	0.00218	1.00380	0.406	0.346	8.99083	9
1	0.14	0.0196	1.0166	0.321	0.525	1.00051	0.04780	0.00228	0.99450	0.892	0.022	8.59649	9
1	0.14	0.0196	0.9594	0.788	0.040	1.00048	0.04630	0.00215	1.01390	0.387	0.383	9.11628	9
1	0.14	0.0196	0.9936	295.000	0.592	1.00048	0.04630	0.00214	1.00210	0.606	0.112	9.15888	9
1	0.14	0.0196	0.9933	0.310	0.550	1.00047	0.04610	0.00213	1.00230	0.456	0.261	9.20188	9
1	0.14	0.0196	1.0143	0.331	509.000	1.00051	0.04800	0.00230	0.99530	1.173	0.005	8.52174	9
1	0.14	0.0196	0.9829	0.397	0.363	1.00050	0.04720	0.00223	1.00580	0.804	0.036	8.78924	9
1	0.14	0.0196	1.0041	0.226	0.814	1.00050	0.04770	0.00227	0.99860	0.693	0.068	8.63436	9
1	0.14	0.0196	0.9660	0.365	0.431	1.00050	0.04760	0.00227	1.01160	0.391	0.374	8.63436	9
1	0.14	0.0196	1.0161	0.545	0.157	1.00046	0.04570	0.00209	0.99470	1.131	0.006	9.37799	9
1	0.14	0.0196	1.0180	0.172	0.928	1.00048	0.04640	0.00216	0.99410	0.490	0.216	9.07407	9
1	0.14	0.0196	0.9776	0.223	0.823	1.00049	0.04710	0.00222	1.00760	0.164	0.942	8.82883	9
1	0.14	0.0196	0.9821	0.272	0.664	1.00046	0.04561	0.00208	1.00600	0.392	0.373	9.42308	9
1	0.14	0.0196	1.0015	0.294	0.594	1.00051	0.04790	0.00229	0.99950	0.841	0.029	8.55895	9
1	0.14	0.0196	0.9922	0.308	0.554	1.00050	0.04744	0.00225	1.00260	0.298	0.583	8.71111	9

$(i)\ \frac{2\sigma^2}{9} = \frac{2(0.14)^2}{9} = \frac{2(0.0196)}{9} = \frac{0.0392}{9} = 0.00436$ $\qquad (ii) E(e_t^*) \approx 1 + \frac{2\sigma^2}{9} \approx 1.00436$

Table 8: Simulation Results when σ = 0.19

$\boldsymbol{X = e_t \sim N(1, \sigma^2), \sigma = 0.19}$ $\qquad \boldsymbol{Y = e_t^* = \frac{1}{\sqrt[3]{e_t}}, e_t \sim N\ (1, \sigma^2), \sigma = 0.19}$

Mean(μ)	**S.D(σ)**	$\boldsymbol{Var(\sigma^2)}$	**Median**	**AD**	**P-value**	**Mean(μ)**	**S.D(σ)**	$\boldsymbol{Var^*(\sigma^2)}$	**Median**	**AD**	**P-value**	$\boldsymbol{\frac{Var(e_t)}{Var(e_t^*)}}$	**≈**
1	0.19	0.0361	0.988594	0.262	0.697	1.00107	0.0692976	0.0048022	1.00383	0.660	0.082	7.5175	8
1	0.19	0.0361	0.988600	0.585	0.124	1.00107	0.0693000	0.0048000	1.00380	1.088	0.007	7.5208	8
1	0.19	0.0361	1.017900	0.124	0.987	1.00127	0.0756000	0.0057100	0.99410	0.902	0.021	6.3222	6
1	0.19	0.0361	0.957400	0.741	0.052	1.00077	0.0589000	0.0034700	1.01460	0.262	0.697	10.4035	10
1	0.19	0.0361	0.982500	0.257	0.714	1.00107	0.0694000	0.0048200	1.00590	1.489	<0.005	7.4896	8

1	0.19	0.0361	1.007100	0.386	0.385	1.00087	0.0625000	0.0039100	0.99760	0.493	0.213	9.2327	9
1	0.19	0.0361	0.976800	0.255	0.723	1.00100	0.0673000	0.0045200	1.00790	0.309	0.553	7.9867	8
1	0.19	0.0361	0.983400	0.219	0.834	1.00104	0.0683000	0.0046700	1.00560	1.152	<0.005	7.7302	8
1	0.19	0.0361	0.989400	0.198	0.885	1.00102	0.0676000	0.0045700	1.00360	0.673	0.077	7.8993	8
1	0.19	0.0361	1.029800	0.381	0.395	1.00091	0.0641000	0.0041100	0.99030	1.513	<0.005	8.7835	9

$(i)\ \frac{2\sigma^2}{9} = 0.00889 (ii)\ E(e_t^*) \approx 1 + \frac{2\sigma^2}{9} \approx 1.00110108$ (iii) The ratio of $Var(e_t)$ to $Var(e_t^*)$ is 20% approximately 9, 70% less than 9 and 10% greater than 9 (iv) 100% of the simulated cases p-value is less than 0.05.

Table 9:Simulation Results when σ=0.22

Mean(μ)	**S.D(σ)**	$Var(\sigma^2)$	**Median**	**AD**	**P-value**	**Mean(μ)**	**S.D(σ)**	$Var^*(\sigma^2)$	**Median**	**AD**	**P-value**	$\frac{Var(e_t)}{Var(e_t^*)}$	≈
1	0.22	0.0484	0.9744	0.124	0.986	1.00154	0.08330	0.00694	1.00870	1.053	0.009	6.97406	7
1	0.22	0.0484	1.0040	0.376	0.407	1.00159	0.08450	0.00714	0.99870	1.020	0.011	6.77871	7
1	0.22	0.0484	1.0078	0.200	0.882	1.00140	0.07950	0.00632	0.99740	1.142	<0.005	7.65823	8
1	0.22	0.0484	1.0062	0.276	0.649	1.00145	0.08090	0.00654	0.99790	1.718	<0.005	7.40061	7
1	0.22	0.0484	0.9640	0.666	0.080	1.00114	0.07160	0.00513	1.01230	0.539	0.163	9.43470	9
1	0.22	0.0484	0.9737	0.262	0.699	1.00170	0.08750	0.00766	1.00890	1.268	<0.005	6.31854	6
1	0.22	0.0484	0.9806	0.230	0.803	1.00209	0.09690	0.00940	1.00650	1.924	<0.005	5.14894	5
1	0.22	0.0484	1.0819	0.395	0.364	1.00124	0.07468	0.00558	0.97409	1.638	<0.005	8.67384	9
1	0.22	0.0484	1.0029	0.456	0.261	1.00186	0.09150	0.00838	0.99900	2.440	<0.005	5.77566	6
1	0.22	0.0484	1.0309	0.206	0.866	1.00161	0.08510	0.00724	0.98990	1.857	<0.005	6.68508	7

$(i)\ \frac{2\sigma^2}{9} = 0.0108 (ii)\ E(e_t^*) \approx 1 + \frac{2\sigma^2}{9} \approx 1.0108$ (iii) The ratio of $Var(e_t)$ to $Var(e_t^*)$ is only 20% approximately 9 and 80% less than 9 (iv) 100% of the simulated cases p-value is less than 0.05.

$$E(Y) = V\begin{bmatrix} \int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}}.du - \int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}}.du \\ + \int_{-\frac{1}{\sigma}}^{\infty} \frac{4\sigma^2 u^2}{18}.e^{-\frac{u^2}{2}}.du - \int_{-\frac{1}{\sigma}}^{\infty} \frac{28\sigma^3 u^3}{162} e^{-\frac{u^2}{2}}.du + \dots \end{bmatrix}$$

$$= \frac{1}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\begin{bmatrix} \int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}}.du - \frac{\sigma}{3}\int_{-\frac{1}{\sigma}}^{\infty} u.e^{-\frac{u^2}{2}}.du + \frac{2\sigma^2}{9}\int_{-\frac{1}{\sigma}}^{\infty} u^2.e^{-\frac{u^2}{2}}.du \\ -\frac{14\sigma^3}{81}\int_{-\frac{1}{\sigma}}^{\infty} u^3.e^{-\frac{u^2}{2}}.du + \cdots \end{bmatrix}$$

$$= \frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[\frac{1}{\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}}.du - \frac{\sigma}{3\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^{\infty} u.e^{-\frac{u^2}{2}}.du + \frac{2\sigma^2}{9\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^{\infty} u^2.e^{-\frac{u^2}{2}}.du - \frac{14\sigma^3}{81\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^{\infty} u^3.e^{-\frac{u^2}{2}}.du + \cdots\right] \quad (17)$$

$$= \frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}[A - B + C - D + \cdots], \quad (18)$$

where; $A = \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}}.du = \Pr\left(u > \frac{-1}{\sigma}\right)$, then

$A = \left[1 - \Phi\left(\frac{-1}{\sigma}\right)\right]$,

$$B = \frac{1}{3\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} u.e^{-\frac{u^2}{2}}.du$$

$$= \frac{\sigma}{6\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} \quad (19)$$

$$C = \frac{2\sigma^2}{9\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} u^2.e^{-\frac{u^2}{2}}.du$$

$$= \frac{2\sigma^2}{9\sqrt{2\pi}}\left[\int_{\frac{-1}{\sigma}}^{\infty} u^2.e^{-\frac{u^2}{2}}.du + \int_0^{\infty} u^2.e^{-\frac{u^2}{2}}.du\right].$$

$$C = \frac{2\sigma^2}{18} - \frac{2\sigma}{9\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}} + \frac{2\sigma^2}{18}.Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right) \quad (20)$$

$$D = \frac{14\sigma^2}{81\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} u^3.e^{-\frac{u^2}{2}}.du.$$

$$= \frac{14\sigma^2}{81\sqrt{2\pi}}\left[\int_{\frac{1}{\sigma^2}}^{\infty} u.y.e^{-\frac{y}{2}}.\frac{dy}{2u}\right] = \frac{14\sigma^3}{81\sqrt{2\pi}}.\frac{u}{2u}\int_{\frac{-1}{\sigma^2}}^{\infty} y.e^{-\frac{y}{2}}.dy = \frac{14\sigma^3}{81\sqrt{2\pi}}\left[\frac{1}{2}\int_{\frac{1}{\sigma^2}}^{\infty} y.e^{-\frac{y}{2}}.dy\right]$$

$$= \frac{14\sigma}{81\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}} + \frac{28\sigma^3}{81\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}} \quad (21)$$

Substitute A, B, C and D of Equation (16), (17), (18) and (19) respectively into Equation (15)

$$E(Y) = \frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\begin{bmatrix} \left(1-\Phi\left(-\frac{1}{\sigma}\right)\right) - \left(\frac{\sigma}{6\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}}\right) + \left(\frac{2\sigma^2}{18} - \frac{2\sigma}{9\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \frac{2\sigma^2}{18}.\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right) \\ -\left(\frac{14\sigma}{81\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}} + \frac{28\sigma^3}{81\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}}\right) + \dots \end{bmatrix}$$

By further simplification and like terms collections

$$E(Y) = \frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\begin{bmatrix} \left(1-\Phi\left(-\frac{1}{\sigma}\right)\right) + \left(\frac{-\sigma}{6\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} - \frac{2\sigma}{9\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \frac{14\sigma}{81\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}}\right) \\ + \left\{\frac{2\sigma^2}{18} + \left(\frac{2\sigma^2}{18}.Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)\right\} - \frac{28\sigma^2}{81\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \cdots \end{bmatrix}$$

$$= \frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[\begin{array}{c}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)-\frac{91\sigma}{162\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} \\ +\left\{\frac{\sigma^2}{9}\left(1+Pr\left(\chi^2_{(1)}<\frac{1}{\sigma^2}\right)\right)\right\}-\frac{28\sigma^3}{81\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}}+\cdots\end{array}\right]$$

$$\therefore E(Y) = 1-\left[\frac{91\sigma e^{-\frac{1}{2\sigma^2}}}{162\sqrt{2\pi}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)}\right]+\left[\frac{\sigma^2\left[1+Pr\left(\chi^2_{(1)}<\frac{1}{\sigma^2}\right)\right]}{9\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)}\right]-\left[\frac{28\sigma^2 e^{-\frac{1}{2\sigma^2}}}{81\sqrt{2\pi}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)}\right]+\cdots \quad (22)$$

Variance of $Y, Var(Y) = E(Y^2)-[E(Y)]^2$ (23)

To find the variance, we have to obtain the second non-central moment

$$E(Y^2) = \int_0^\infty y^2\, g(y)dy \quad (24)$$

$$E(Y^2) = \int_0^\infty y^2 . \frac{3}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}.e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^3}-1\right)^2}.\frac{1}{y^4}dy$$

$$= \frac{3}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\int_0^\infty y^{-2}.e^{-\frac{1}{2}\left(\frac{\frac{1}{y^3}-1}{\sigma}\right)^2}dy$$

$$Let\ u = \frac{\frac{1}{y^3}-1}{\sigma} \Longrightarrow \frac{1}{y^3}-1 = \sigma u \Longrightarrow \frac{1}{y^3} = \sigma u+1 \ \Longrightarrow\ y^3 = (\sigma u+1)^{-1},$$

$y = (\sigma u+1)^{-\frac{1}{3}} \ \Rightarrow \frac{dy}{du} = \frac{dy}{dm}.\frac{dm}{du}$. Where $m = \sigma u+1 \Longrightarrow \frac{dm}{du} = \sigma$ and

Thus, $\frac{dy}{du} = \frac{-\sigma(\sigma u+1)^{-\frac{4}{3}}}{3}$, $\quad (-\frac{1}{\sigma}<u<\infty)$

By substitution

$$E(Y^2) = \frac{1}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\int_{-\frac{1}{\sigma}}^\infty(\sigma u+1)^{-\frac{2}{3}}.e^{-\frac{u^2}{2}}du \quad (25)$$

Consequently,

$$(1+\sigma u)^{-\frac{2}{3}} = 1+\frac{\left(-\frac{2}{3}\right)(\sigma u)}{1!}+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)(\sigma u)^2}{2!}+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)(\sigma u)^3}{3!}+\cdots$$

$$= 1+\frac{\left(\frac{-2}{3}\right)\sigma u}{1}+\frac{\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)(\sigma u)^2}{2\times1}+\frac{\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{-8}{3}\right)(\sigma u)^3}{3\times2\times1}+\cdots$$

$$(1+\sigma u)^{-\frac{2}{3}} = 1-\frac{2\sigma u}{3}+\frac{10(\sigma u)^2}{18}-\frac{80(\sigma u)^3}{162}+\cdots \quad (26)$$

From Equation (12), let $V = \frac{1}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}$, Then

$$E(Y^2) = V\int_{-\frac{1}{\sigma}}^\infty\left[1-\left(\frac{2\sigma u}{3}\right)+\left(\frac{10(\sigma u)^2}{18}\right)-\left(\frac{80(\sigma u)^3}{162}\right)+\cdots\right]e^{-\frac{u^2}{2}}du$$

$$= V\left[\int_{-\frac{1}{\sigma}}^\infty e^{-\frac{u^2}{2}}du-\int_{-\frac{1}{\sigma}}^\infty\frac{2\sigma u}{3}.e^{-\frac{u^2}{2}}du+\int_{-\frac{1}{\sigma}}^\infty\frac{10(\sigma u)^2}{18}.e^{-\frac{u^2}{2}}du-\int_{-\frac{1}{\sigma}}^\infty\frac{80(\sigma u)^3}{162}.e^{-\frac{u^2}{2}}du+\cdots\right]$$

Now replacing V, we have

$$E(Y^2) = \frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[\begin{array}{c}\frac{1}{\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^\infty e^{-\frac{u^2}{2}}du-\frac{2\sigma}{3\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^\infty u.e^{-\frac{u^2}{2}}du \\ +\frac{5\sigma^2}{9\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^\infty u^2.e^{-\frac{u^2}{2}}du-\frac{40\sigma^3}{81}\int_{-\frac{1}{\sigma}}^\infty u^3.e^{-\frac{u^2}{2}}du+\cdots\end{array}\right] \quad (27)$$

Where $E(Y^2) = \frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}[P-Q+R-S+\cdots]$ (28)

$$P = \frac{1}{\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^\infty e^{-\frac{u^2}{2}}du = Pr\left(u>^{-\frac{1}{\sigma}}\right) = \left[1-\Phi\left(-\frac{1}{\sigma}\right)\right] \quad (29)$$

$Q = \frac{2\sigma}{3\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^{\infty} u.e^{-\frac{u^2}{2}}du = \frac{2\sigma}{3\sqrt{2\pi}}\left[e^{-\frac{1}{2\sigma^2}}\right] = \frac{2\sigma}{3\sqrt{2\pi}}\left(\frac{1}{2}.e^{-\frac{1}{2\sigma^2}}\right) = \frac{2\sigma}{6\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}}$

$R = \frac{5\sigma^2}{9\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^{\infty} u^2.e^{-\frac{u^2}{2}}.du$,

$= \frac{5\sigma^2}{9\sqrt{2\pi}}\left[\sqrt{\frac{\pi}{2}} - \frac{1}{\sigma}.e^{-\frac{1}{2\sigma^2}} + \frac{1}{2}\int_0^{\frac{1}{\sigma^2}} k^{-\frac{1}{2}}.e^{-\frac{k}{2}}.dk\right] \qquad = \frac{5\sigma^2}{9\sqrt{2\pi}}\left(\sqrt{\frac{\pi}{2}}\right) - \frac{5\sigma^2}{9\sqrt{2\pi}}\left(\frac{1}{\sigma}.e^{-\frac{1}{2\sigma^2}}\right) + \frac{5\sigma^2}{9\sqrt{2\pi}}\left(\frac{1}{2}\int_0^{\frac{1}{\sigma^2}} k^{-\frac{1}{2}}.e^{-\frac{k}{2}}\,dk\right)$

$= \frac{5\sigma^2\times\sqrt{\pi}}{9\times\sqrt{4}\times\sqrt{\pi}} - \frac{5\sigma}{9\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \frac{5\sigma^2}{9\sqrt{\pi}}.\frac{\sqrt{2\pi}}{2}.Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)$

$= \frac{5\sigma^2}{18} - \frac{5\sigma}{9\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \frac{5\sigma^2}{18}.Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)$ (30)

In accordance with Equation (21),

$S = \frac{40\sigma^3}{81\sqrt{2\pi}}\left(\frac{1}{\sigma^2}.e^{-\frac{1}{2\sigma^2}} + 2.e^{-\frac{1}{2\sigma^2}}\right) = \frac{40\sigma^3 e^{-\frac{1}{2\sigma^2}}}{\sigma^2 81\sqrt{2\pi}} + \frac{80\sigma^3 e^{-\frac{1}{2\sigma^2}}}{81\sqrt{2\pi}}$

$S = \frac{40\sigma}{81\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \frac{80\sigma^3}{81\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}}$ (31)

Substituting P, Q, R, and S into Equation (28),

$$E(Y^2) = \frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right] - \left[\frac{\sigma}{3\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}}\right] + \left[\frac{5\sigma^2}{18} - \frac{5\sigma}{9\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \frac{5\sigma^2}{18}.Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] - \left[\frac{40\sigma}{81\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \frac{80\sigma^3}{81\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}}\right] + \cdots\right]$$

$$= \frac{1}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right] + \left(\frac{-\sigma}{3\sqrt{2\pi}} - \frac{5\sigma}{9\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}} + \frac{40\sigma}{81\sqrt{2\pi}}\right).e^{-\frac{1}{2\sigma^2}} + \left\{\frac{5\sigma^2}{18} + \frac{5\sigma^2}{18}.\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right\} - \left(\frac{80\sigma^3}{81\sqrt{2\pi}}.e^{-\frac{1}{2\sigma^2}}\right)\right] + \cdots$$

$$E(Y^2) = \left[\frac{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}{\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\right] + \left[\left(\frac{-27\sigma - 45\sigma - 40\sigma}{81\sqrt{2\pi}\ \left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\right).e^{-\frac{1}{2\sigma^2}}\right]$$

$$+ \left[\frac{5\sigma^2}{18\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left(1 + \Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)\right] - \left[\frac{80\sigma^3}{81\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}.e^{-\frac{1}{2\sigma^2}}\right] + \cdots$$

Therefore, $E(Y^2) = 1 - \left[\left(\frac{112}{81\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\right).e^{-\frac{1}{2\sigma^2}}\right] + \left[\frac{5\sigma^2}{18\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left(1 + \Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)\right]$
$- \left[\frac{80\sigma^3}{81\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}.e^{-\frac{1}{2\sigma^2}}\right] + \cdots$ (32)

Observations:

I. It is observed that subsequent terms on Equation (22) and (32) for $E(Y)$ and $E(Y^2)$ have $e^{-\frac{1}{2\sigma^2}}$ as a common factor respectively.

II. Also $e^{-\frac{1}{2\sigma^2}} = 0$ for $\sigma \leq 0.220$ correct to four decimal places.

III. Condition (I) and (II) implies that all subsequent terms for $E(Y)$ and $E(Y^2)$ are all zeros for $\sigma \leq 0.220$.

Therefore, for $\sigma \leq 0.220$,

$E\,(Y) = 1 + \left[\frac{\sigma^2}{9\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}.\left(1 + \Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)\right]$ (33)

and

$$E(Y^2) = 1 + \left[\frac{5\sigma^2}{18\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\cdot\left(1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)\right] \quad (34)$$

From Equation (23) in line with Equation (33) and Equation (34),

$$Var\,(Y) = \left[1 + \frac{5\sigma^2}{18\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\cdot\left(1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)\right] - \left[1 + \frac{\sigma^2}{9\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\cdot\left(1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)\right]^2$$

$$Var\,(Y) = \left(1 + \frac{5\sigma^2 B}{18A}\right) - \left(1 + \frac{\sigma^2 B}{9A}\right)^2 \quad (35)$$

$$= \frac{\sigma^2 B}{18A} - \left(\frac{\sigma^2 B}{9A}\right)^2 \quad (36)$$

Replacing the actual values of A and B in Equation (36), it becomes

$$Var\,(Y) = \frac{\sigma^2\left[1+\Pr\left(\chi^2_{(1)}<\frac{1}{\sigma^2}\right)\right]}{18\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} - \frac{\sigma^2\left[1+\Pr\left(\chi^2_{(1)}<\frac{1}{\sigma^2}\right)\right]}{9\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} \quad (37)$$

2.3 Numerical Computations of Mean and Variance of The Probability Density Function $g(Y)$ of The Inverse Cube Root Transformed Error Term $(e_t^* = Y)$

Equation (33) and Equation (37) shall be used to compute values of $E(Y)$ and $Var(Y)$ respectively for $\sigma\epsilon[0.01, 0.40]$

Table 10 comprises values of the mean and variance of the probability density function of Y when A = 1 and B = 2.

3.0 Results and Discussion

It can be deduced from the paper that:

1) The curve shapes are bell-shaped and approximately symmetrical about $mean \approx 1\ for\ \sigma < 0.380$
2) By Rolle's Theorem:
   $Mean\ \approx\ mode\ \approx\ 1\ for;$
   $0 < \sigma < 0.110\ \ correct\ to\ two\ (2) decimal\ places$
   $0 < \sigma < 0.385\ correct\ to\ one\ (1)\ decimal\ places$
3) Using simulated Data on the error terms:
   i. $Median\ \approx mean \approx 1\ when\ \sigma \leq 0.20$
   ii. $E(Y) =\ E(e_t^*) \approx 1 + \frac{2\sigma^2}{9}\ \ when\ \sigma \leq 0.20$
   iii. $Var\ (e_t^*) \approx\ \frac{1}{9}Var\ (e_t)\ or\ Var\ (e_t) \approx\ 9Var\ (e_t^*)\ when\ \sigma\ \leq 0.14\ or$ when $\sigma < 0.20$
   iv. The P-value of the A.D. test statistic strongly supports the normality of e_t^* at $\sigma < 0.20$
4) Using the moment of $Y =\ e_t^*$, functional expressions for mean and variance:
   i. $E(Y) = E(e_t^*) \approx 1$
      Correct to 2 decimal places for $\sigma \leq 0.045$
      Correct to 1 decimal place for $\sigma \leq 0.010$
   ii. $\frac{Var\ (e_t)}{Var\ (e_t^*)} = \frac{Var\ (X)}{Var\ (Y)} \approx 9$
      Correct to 2 decimal places for $\sigma \leq 0.035$
      Correct to 1 decimal place for $\sigma \leq 0.110$

3.1 Real Life Data Description and Plotting

Real life data appropriate for this time series work on inverse cube root transformation were obtained from the Central Bank of Nigeria Statistics Database. The data obtained were data on monthly CBN survey (Naira Million) on Net Foreign Assets between the year 2014 and 2018 with 60 data points.

The time plot of the sixty data points of the data collected is shown on Fig. 5.

Table 10: Computations of $E^*(X)$, $E(Y)$, $Var^*(X)$ and $Var(Y)$ for $\sigma \in [0.01,\ 0.50]$

σ	σ^2	$e^{-\frac{1}{2\sigma^2}}$	A	B	$E^*(X)$	$E(Y)$	$Var^*(X)$	$Var(Y)$	$\frac{Var^*(X)}{Var(Y)}$
0.010	0.000100	0.000000	1.00000	2.00000	1.00000	1.00002	0.000100	0.0000111	9.00040
0.015	0.000225	0.000000	1.00000	2.00000	1.00000	1.00005	0.000225	0.0000250	9.00090
0.020	0.000400	0.000000	1.00000	2.00000	1.00000	1.00009	0.000400	0.0000444	9.00160
0.025	0.000625	0.000000	1.00000	2.00000	1.00000	1.00014	0.000625	0.0000694	9.00250
0.030	0.000900	0.000000	1.00000	2.00000	1.00000	1.00020	0.000900	0.0001000	9.00360
0.035	0.001225	0.000000	1.00000	2.00000	1.00000	1.00027	0.001225	0.0001360	9.00490
0.040	0.001600	0.000000	1.00000	2.00000	1.00000	1.00036	0.001600	0.0001777	9.00640
0.045	0.002025	0.000000	1.00000	2.00000	1.00000	1.00045	0.002025	0.0002248	9.00811
0.050	0.002500	0.000000	1.00000	2.00000	1.00000	1.00056	0.002500	0.0002775	9.01001
0.055	0.003025	0.000000	1.00000	2.00000	1.00000	1.00067	0.003025	0.0003357	9.01212
0.060	0.003600	0.000000	1.00000	2.00000	1.00000	1.00080	0.003600	0.0003994	9.01442
0.065	0.004225	0.000000	1.00000	2.00000	1.00000	1.00094	0.004225	0.0004686	9.01693
0.070	0.004900	0.000000	1.00000	2.00000	1.00000	1.00109	0.004900	0.0005433	9.01964
0.075	0.005625	0.000000	1.00000	2.00000	1.00000	1.00125	0.005625	0.0006234	9.02256
0.080	0.006400	0.000000	1.00000	2.00000	1.00000	1.00142	0.006400	0.0007091	9.02567
0.085	0.007225	0.000000	1.00000	2.00000	1.00000	1.00161	0.007225	0.0008002	9.02899
0.090	0.008100	0.000000	1.00000	2.00000	1.00000	1.00180	0.008100	0.0008968	9.03252
0.095	0.009025	0.000000	1.00000	2.00000	1.00000	1.00201	0.009025	0.0009988	9.03625
0.100	0.010000	0.000000	1.00000	2.00000	1.00000	1.00222	0.010000	0.0011062	9.04018
0.105	0.011025	0.000000	1.00000	2.00000	1.00000	1.00245	0.011025	0.0012190	9.04432
0.110	0.012100	0.000000	1.00000	2.00000	1.00000	1.00269	0.012100	0.0013372	9.04866
0.115	0.013225	0.000000	1.00000	2.00000	1.00000	1.00294	0.013225	0.0014608	9.05321
0.120	0.014400	0.000000	1.00000	2.00000	1.00000	1.00320	0.014400	0.0015898	9.05797
0.125	0.015625	0.000000	1.00000	2.00000	1.00000	1.00347	0.015625	0.0017241	9.06294
0.130	0.016900	0.000000	1.00000	2.00000	1.00000	1.00376	0.016900	0.0018637	9.06811
0.135	0.018225	0.000000	1.00000	2.00000	1.00000	1.00405	0.018225	0.0020086	9.07350
0.140	0.019600	0.000000	1.00000	2.00000	1.00000	1.00436	0.019600	0.0021588	9.07909

0.145	0.021025	0.000000	1.00000	2.00000	1.00000	1.00467	0.021025	0.0023143	9.08489
0.150	0.022500	0.000000	1.00000	2.00000	1.00000	1.00500	0.022500	0.0024750	9.09091
0.155	0.024025	0.000000	1.00000	2.00000	1.00000	1.00534	0.024025	0.0026409	9.09714
0.160	0.025600	0.000000	1.00000	2.00000	1.00000	1.00569	0.025600	0.0028121	9.10358
0.165	0.027225	0.000000	1.00000	2.00000	1.00000	1.00605	0.027225	0.0029884	9.11023
0.170	0.028900	0.000000	1.00000	2.00000	1.00000	1.00642	0.028900	0.0031699	9.11710
0.175	0.030625	0.000000	1.00000	2.00000	1.00000	1.00681	0.030625	0.0033565	9.12419
0.180	0.032400	0.000000	1.00000	2.00000	1.00000	1.00720	0.032400	0.0035482	9.13149
0.185	0.034225	0.000000	1.00000	2.00000	1.00000	1.00761	0.034225	0.0037449	9.13901
0.190	0.036100	0.000001	1.00000	2.00000	1.00000	1.00802	0.036100	0.0039468	9.14674
0.195	0.038025	0.000002	1.00000	2.00000	1.00000	1.00845	0.038025	0.0041536	9.15468
0.200	0.040000	0.000004	1.00000	2.00000	1.00000	1.00889	0.040000	0.0043654	9.16283
0.205	0.042025	0.000007	1.00000	2.00000	1.00000	1.00934	0.042024	0.0045822	9.17118
0.210	0.044100	0.000012	1.00000	2.00000	1.00000	1.00980	0.044099	0.0048040	9.17972
0.215	0.046225	0.000020	1.00000	2.00000	1.00000	1.01027	0.046223	0.0050306	9.18844
0.220	0.048400	0.000033	1.00000	1.99999	1.00000	1.01076	0.048397	0.0052621	9.19731
0.225	0.050625	0.000051	1.00000	1.99999	1.00000	1.01125	0.050620	0.0054984	9.20632
0.230	0.052900	0.000079	0.99999	1.99999	1.00001	1.01176	0.052893	0.0057396	9.21544
0.235	0.055225	0.000117	0.99999	1.99998	1.00001	1.01227	0.055214	0.0059855	9.22463
0.240	0.057600	0.000170	0.99998	1.99997	1.00002	1.01280	0.057584	0.0062362	9.23385
0.245	0.060025	0.000241	0.99998	1.99996	1.00002	1.01334	0.060001	0.0064915	9.24305
0.250	0.062500	0.000335	0.99997	1.99994	1.00003	1.01389	0.062467	0.0067515	9.25219
0.255	0.065025	0.000458	0.99996	1.99991	1.00005	1.01445	0.064978	0.0070162	9.26120
0.260	0.067600	0.000613	0.99994	1.99988	1.00006	1.01502	0.067536	0.0072854	9.27004
0.265	0.070225	0.000809	0.99992	1.99984	1.00009	1.01561	0.070139	0.0075592	9.27864
0.270	0.072900	0.001050	0.99989	1.99979	1.00011	1.01620	0.072787	0.0078376	9.28693
0.275	0.075625	0.001345	0.99986	1.99972	1.00015	1.01681	0.075477	0.0081204	9.29485
0.280	0.078400	0.001699	0.99982	1.99964	1.00019	1.01742	0.078210	0.0084076	9.30234
0.285	0.081225	0.002121	0.99977	1.99955	1.00024	1.01805	0.080984	0.0086992	9.30933
0.290	0.084100	0.002618	0.99972	1.99944	1.00030	1.01869	0.083797	0.0089952	9.31577

0.295	0.087025	0.003197	0.99965	1.99930	1.00038	1.01934	0.086648	0.0092955	9.32160
0.300	0.090000	0.003866	0.99957	1.99914	1.00046	1.02000	0.089537	0.0096000	9.32676
0.305	0.093025	0.004631	0.99948	1.99896	1.00056	1.02067	0.092461	0.0099088	9.33121
0.310	0.096100	0.005501	0.99937	1.99874	1.00068	1.02136	0.095419	0.0102217	9.33491
0.315	0.099225	0.006480	0.99925	1.99850	1.00081	1.02205	0.098409	0.0105388	9.33782
0.320	0.102400	0.007576	0.99911	1.99822	1.00097	1.02276	0.101431	0.0108600	9.33991
0.325	0.105625	0.008794	0.99895	1.99791	1.00114	1.02347	0.104482	0.0111852	9.34115
0.330	0.108900	0.010139	0.99878	1.99756	1.00134	1.02420	0.107562	0.0115144	9.34153
0.335	0.112225	0.011616	0.99858	1.99716	1.00155	1.02494	0.110668	0.0118475	9.34104
0.340	0.115600	0.013230	0.99837	1.99673	1.00180	1.02569	0.113799	0.0121845	9.33966
0.345	0.119025	0.014984	0.99813	1.99625	1.00207	1.02645	0.116955	0.0125254	9.33740
0.350	0.122500	0.016880	0.99786	1.99573	1.00236	1.02722	0.120132	0.0128701	9.33425
0.355	0.126025	0.018921	0.99758	1.99515	1.00269	1.02801	0.123332	0.0132185	9.33025
0.360	0.129600	0.021110	0.99726	1.99453	1.00304	1.02880	0.126551	0.0135706	9.32538
0.365	0.133225	0.023446	0.99693	1.99385	1.00342	1.02961	0.129789	0.0139263	9.31969
0.370	0.136900	0.025931	0.99656	1.99312	1.00384	1.03042	0.133044	0.0142856	9.31318
0.375	0.140625	0.028566	0.99617	1.99234	1.00429	1.03125	0.136317	0.0146484	9.30588
0.380	0.144400	0.031348	0.99575	1.99150	1.00477	1.03209	0.139605	0.0150147	9.29783
0.385	0.148225	0.034278	0.99530	1.99061	1.00529	1.03294	0.142907	0.0153845	9.28906
0.390	0.152100	0.037354	0.99483	1.98966	1.00584	1.03380	0.146224	0.0157576	9.27960
0.395	0.156025	0.040575	0.99432	1.98865	1.00643	1.03467	0.149553	0.0161339	9.26948
0.400	0.160000	0.043937	0.99379	1.98758	1.00706	1.03556	0.152895	0.0165136	9.25875
0.405	0.164025	0.047439	0.99323	1.98646	1.00772	1.03645	0.156248	0.0168964	9.24744
0.410	0.168100	0.051077	0.99264	1.98527	1.00842	1.03736	0.159613	0.0172823	9.23560
0.415	0.172225	0.054849	0.99202	1.98403	1.00915	1.03827	0.162987	0.0176713	9.22325
0.420	0.176400	0.058750	0.99137	1.98273	1.00993	1.03920	0.166372	0.0180634	9.21046
0.425	0.180625	0.062777	0.99069	1.98137	1.01074	1.04014	0.169766	0.0184583	9.19725
0.430	0.184900	0.066926	0.98998	1.97996	1.01160	1.04109	0.173168	0.0188561	9.18366
0.435	0.189225	0.071193	0.98924	1.97849	1.01249	1.04205	0.176580	0.0192568	9.16974
0.440	0.193600	0.075574	0.98848	1.97696	1.01342	1.04302	0.179999	0.0196602	9.15552

0.445	0.198025	0.080064	0.98769	1.97537	1.01439	1.04401	0.183427	0.0200663	9.14106
0.450	0.202500	0.084658	0.98687	1.97373	1.01540	1.04500	0.186862	0.0204750	9.12637
0.455	0.207025	0.089352	0.98602	1.97204	1.01645	1.04601	0.190305	0.0208863	9.11150
0.460	0.211600	0.094142	0.98514	1.97029	1.01754	1.04702	0.193756	0.0213000	9.09650
0.465	0.216225	0.099023	0.98424	1.96849	1.01866	1.04805	0.197213	0.0217162	9.08138
0.470	0.220900	0.103989	0.98332	1.96663	1.01983	1.04909	0.200678	0.0221347	9.06619
0.475	0.225625	0.109037	0.98237	1.96473	1.02103	1.05014	0.204149	0.0225555	9.05097
0.480	0.230400	0.114162	0.98139	1.96278	1.02228	1.05120	0.207628	0.0229786	9.03573
0.485	0.235225	0.119358	0.98039	1.96078	1.02356	1.05227	0.211114	0.0234037	9.02052
0.490	0.240100	0.124623	0.97937	1.95873	1.02487	1.05336	0.214607	0.0238310	9.00537
0.495	0.245025	0.129950	0.97832	1.95664	1.02623	1.05445	0.218106	0.0242602	8.99029
0.500	0.250000	0.135335	0.97725	1.95450	1.02762	1.05556	0.221613	0.0246914	8.97533

Table 11: CBN Survey (Naira million), Year: 2014 to 2018, Descriptor: Net Foreign Assets

Year(Period)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2014	5,285,636.55	4,890,638.14	4,745,014.61	4,985,765.90	5,092,240.72	5,993,875.84	6,320,812.94	6,095,161.10	6,101,323.34	5,645,733.15	5,764,692.09	5,889,351.63
2015	5,340,540.46	5,653,706.71	5,752,368.26	5,705,837.31	5,481,799.43	5,383,005.19	5,134,759.13	5,221,984.00	4,830,747.17	4,304,354.99	4,838,743.05	5,328,202.49
2016	5,016,070.91	5,099,675.75	4,813,508.19	4,326,645.22	4,572,100.70	6,098,753.16	6,978,039.06	6,488,563.02	7,152,376.26	6,720,228.34	7,138,216.39	7,695,924.17
2017	8,176,637.74	7,066,062.68	6,971,662.02	6,431,702.52	7,301,006.67	7,307,268.26	7,644,241.58	8,363,938.92	8,430,443.77	10,323,497.67	11,291,865.19	10,825,680.36
2018	10,620,279.72	10,630,792.14	10,203,308.02	9,005,291.44	11,755,304.13	11,572,165.28	11,086,656.16	13,275,772.78	12,635,873.02	12,765,452.46	13,369,301.01	12,126,988.18

(Source: Central Bank of Nigeria Statistical Database)

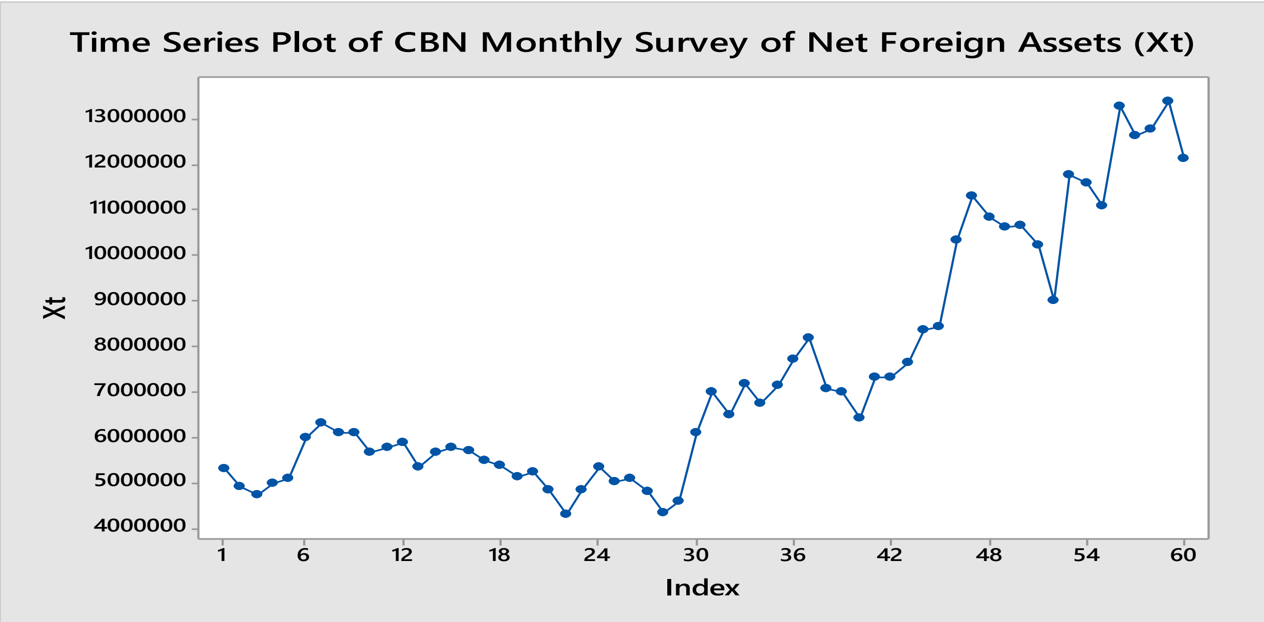

Fig. 5: Time plot of CBN Survey-Monthly (Naira million) of Net Foreign Assets from January 2014 to December 2018

3.2 Validation of the Choice of Multiplicative Time Series Model

Only logarithm transformation is additive on the list of power transformation, as such to carry out inverse cube root transformation, the appropriate model as stated in 1.2 (Choice of model) is the multiplicative model. On that note, to further ascertain the choice of Multiplicative time series model type, a graph of the periodic means against period (year) and a graph of periodic standard deviations still against period were plotted (see Fig. 4.2 and 4.3), it was observed that the two graphs plotted followed the same pattern of increasing and decreasing in most periods, as such, it is in conformity with Iwueze and Nwogu (2004). This increase / decrease of the periodic mean alongside that of the periodic standard deviation justify and validate the choice of multiplicative time series model type over additive time series type (See Table 4.3 for their respective values).

Table 12: Buys-Ballot Table of CBN Survey on Net Foreign Assets (Naira million)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total	Mean	StDev
2014	5285637	4890638	4745015	4985766	5092241	5993876	6320813	6095161	6101323	5645733	5764692	5889352	66810246	5567521	542689
2015	5340540	5653707	5752368	5705837	5481799	5383005	5134759	5221984	4830747	4304355	4838743	5328202	62976048	5248004	422336
2016	5016071	5099676	4813508	4326645	4572101	6098753	6978039	6488563	7152376	6720228	7138216	7695924	72100101	6008342	1177364
2017	8176638	7066063	6971662	6431703	7301007	7307268	7644242	8363939	8430444	10323498	11291865	10825680	100134007	8344501	1613256
2018	10620280	10630792	10203308	9005291	11755304	11572165	11086656	13275773	12635873	12765452	13369301	12126988	139047184	11587265	1332005
Total	34439165	33340875	32485861	30455242	34202452	36355068	37164509	39445420	39150764	39759267	42402818	41866147	441067588		
Mean	6887833	6668175	6497172	6091048	6840490	7271014	7432902	7889084	7830153	7951853	8480564	8373229		7351126	
StDev	2452387	2372215	2259430	1809144	2933235	2503551	2242749	3222380	2995829	3497670	3682522	3000242			562746

Table 13: Period (Year), Periodic Means and Periodic Standard Deviations to the Nearest Whole Number

Year	*Periodic Means* ($\bar{X}_{i.}$)	*Periodic Standard Deviations* ($\hat{\sigma}_{i.}$)
2014	5567521	542689
2015	5248004	422336
2016	6008342	1177364
2017	8344501	1613256
2018	11587265	1332005

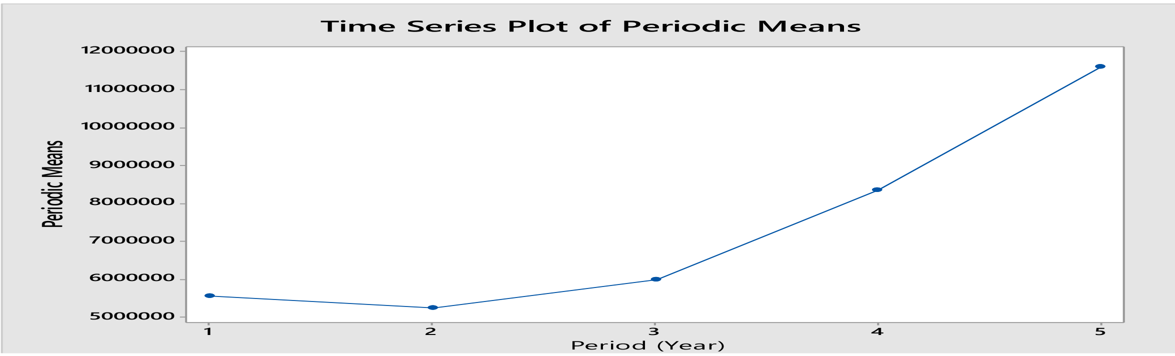

Fig. 6: Periodic Means Time Series Graph

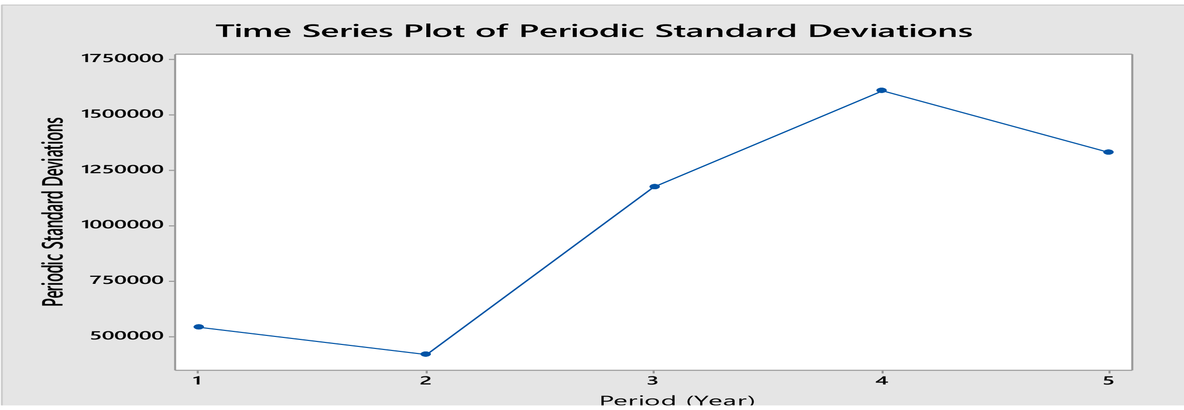

Fig. 7: Periodic Standard Deviation time series graph

3.3 Determination of Appropriate Transformation

For the appropriate transformation to be determined or identified for the real life data generated from the CBN Statistics database, the natural logarithm of the periodic means (LOGMEAN) and the natural logarithm of the periodic standard deviations (LOGSTD) were obtained (See Table 14). From which the fitted linear regression line plot of LOGSTD against that of the LOGMEAN was obtained using Minitab 17 and Introstat software application (see Fig. 8).

Table 14: Natural Log of the Periodic Means and Standard Deviations

Year (Period)	ln mean (LOGMEAN) ($\ln\bar{X}_{i.}$)	ln StD (LOGSTD)($\ln\hat{\sigma}_{i.}$)
2014	15.53	13.20
2015	15.47	12.95
2016	15.61	13.98
2017	15.94	14.29
2018	16.27	14.10

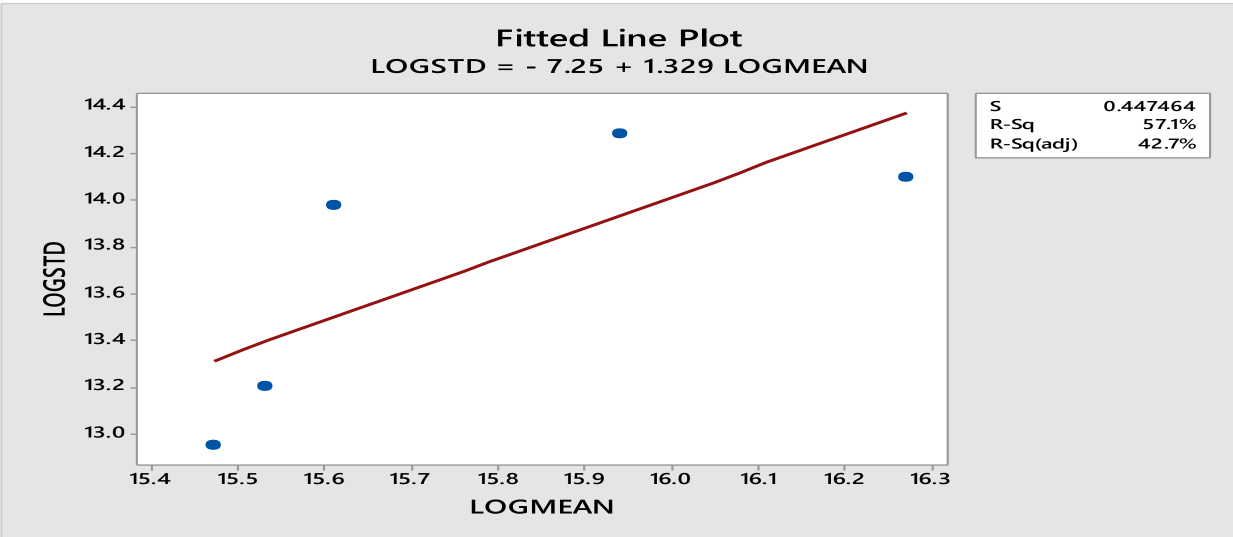

Fig. 8: Fitted Regression Linear Line Plot of LOGSTD against LOGMEAN

3.4 Inverse Cube Root Transformation of the Real Life Data

Here, the original data collected was transformed by obtaining the inverse cube root of each of the observations ($Y_t = \frac{1}{\sqrt[3]{X_t}}$).

Table 15: Inverse Cube Root Transformed Data of the CBN Monthly Survey of Net Foreign Assets

Year(Period)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2014	0.0057407	0.0058913	0.0059510	0.0058536	0.0058125	0.0055051	0.0054085	0.0054744	0.0054726	0.0056160	0.0055771	0.0055375
2015	0.0057210	0.0056134	0.0055811	0.0055962	0.0056714	0.0057059	0.0057964	0.0057640	0.0059156	0.0061475	0.0059123	0.0057254
2016	0.0058418	0.0058097	0.0059226	0.0061369	0.0060251	0.0054733	0.0052331	0.0053615	0.0051902	0.0052991	0.0051936	0.0050650
2017	0.0049637	0.0052112	0.0052347	0.0053772	0.0051547	0.0051533	0.0050764	0.0049264	0.0049134	0.0045926	0.0044574	0.0045204
2018	0.0045494	0.0045479	0.0046106	0.0048066	0.0043980	0.0044211	0.0044847	0.0042232	0.0042934	0.0042788	0.0042134	0.0043526

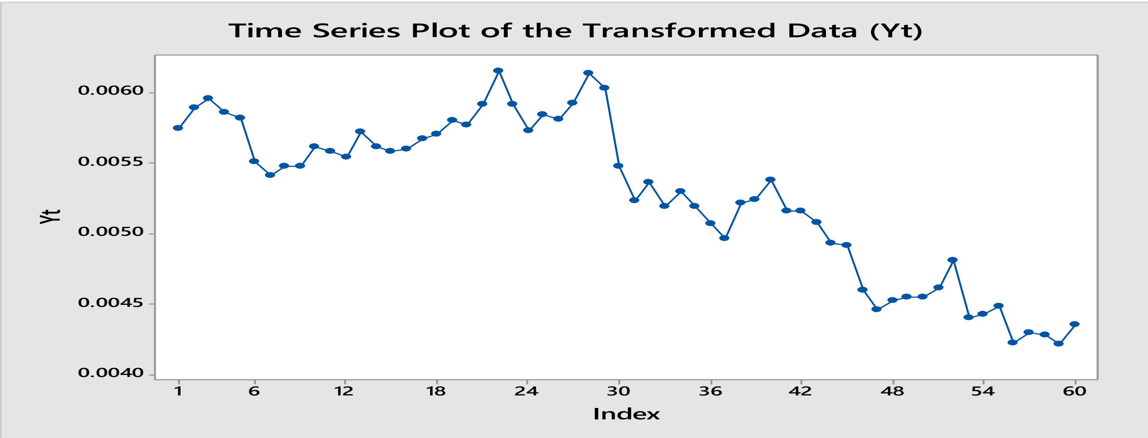

Fig. 9: Time Plot of the Inverse Cube Root Transformed Data (Y_t)

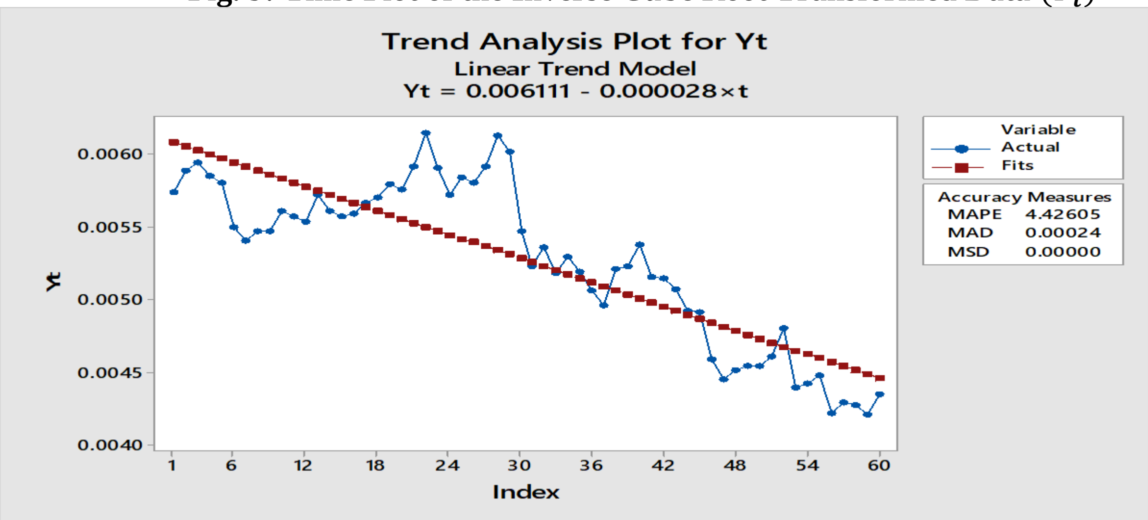

Fig. 10: Linear Trend fit for the Data on Monthly Central Bank of Nigeria Survey on Net Foreign Assets (Naira million)

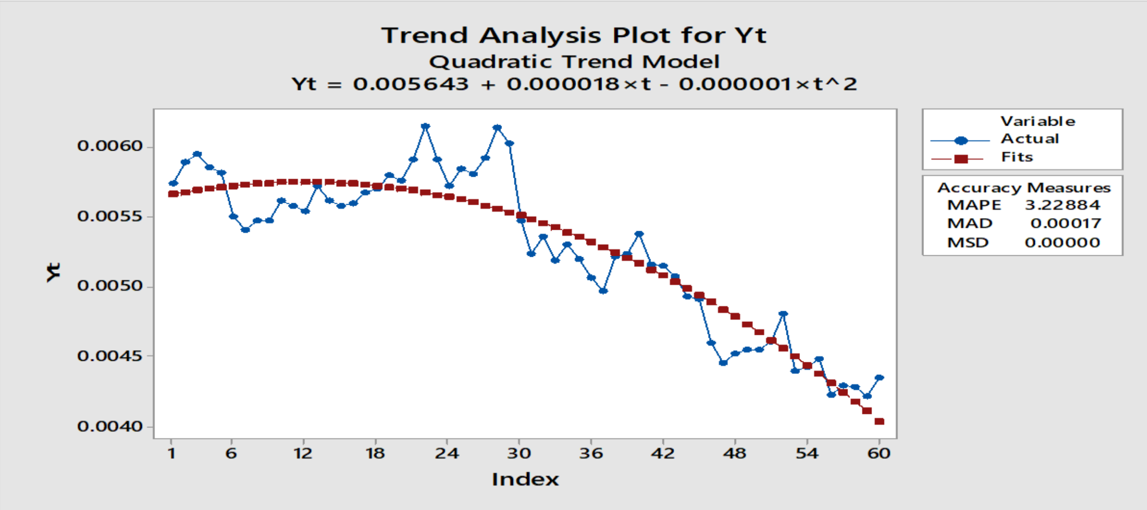

Fig. 11: Quadratic Trend Curve Plot for the Inverse Cube Root Transformed Data

In order, to isolate the error component from the other components, the transformed time series data (Y_t) is divided by the corresponding trend-cycle values and seasonal indices.

Table 15: Decomposition of the Inverse Cube Root Transformed Data (Y_t) on the CBN Monthly Survey Net Foreign Assets

t	X_t	Y_t	M_t^*	S_t^*	e_t^*
1	5285637	0.0057407	0.0056600	1.00000	1.01427
2	4890638	0.0058913	0.0056755	1.00001	1.03801
3	4745015	0.0059510	0.0056896	1.00002	1.04592
4	4985766	0.0058536	0.0057022	1.00003	1.02653
5	5092241	0.0058125	0.0057132	1.00004	1.01733
6	5993876	0.0055051	0.0057229	1.00005	0.96190
7	6320813	0.0054085	0.0057310	0.99996	0.94377
8	6095161	0.0054744	0.0057376	0.99996	0.95416
9	6101323	0.0054726	0.0057428	0.99997	0.95298
10	5645733	0.0056160	0.0057464	0.99998	0.97732
11	5764692	0.0055771	0.0057486	0.99998	0.97018
12	5889352	0.0055375	0.0057493	0.99999	0.96316
13	5340540	0.0057210	0.0057485	1.00000	0.99521
14	5653707	0.0056134	0.0057463	1.00001	0.97686
15	5752368	0.0055811	0.0057425	1.00002	0.97187
16	5705837	0.0055962	0.0057373	1.00003	0.97538
17	5481799	0.0056714	0.0057306	1.00004	0.98964
18	5383005	0.0057059	0.0057223	1.00005	0.99708
19	5134759	0.0057964	0.0057126	0.99996	1.01471
20	5221984	0.0057640	0.0057015	0.99996	1.01100
21	4830747	0.0059156	0.0056888	0.99997	1.03989
22	4304355	0.0061475	0.0056746	0.99998	1.08335

23	4838743	0.0059123	0.0056590	0.99998	1.04477
24	5328202	0.0057254	0.0056419	0.99999	1.01481
25	5016071	0.0058418	0.0056233	1.00000	1.03885
26	5099676	0.0058097	0.0056032	1.00001	1.03684
27	4813508	0.0059226	0.0055816	1.00002	1.06107
28	4326645	0.0061369	0.0055586	1.00003	1.10401
29	4572101	0.0060251	0.0055340	1.00004	1.08869
30	6098753	0.0054733	0.0055080	1.00005	0.99366
31	6978039	0.0052331	0.0054805	0.99996	0.95490
32	6488563	0.0053615	0.0054515	0.99996	0.98353
33	7152376	0.0051902	0.0054210	0.99997	0.95745
34	6720228	0.0052991	0.0053890	0.99998	0.98335
35	7138216	0.0051936	0.0053556	0.99998	0.96978
36	7695924	0.0050650	0.0053206	0.99999	0.95197
37	8176638	0.0049637	0.0052842	1.00000	0.93935
38	7066063	0.0052112	0.0052463	1.00001	0.99331
39	6971662	0.0052347	0.0052069	1.00002	1.00531
40	6431703	0.0053772	0.0051660	1.00003	1.04086
41	7301007	0.0051547	0.0051236	1.00004	1.00603
42	7307268	0.0051533	0.0050798	1.00005	1.01441
43	7644242	0.0050764	0.0050344	0.99996	1.00837
44	8363939	0.0049264	0.0049876	0.99996	0.98776
45	8430444	0.0049134	0.0049393	0.99997	0.99478
46	10323498	0.0045926	0.0048895	0.99998	0.93929
47	11291865	0.0044574	0.0048383	0.99998	0.92129
48	10825680	0.0045204	0.0047855	0.99999	0.94462
49	10620280	0.0045494	0.0047312	1.00000	0.96157
50	10630792	0.0045479	0.0046755	1.00001	0.97270
51	10203308	0.0046106	0.0046183	1.00002	0.99830
52	9005291	0.0048066	0.0045596	1.00003	1.05413
53	11755304	0.0043980	0.0044994	1.00004	0.97742
54	11572165	0.0044211	0.0044377	1.00005	0.99619
55	11086656	0.0044847	0.0043746	0.99996	1.02521
56	13275773	0.0042232	0.0043099	0.99996	0.97992
57	12635873	0.0042934	0.0042438	0.99997	1.01171
58	12765452	0.0042788	0.0041762	0.99998	1.02459
59	13369301	0.0042134	0.0041071	0.99998	1.02589
60	12126988	0.0043526	0.0040365	0.99999	1.07831

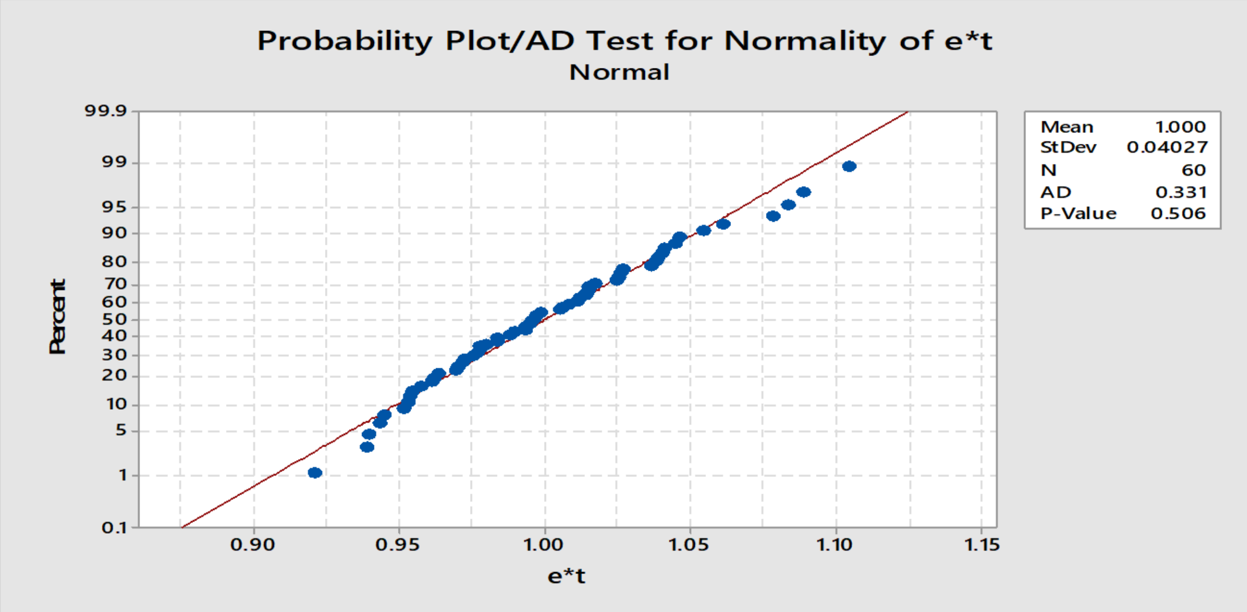

Fig. 11: Normal Probability Plot/ AD Test for Normality of the Transformed Data (e_t^*)

4.0 Validation of Results

To validate the results of this transformation, the functional expressions obtained above for the mean and the variance of the inverse cube root transformed error term in Equations (38) and (39) respectively are used to obtain the descriptive statistics of the inverse cube root transformed error component.

i. Computations of the Characteristics of the Inverse Cube Root Transformed Error Component from the Appropriate

The results of the decomposition of the time series data on the Central Bank of Nigeria monthly Survey Net Foreign Assets (Naira Million) into the trend- cycle component (M_t), seasonal component (S_t) and irregular/ residual components (e_t) for the **untransformed** (real life) data with the mean, median, standard deviation and variance of the irregular component/residual series are: Mean = $[E(e_t)]$ = 1.0003 ≈ 1 correct to 3 decimal places, Median = 1.0102 ≈ 1 correct to 1 decimal place, Standard Deviation (σ) = 0.1121 and Variance (σ^2) = 0.0126. The irregular/error/residual component was assessed for normality using the Anderson Darling (AD) test of normality.

Similarly, the results of the decomposition of the time series data on the Central Bank of Nigeria monthly Survey Net Foreign Assets into the trend-cycle(M_t^*), seasonal $(S_t^*)$and irregular (e_t^*) components are shown in Table 11 for the transformed data (Y_t) . The mean, median, standard deviation and variance of the Irregular component/Residual series are: Mean = $[E(e_t)]$ = 1.0001 ≈ 1 correct to 3 decimal places, Median = 0.9957 ≈ 1 correct to 2 decimal places, Standard Deviation (σ) = 0.0403 and Variance (σ^2) = 0.00162. The irregular/error component was assessed for normality using the Anderson Darling (AD) test of normality.

ii. Computations of the Characteristics of the Inverse Cube Root Transformed Error Component Using the Derived Formulae

1.) From the derivation in Equation (3.8),

$$y_{max} = \sqrt[3]{\frac{-3 + \sqrt{9 + 48\sigma^2}}{8\sigma^2}}$$

$with\ y_{max} = Mode$

Where $\sigma = 0.0112\ and\ \sigma^2 = 0.012544$ (From the untransformed standard deviation)

By substitution, $y_{max} = \sqrt[3]{\frac{-3+\sqrt{9+48(0.012544)}}{8(0.012544)}} = \sqrt[3]{\frac{-3+\sqrt{9.602112}}{0.100353}}$

$= \sqrt[3]{\frac{-3+3.09873}{0.100353}} = \sqrt[3]{\frac{0.09873}{0.100353}} = \sqrt[3]{0.9838} =$ $(0.9838)^{\frac{1}{3}} = 0.99457$

Therefore, $Mode = 0.99457 \approx 1\ correct\ to\ 1\ decimal\ place$

2.) From the derivation in Equation (3.36), $E\ (e_t^*) = 1 + \frac{2\sigma^2}{9} = 1 + \frac{2(0.012544)}{9} = 1 + \frac{0.025088}{9} = 1 + 0.0027876 = 1.0027876$

$\therefore E\ (e_t^*) \approx 1$ correct to 2 decimal places

3.) From the derivation in Equation (3.37), $Var\ (e_t^*) = \frac{\sigma^2}{9} - \left(\frac{2\sigma^2}{9}\right)^2 = \frac{(0.012544)}{9} - \left(\frac{2(0.012544)}{9}\right)^2$

$= 0.0013938 - (0.0027876)^2 = 0.0013938 - 0.00000777$

$\therefore Var\ (e_t^*) = 0.00138603 \approx 0.00136\ to\ three\ significant\ figure$

The variance ratio of the untransformed error component $[Var\ (e_t)]$ to the transformed error component $[Var\ (e_t^*)] = \frac{Var\ (e_t)}{Var\ (e_t^*)} = \frac{0.0125}{0.00136} = 9.19118$

Therefore, $\frac{Var\ (e_t)}{Var\ (e_t^*)} \approx 9$ correct to the nearest whole number (integer). Which is the same as $Var\ (e_t) \approx 9Var\ (e_t^*)$.

4.0 Conclusion

This study derived and study the properties of the distribution of the inverse cube root transformed left-truncated normally distributed errors based on the multiplicative time series decomposition model. The variance of the untransformed error term is empirically determined to be nine (9) times that of the transformed error term. This implies improved efficiency due to transformation. For a successful inverse cube root transformation, this study recommends the value of the standard deviation σ satisfying $\sigma < 0.38$

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