Communication in Physical Sciences, 2020, 7(3): 147-163

A Type I Half Logistic Exponentiated-G Family of Distributions: Properties and Application

Olalekan Akanji Bello*, Sani Ibrahim Doguwa, Abubakar Yahaya and Haruna Mohammed Jibril
Received: 12 June 2021/Accepted 09 September 2021/Published online: 14 September 2021

Abstract: *Several new improved, generalized, and extended families of distributions have been discovered in recent years from families of distributions to aid their application in a variety of fields. The Type I half-logistic exponentiated-G family of distributions which generalizes and extends the Type I half-logistic family of distributions, with two extra positive shape parameters is investigated and proposed. We discuss some of the statistical properties of the proposed family such as explicit expressions for the quantile function, ordinary and incomplete moments, generating function, reliability and order statistics. Some of the new family's sub-models are discussed. We discuss the estimation of the model parameters by method of maximum likelihood. Two real data sets are employed to show the applicability and flexibility of the new family*.

Keywords: *Hazard rate, Reliability, Exponentiated-G, Type I Half Logistic G, Maximum likelihood, Order Statistics.*

Olalekan Akanji Bello*
Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria
Email: olalekan4sure@gmail.com
Orcid id: 0000-0002-2209-9035

Sani Ibrahim Doguwa
Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria.
Email: sidoguwa@gmail.com
Orcid id: 0000-0002-5779-2358

Abubakar Yahaya
Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria.
Email: ensiliyu2@yahoo.co.uk
Orcid id: 0000-0002-1453-7955

Haruna Mohammed Jibril
Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria
Email: alharun2004@yahoo.com
Orcid id: 0000-0001-7869-1807

1.0 Introduction

Statistical distributions are frequently used to describe real-life events and the theory of statistical distributions can be ~~is~~ intensively studied to obtain novel distributions for optimum utility. In statistics, there is a significant desire to construct more flexible statistical distributions. Many different types of generalized distributions have been devised and applied to various phenomena. Several continuous univariate distributions have been extensively used for modeling data in many areas such as economics, engineering, biological studies and environmental sciences (Johnson *et al*., 1994). However, applications in areas such as finance, lifetime analysis and insurance clearly require extended forms of these distributions. Consequently, several classes of distributions have been constructed by extending common families of continuous distributions.

Generated family of continuous distributions is a new improvement for creating and extending the usual classical distributions. The newly generated families have been broadly studied in several areas and have been observed to generate more flexibility in applications. For example, Gupta *et.al.,* (1998) pioneered the exponentiated-G (E-G) class, Eugene *et.al*., (2002) pioneered beta-G, Marshall-Olkin-G by Marshall and Olkin (1997), gamma G distributions by Zografos and Balakrishnan (2009), Kumaraswamy Weibull G family by Cordeiro *et.al*, (2010). Ristic and Balakrishnan (2011) proposed an Alternative Gamma G distribution, Kumaraswamy-G family by Cordeiro and Castro (2011), Kummer beta generalized family by Cordeiro *et.al*., (2012),a new methods for generating families of continuous distributions by Alzaatreh *et.al*., (2013).

Exponentiated T-X family was proposed by Alzaghal *et.al*., (2013), Weibull-G family by Silva *et.al*., (2014), Exponentiated half-logistic family by Cordeiro *et.al*., (2014a), Lomax Generator by Cordeiro *et.al*., (2014b), Logistic-generated distribution by Torabi and Montazari (2014), Type I half logistic family of distribution by Cordeiro *et.al*., (2015), beta Marshall-Olkin family of distributions by Alizadeh *et.al*., (2015a), Kumaraswamy Odd Log-Logistic-G by Alizadeh *et.al*., (2015b), the logistic-X family by Tahir *et.al*., (2016), the Topp-Leone family of distributions by Al-Shomrani *et.al*., (2016), Exponentiated Marshal-Olkin family of distributions by Dias *et.al*., (2016), Generalized Burr-G family of distributions by Nasir *et.al*.,(2017), Odd Exponentiated Half-Logistic-G family by Afify *et.al*., (2017), Weibull-X family of distributions by Ahmad *et.al*., (2018), Marshall-Olkin generalized-G (MOG-G) family of distribution by Yousof *et.al*., (2018), Exponentiated Kumaraswamy-G by Silva *et.al*., (2019).

Recently, Topp-Leone exponentiated-G (TLEx-G) family of distributions was proposed by Ibrahim *et.al*., (2020a), Topp Leone Kumaraswamy-G family of distribution by Ibrahim *et.al*., (2020b), Burr X Exponential-G Family by Sanusi *et.al*., (2020a), Topp-Leone Exponential-G family of distributions by Sanusi *et.al*., (2020b), Kumaraswamy-Odd Rayleigh-G family by Falgore and Doguwa (2020a), Inverse Lomax-G (IL-G) family of distributions by Falgore and Doguwa (2020b), Inverse Lomax Exponentiated G family of distributions by Falgore and Doguwa (2020c). A new generalized Weibull-Odd Frechet family of distributions has also been proposed by Usman *et.al*., (2020). Others include Type II Exponentiated Half-logistic family of distributions by Al-Moeh *et.al*., (2020), Odd Weibull-Topp-Leone-G power series family of distributions by Broderick *et.al*., (2021), and Exponentiated Half Logistic Odd Lindley-G (EHLOL-G) distribution by Whatmore *et.al*.,(2021).

The ability to model both monotonically and non-monotonically increasing, decreasing and constant or more importantly with bathtub-shaped failure rates, even if the baseline failure rate is monotonic, is the motivation for extending distributions for modeling lifetime data. The following are the basic vindications for creating a new family of distributions in practice: to generate skewness for symmetrical models; to generate distributions with negatively skewed, positively skewed, and symmetric; to define special models with all types of hazard rate functions; to make the kurtosis more flexible than that of the baseline distribution, to construct tail weight distributions for modeling various real data sets; to provide consistently better fits than other generated distributions with the same underlying model.

The objective of this paper is to develop and investigate the Type I Half-Logistic Exponentiated-G

family of distributions (TIHLEt-G). The layout of the paper is in 8 sections: Section 1 covers the introduction; Section 2 defines the T1HLEt-G family of probability distributions.

We obtained a very useful and important representation for the TIHLEt-G cumulative distribution function in Section 3. Section 4 derives some statistical properties of the new family such as the survivor function, the hazard function, the quantile function and the order statistics. Two sub-models of the new family are discussed in Section 5. The parameters of the new family were estimated using the maximum likelihood estimation approach in Section 6. The application of the sub-model to two real data sets was shown in Section 7 to demonstrate the use of the new family. Finally, Section 8 concludes the paper.

2.0 Type I Half-Logistic Exponentiated-G Family

The Exponentiated G (Et-G) family of distributions as defined in Ibrahim *et.al.*, (2020) has cumulative distribution function (cdf) given as

$$F_{EG}(x;\alpha,\boldsymbol{\beta}) = [H(x;\boldsymbol{\beta})]^{\alpha} \qquad (1)$$

and its corresponding probability density function (pdf) is defined as

$$f_{EG}(x;\ \alpha,\boldsymbol{\beta}) = \alpha h(x,\boldsymbol{\beta})\left[H(x;\boldsymbol{\beta})\right]^{\alpha-1},x>0 \qquad (2)$$

where $\alpha > 0$ is the shape parameter and $h(x;\ \boldsymbol{\beta})$ and $H(x;\ \boldsymbol{\beta})$ are the probability density function (pdf) and cumulative distribution function (cdf) of the baseline distribution with parameter vector $\boldsymbol{\beta}$.

Cordeiro, *et.al.* (2015) proposed the Type I half logistic family (TIHL) of distributions with cdf defined as:

$$F_{TIHL}(x;\lambda,\xi) = \frac{1-[1-G(x;\xi)]^{\lambda}}{1+[1-G(x;\xi)]^{\lambda}} \qquad (3)$$

and its corresponding pdf is

$$f_{TI\,HL}(x;\lambda,\xi) = \frac{2\lambda g(x;\xi)\left[1-G(x;\xi)\right]^{\lambda-1}}{\left[1+\left[1-G(x;\xi)\right]^{\lambda}\right]^{2}} \qquad (4)$$

where $\lambda > 0$, $x > 0$ and $G(x;\ \xi)$ and $g(x;\ \xi)$ are the cdf and pdf of the baseline distribution with parameter vector ξ.

Proposition

The cdf of a new family of distribution that extends the TIHL family called Type I Half-Logistic Exponentiated-G Family of distributions is given as

$$F_{TIHLEt\_G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{1-\left[1-H^{\alpha}(x;\boldsymbol{\beta})\right]^{\lambda}}{1+\left[1-H^{\alpha}(x;\boldsymbol{\beta})\right]^{\lambda}} \qquad (5)$$

and the pdf is derived as

$$f_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{\partial F_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta})}{\partial x}$$

$$= \frac{2\lambda\alpha h(x;\boldsymbol{\beta})H^{\alpha}(x;\boldsymbol{\beta})\left[1-H^{\alpha}(x;\boldsymbol{\beta})\right]^{\lambda-1}}{\left[1+\left[1-H^{\alpha}(x;\boldsymbol{\beta})\right]^{\lambda}\right]^{2}},\ x>0,\ \lambda,\alpha>0\ \ \text{and } \boldsymbol{\beta} \text{ is parameter vector} \qquad (6)$$

Proof:

Let the Exponentiated G family be the baseline family with cdf and pdf given in equations (1) and (2) respectively, then the proposed Type I Half-Logistic Exponentiated-G family of probability distributions has cdf given as:

$$F_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) \int_{0}^{[H(x;\boldsymbol{\beta})]^{\alpha}} \frac{2\lambda\alpha h(t;\boldsymbol{\beta})H^{\alpha}(t;\boldsymbol{\beta})\left[1-H^{\alpha}(t;\boldsymbol{\beta})\right]^{\lambda-1}}{\left[1+\left[1-H^{\alpha}(t;\boldsymbol{\beta})\right]^{\lambda}\right]^{2}}dt$$

Let $y = \left[H(t;\boldsymbol{\beta})\right]^{\alpha}$; when $t = 0$, $y = 0$; when $t = x$, $y = \left[H(x;\boldsymbol{\beta})\right]^{\alpha}$; $\partial y = \alpha h(t;\boldsymbol{\beta})\left[H(t;\boldsymbol{\beta})\right]^{\alpha-1}$

Then

$$F_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \int_{0}^{[H(x;\boldsymbol{\beta})]^{\alpha}} \frac{2\lambda\left[1-y\right]^{\lambda-1}}{\left[1+\left[1-y\right]^{\lambda}\right]^{2}}\,dy$$

Let

$$m = \left[1+\left[1-y\right]^{\lambda}\right]^{2},\ \ k = 1+\left[1-y\right]^{\lambda}\ \ when\ y=0\ \ m=4;\ when\ y=\left[H(x;\boldsymbol{\beta})\right]^{\alpha},\ m=\left[1+\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}\right]^{2}$$

$$\frac{\partial m}{\partial k} = 2\left[1+\left[1-y\right]^{\lambda}\right]\frac{\partial k}{\partial y}\ \ but\ \frac{\partial k}{\partial y} = -\lambda\left[1-y\right]^{\lambda-1}$$

$$\therefore\ \ \partial m = -2\lambda\left[1+\left[1-y\right]^{\lambda}\right]\left[1-y\right]^{\lambda-1}\partial y$$

$$so\ \ \partial y = -\frac{1}{2\lambda\left[1+\left[1-y\right]^{\lambda}\right]\left[1-y\right]^{\lambda-1}}\,\partial y$$

and

$$F_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = -\int_{4}^{\left[1+\left[1-\left[H(x;\boldsymbol{\beta})\right]\alpha\right]\lambda\right]^{2}} \frac{1}{m^{\frac{3}{2}}}\,\partial m\ = \left[-\frac{m^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_{4}^{\left[1+\left[1-\left[H(x;\boldsymbol{\beta})\right]\alpha\right]\lambda\right]^{2}}$$

$$= \left[2m^{-\frac{1}{2}}\right]_{4}^{\left[1+\left[1-\left[H(x;\boldsymbol{\beta})\right]\alpha\right]\lambda\right]^{2}} = 2\left[\left[1+\left[1-\left[H(x;\boldsymbol{\beta})\right]\alpha\right]\lambda\right]^{2}\right]^{-\frac{1}{2}} - 2\left[4\right]^{-\frac{1}{2}} = \frac{2}{1+\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}} - \frac{2}{2}$$

$$= \frac{2-\left[1+\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}\right]}{1+\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}}$$

$$F_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{1-\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}}$$

3. 0 Important Representation

In this section, we derived a useful representation for the TIHLEt−G cdf. The cdf defined in equation (5) can be expressed as

$$F(x) = \underbrace{\left[1-\left[1-H^{\alpha}(x)\right]^{\lambda}\right]}_{A}\underbrace{\left[1+\left[1-H^{\alpha}(x)\right]^{\lambda}\right]^{-1}}_{B}$$

Using the generalized binomial series, we have

$$A=\sum_{i-0}^{\infty}(-1)^i\binom{-1}{i}\left[1-\left[H^{\alpha}(x)\right]\right]^{\lambda i} \quad and \quad B=\sum_{j=0}^{\infty}(-1)^j\binom{-1}{j}\left[1-\left[H^{\alpha}(x)\right]\right]^{\lambda j}$$

Combining A and B, we obtain

$$F(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^i(-1)^j\left[1-\left[H^{\alpha}(x)\right]\right]^{\lambda(i+j)}$$

Consider

$$\left[1-\left[H^{\alpha}(x)\right]\right]^{\lambda(i+j)}=\sum_{k=0}^{\infty}(-1)^k\binom{\lambda(i+j)}{k}\left[H(x)\right]^{\alpha k}$$

$$F(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i(-1)^j(-1)^k\binom{-1}{i}\binom{-1}{j}\binom{\lambda(i+j)}{k}\left[H(x)\right]^{\alpha k}$$

According to Jamal, *et.al.,* (2018)

$$\left[H(x)\right]^{\alpha}=\sum_{q=0}^{\infty}\sum_{q=k}^{\infty}\binom{\alpha k}{m}\binom{m}{q}(-1)^{q+m}\left[H(x)\right]^{q}$$

Therefore

$$F(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i(-1)^j(-1)^k\binom{-1}{i}\binom{-1}{j}\binom{\lambda(i+j)}{k}\sum_{q=0}^{\infty}\sum_{q=k}^{\infty}\binom{\alpha k}{m}\binom{m}{q}(-1)^{q+m}\left[H(x)\right]^{q}$$

$$F(x)=\sum_{q=0}^{\infty}t_q H_q(x) \qquad (7)$$

where $t_q=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i(-1)^j(-1)^k\binom{-1}{i}\binom{-1}{j}\binom{\lambda(i+j)}{k}\sum_{q=k}^{\infty}\binom{\alpha k}{m}\binom{m}{q}(-1)^{q+m}$

Also, we have an expansion for the pdf as

$$f(x)=\sum_{q=0}^{\infty}t_q h_{q-1}(x) \qquad (8)$$

where $h_{q-1}(x)=qh\left[H(x)\right]^{q-1}$

4.0 Statistical Properties

In this section, we derived statistical properties of the new family of distribution.

4.1 Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the rth moment for the new family.

$$\mu_r^{'}=E(x^r)=\int_0^{\infty}x^r f(x)dx \qquad (9)$$

By using the important representation of the pdf in equation (8), we have

$$M_x(t)=\sum_{q=0}^{\infty}t_q M_{q-1}(t) \qquad (12)$$

$$\mu_r^{'}=\sum_{q=0}^{\infty}t_q\delta_r^{'} \qquad (10)$$

Where $\delta_r^{'}=\int_0^{\infty}x^r h_{q-1}(x)dx$

4.2 Moment generating function (mgf)

The Moment Generating Function of x is given as:

$$M_x(t)=\int_0^{\infty}e^{tx}f(x)dx \qquad (11)$$

Also, by using the important representation of the pdf in equation (8), we have

where $M_{q-1}(t)=\int_{0}^{\infty} e^{tx} h_{q-1}(x)dx$

4.3 *Incomplete moment*

The sth incomplete moments, say $\varphi(t)$ is given by

$$\varphi_s(t)=\mu_s^{'}=\int_{0}^{x} x^s f(x)dx \qquad (13)$$

From the important representation of the pdf in equation (8), we have

$$\mu_s^{'}=\sum_{q=0}^{\infty} t_q \phi_s^{'} \qquad (14)$$

where $\phi_s^{'}=\int_{0}^{x} x^s h_{q-1}(x)dx$

The first incomplete moment of the TIHLEt-G family $\varphi_s(t)$ can be obtained by setting s = 1.

4.4 Mean deviation

The mean deviation about the mean $\left[\sigma_2=E(|x-\mu_1^{'}|)\right]$ and about the median $\left[\sigma_1=E(|x-M|)\right]$ of x given as

$$\sigma_1=2\mu_1^{'}F(\mu^{'})-2J(\mu_1^{'}) \qquad (15)$$

$$\sigma_2=\mu_1^{'}-2J(M) \qquad (16)$$

Where $\mu_1^{'}=E(x)$, is the mean, M = Median(x) = Q(0.5), is the median and $j(c)=\int_{0}^{x} xf(x)dx$

From the important representation of the pdf in equation (3.2), we have

$$j(c)=\sum_{q=0}^{\infty} t_q \phi_{q-1}^{'}(c) \qquad (17)$$

Where $\phi_{q-1}^{'}(c)=\int_{0}^{c} xh_{q-1}(x)dx$ the integral depends on H(x) and h(x)

4.5 Reliability function

The reliability function which is also known as survivor function, that gives the probability that a patient will survive longer than specified period of time. It is defined as

$$R(x;\lambda,\alpha,\boldsymbol{\beta})=1-F(x;\lambda,\alpha,\boldsymbol{\beta})$$

$$R(x;\lambda,\alpha,\boldsymbol{\beta})=\frac{2\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}{1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}} \qquad (18)$$

Obtaining survival probabilities for different values of time x provides crucial summary information from survival data.

4.6 Hazard function

The hazard function is the probability of an event of interest occurring within a relatively short time frame and is defined as:

$$T(x;\lambda,\alpha,\boldsymbol{\beta})=\frac{\lambda\alpha h(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})}{\left[1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]\right]\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]} \qquad (19)$$

The hazard function also known as conditional failure rate, gives the instantaneous potential per unit time for the event of interest to occur, given that the individual has survived up to time x.

4.7 Quantile Function

The quantile function is a vital tool to create random variables from any continuous probability distribution. As a result, it has a significant position in probability theory. For x, the quantile function is F(x) = u, where u is distributed as U(0,1).The TIHLEt-G family is easily simulated by inverting equation (2.5) which yields the Quantile function Q(u) defined as:

$$Q(u) = H^{-1}\left\{1 - \left(\frac{1-u}{1+u}\right)^{1/\lambda}\right\}^{1/\alpha} \tag{20}$$

where H^{-1} is the quantile function of the baseline cdf H(x; β). The first quartile, the median and the third quartile are obtained by putting u = 0.25, 0.5 and 0.75, respectively in equation (20).

4.8 Order Statistics

Many areas of statistics including reliability and life testing, have made substantial use of order statistics. Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with their corresponding continuous distribution function F (x). Let $X_1, X_2,.., X_n$ be n independently distributed and continuous random variables from the TIHLE-G family of distribution. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, r = 1, 2, 3, ..., n denote the cdf and pdf of the rth order statistics $X_{r:n}$ respectively. David (1970) gave the probability density function of $X_{r:n}$ as:

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} F^{r-1}(x)\left[1-F(x)\right]^{n-r} f(x) \tag{21}$$

By substituting equation (5) and equation (6) into equation (21), we have,

$$f_{r:n}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{1}{B(r,n-r+1)}\left[\frac{1-\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}{1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}\right]^{r-1}\left[\frac{2\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}{1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}\right]^{n-r}$$
$$\times\frac{2\lambda\alpha h(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha-1}(x;\lambda,\alpha,\boldsymbol{\beta})\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda-1}}{\left[1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}\right]^{2}} \tag{22}$$

The equation above is called the rth order statistics for the TIHLEt-G family of distributions.
Let r = n, then the probability density function of the maximum order statistics is

$$f_{n:n}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2n\lambda\alpha h(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha-1}(x;\lambda,\alpha,\boldsymbol{\beta})\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda-1}}{\left[1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}\right]^{2}}\left[\frac{\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}{1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}\right]^{n-1} \tag{23}$$

Also, let r = 1, then the probability density function of the minimum order statistics is

$$f_{1:n}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2n\lambda\alpha h(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha-1}(x;\lambda,\alpha,\boldsymbol{\beta})\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda-1}}{\left[1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}\right]^{2}}\left[\frac{2\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}{1+\left[1-H^{\alpha}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^{\lambda}}\right]^{n-1} \tag{24}$$

5.0 Sub-Models

In this section, we describe two sub-models of the TIHLEt-G family namely, TIHLEt-Exponential and TIHLEt-Log-logistic respectively.

5.1 Type I Half-Logistic Exponentiated Exponential (TIHLEtE) Distribution

The cdf and pdf of Exponential distribution which is our baseline distribution with parameter θ are:

$$H(x;\theta)=1-e^{-\theta x} \tag{25}$$

and

$$h(x;\theta)=\theta e^{-\theta x}\,, \quad x>0,\theta>0 \tag{26}$$

The TIHLEtE has cdf and pdf as follows:

$$F_{TIHLEtE}(x;\lambda,\alpha,\theta)=\frac{1-\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda}}\ ,\ x>0,\lambda,\alpha,\theta>0 \tag{27}$$

And

$$f(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda}\right]^{2}},\ x>0,\ \lambda,\alpha,\theta>0 \tag{28}$$

Furthermore, the following are the reliability function, hazard rate function and the quantile function respectively:

$$R(x;\lambda,\alpha,\theta)=\frac{2\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda}} \tag{29}$$

$$T(x;\lambda,\alpha,\theta)=\frac{\lambda\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{\alpha-1}}{\left[1+\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda}\right]\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]} \tag{30}$$

$$x=\frac{-1}{\theta}\ln\left[1-\left[1-\left[\frac{1-u}{u+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right] \tag{31}$$

5.2 Type I Half-Logistic Exponentiated Log-Logistic (TIHLEtLL) Distribution

cdf and pdf of Log-Logistic distribution with parameter θ as our baseline distribution are:

$$H(x;\theta)=\frac{x^{\theta}}{1+x^{\theta}} \tag{32}$$

$$h(x;\theta)=\frac{\theta x^{\theta-1}}{(1+x^{\theta})^{2}}\,, \quad x>0,\theta>0 \tag{33}$$

Now

$$H^{\alpha}(x;\theta,\alpha)=\left[\frac{x^{\theta}}{1+x^{\theta}}\right]^{\alpha}=\left[x^{-\theta}+1\right]^{-\alpha}$$

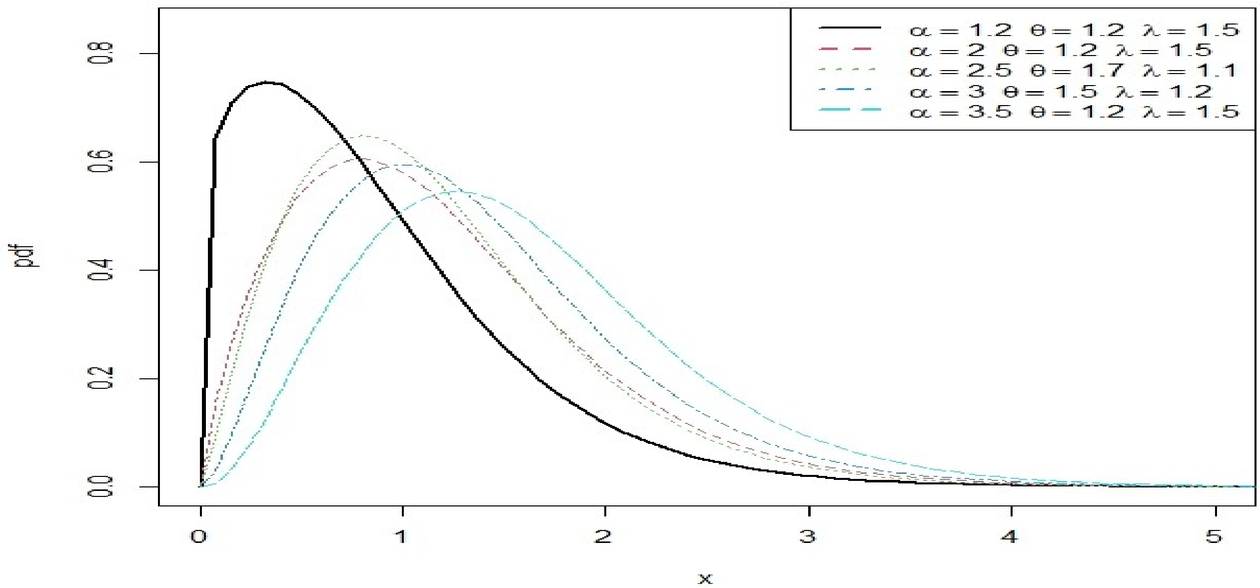

Fig. 1: Plots of Pdf of the TIHLEtE distribution for different parameter values.

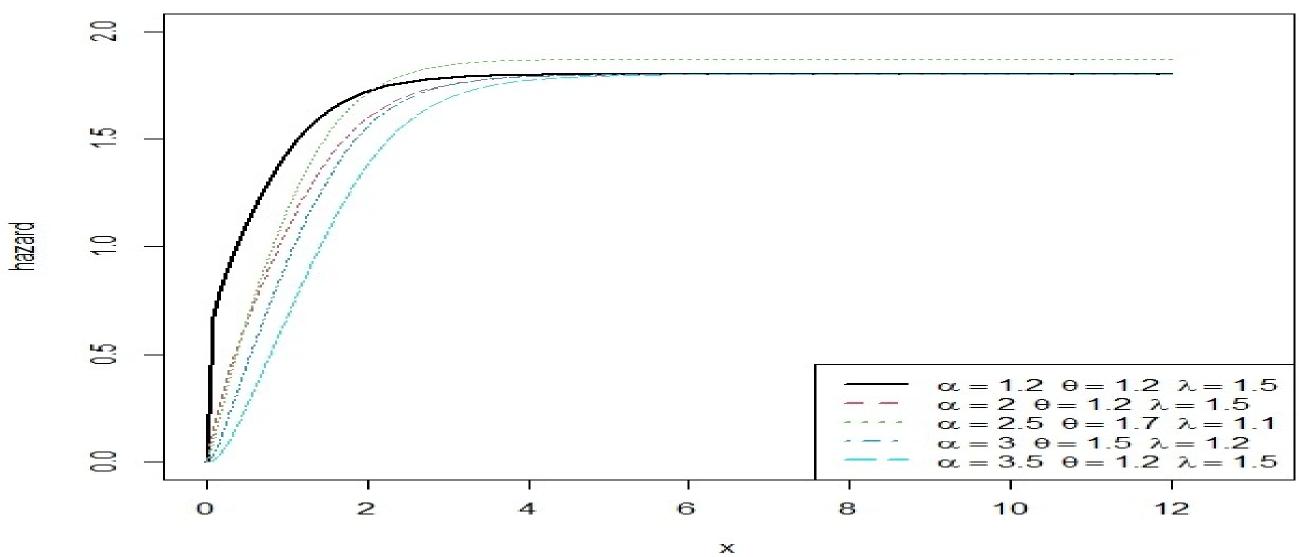

Fig. 2: Plots of hazard of the TIHLEtE distribution for different parameter values

The TIHLEtLL distribution has CDF and PDF given as;

$$F_{TIHLEtLL}(x;\lambda,\alpha,\theta)=\frac{1-\left[1-\left[x^{-\theta}+1\right]^{-\alpha}\right]^{\lambda}}{1+\left[1-\left[x^{-\theta}+1\right]^{-\alpha}\right]^{\lambda}},x>0,\lambda,\alpha,\theta>0 \tag{34}$$

$$f_{TIHLEt\,L\,L}(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta x^{\theta-1}\left[x^{-\theta}+1\right]^{1-\alpha}\left[1+x^{\theta}\right]^{-2}\left[1-\left[x^{-\theta}+1\right]^{-\alpha}\right]^{\lambda}}{\left[1+\left[1-\left[x^{-\theta}+1\right]^{-\alpha}\right]^{\lambda}\right]^{2}} \tag{35}$$

Moreso, the following are the reliability function, hazard rate function and the quantile function respectively:

$$R(x;\lambda,\alpha,\theta)=\frac{2\left[1-\left[x^{-\theta}+1\right]^{-\alpha}\right]^{\lambda}}{1+\left[1-\left[x^{-\theta}+1\right]^{-\alpha}\right]^{\lambda}} \tag{36}$$

$$T(x;\lambda,\alpha,\theta)=\frac{\lambda\alpha\theta x^{\theta-1}\left[1+x^{\theta}\right]^{-2}\left[x^{-\theta}+1\right]^{1-\alpha}}{1+\left[1-\left[x^{-\theta}+1\right]^{-\alpha}\right]^{\lambda}\left[1-\left[x^{-\theta}+1\right]^{-\alpha}\right]} \tag{37}$$

$$x=\left[\left[1-\left[\frac{1-u}{u+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{-1}{\alpha}}-1\right]^{\frac{-1}{\theta}} \tag{38}$$

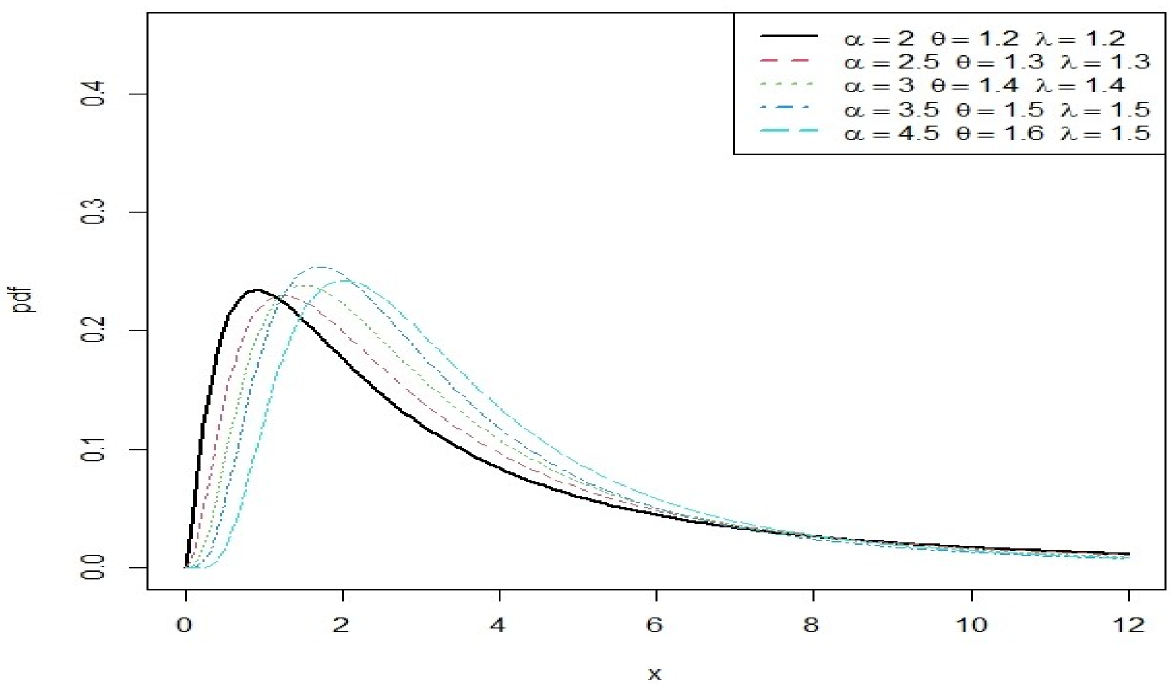

Fig. 3: Plots of Pdf of the TIHLEtLL distribution for different parameter values.

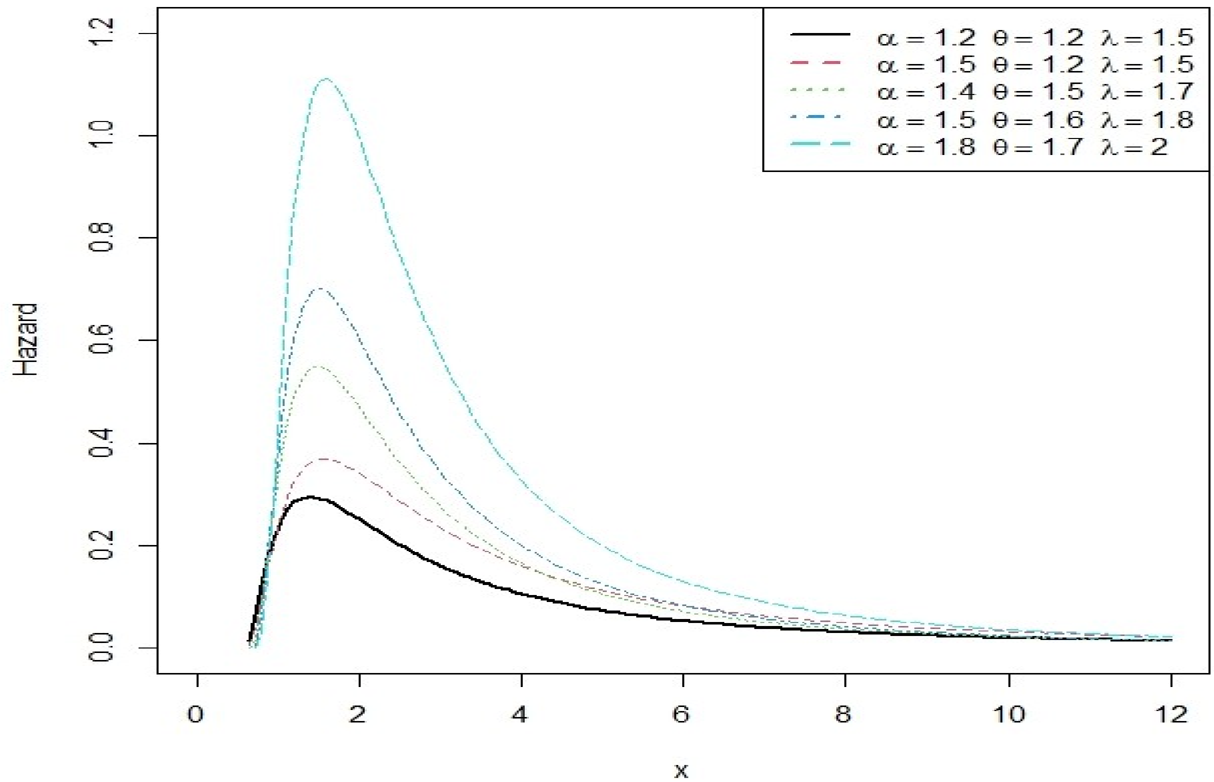

Fig. 4: Plots of hazard of the TIHlEtLL distribution for different parameter values.

6.0 Parameter Estimation

In this paper, we explore the maximum likelihood technique to estimate the unknown parameters of the TIHLEt-G family for complete data. Maximum likelihood estimates (MLEs) have appealing qualities that may be used to generate confidence ranges and provide simple approximations that function well in finite samples. In distribution theory, the resulting approximation for MLEs is easily handled, either analytically or numerically. Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size n from the TIHLEt-G family. Then, the likelihood function based on observed sample for the vector of parameter $(\lambda, \alpha, \beta)^T$ is given by

$$L(\phi) = n\log(2) + n\log(\lambda) + n\log(\alpha) + \sum_{i=0}^{n} \log h(x_i\beta) + (\alpha-1)\sum_{i=0}^{n} \log\left[H(x_i\beta)\right]$$

$$+ (\lambda-1)\sum_{i=0}^{n} \log\left[1 - H^{\alpha}(x_i\beta)\right] - 2\sum_{i=0}^{n} \log\left[1 + \left[1 - H^{\alpha}(x_i,\beta)\right]^{\lambda}\right] \tag{39}$$

The components of score vector $U = (U_\lambda, U_\alpha, U_\beta)^T$ are given as

$$U_\lambda = \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left[1 - H^{\alpha}(x_i,\beta)\right] - 2\sum_{i=1}^{n} \frac{\left[1 - H^{\alpha}(x_i,\beta)\right]^{\lambda} \log\left[1 - H^{\alpha}(x_i,\beta)\right]}{1 + \left[1 - H^{\alpha}(x_i,\beta)\right]^{\lambda}} \tag{40}$$

$$U_{\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[H(x_i,\beta)\right] + (\lambda-1)\sum_{i=1}^{n} \frac{H^{\alpha}(x_i,\beta)\log\left[H(x_i,\beta)\right]}{1-\left[H^{\alpha}(x_i,\beta)\right]}$$
$$-2\sum_{i=1}^{n} \frac{\left[1-H^{\alpha}(x_i,\beta)\right]^{\lambda-1} H^{\alpha}(x_i,\beta)\log\left[H(x_i,\beta)\right]}{1+\left[1-H^{\alpha}(x_i,\beta)\right]^{\lambda}} \tag{41}$$

$$U_{\beta} = \sum_{i=1}^{n}\left[\frac{h(x_i,\beta)^{\beta}}{h(x_i,\beta)}\right] + (\alpha-1)\sum_{i=1}^{n}\left[\frac{H(x_i,\beta)^{\beta}}{H(x_i,\beta)}\right] + \alpha(\lambda-1)\sum_{i=1}^{n}\left[\frac{H^{\alpha-1}(x_i,\beta)H(x_i,\beta)^{\beta}}{1-H^{\alpha}(x_i,\beta)}\right]$$
$$+2\lambda\alpha\sum_{i=1}^{n}\left[\frac{\left[1-H^{\alpha}(x_i,\beta)\right]^{\lambda-1} H^{\alpha-1}(x_i,\beta)H(x_i,\beta)^{\beta}}{1+\left[1-H^{\alpha}(x_i,\beta)\right]^{\lambda}}\right] \tag{42}$$

The MLEs are obtained by setting U_{λ}, U_{α} and U_{β} to zero and solving these equations simultaneously. These Equations cannot be solved analytically, necessitating the use of analytical tools to solve them numerically.

7. 0 Applications to Real Data

In this section, we fit the TIHLEtE distribution to two real data sets and give a comparative study with the fits of the Type I half logistic exponential (TIHLE) by Almarashi *et.al.,* (2019), Topp-Leone exponential distribution (TLEx) by Al-Shomrani *et.al.*, (2016), Kumaraswamy exponential distribution (KEx) by Adepoju and Chukwu (2015), Exponentiated exponential Distribution (ExEx) by Gupta and Kundu (1999), and Logistic-X exponential distribution (LoEx) by Oguntunde *et.al.,* (2018), as comparator distributions for illustrative purposes.

The TIHLE distribution proposed by Almarashi *et.al.*, (2019) has probability density function given as:

$$f(x;\lambda,\alpha) = \frac{2\lambda\alpha e^{-\alpha\lambda x}}{\left[1+e^{-\alpha\lambda x}\right]^{2}} \tag{43}$$

The TLEx distribution proposed by Al-Shomrani *et.al.*, (2016) has pdf defined as:

$$f(x;\alpha,\theta) = 2\alpha\theta e^{-2\alpha x}\left[1-e^{-2\alpha x}\right]^{\theta-1} \tag{44}$$

The KE distribution developed by Adepoju and Chukwu (2015) has pdf defined as:

$$f(x;\alpha,\lambda,\theta) = \alpha\lambda\theta e^{-\alpha x}(1-e^{-\alpha x})\left[1-\left[1-e^{-\alpha x}\right]^{\alpha}\right]^{\lambda-1} \tag{45}$$

The ExEx distribution pioneered Gupta and Kundu (1999) has pdf given as:

$$f(x;\alpha,\theta) = \alpha\theta e^{-\alpha x}\left[1-e^{-\alpha x}\right]^{\theta-1} \tag{46}$$

And the LoEx distribution developed by Oguntunde *et.al.*, (2018) has pdf given as:

$$f(x;\alpha,\lambda) = \frac{\lambda}{\alpha^{\lambda}} x^{-(\lambda+1)}\left[1+(\alpha x)^{-\lambda}\right]^{-2} \tag{47}$$

The two datasets that were used as examples in the application demonstrate the new family of distributions' flexibility and 'best fit' compared to the above comparator distributions in modeling the data sets experimentally. The R programming language is used to carry out all of the computations.

Data set 1

The first data set as listed below represents the COVID-19 positive cases record in Pakistan from March 24 to April 28, 2020, previously used by Al-Marzouki, *et.al*., (2020): 2, 2, 3, 4, 26, 24, 25, 19, 4, 40, 87, 172, 38, 105, 155, 35, 264, 69, 283, 68, 199, 120, 67, 36, 102, 96, 90, 181, 190, 228, 111, 163, 204, 192, 627, 263.

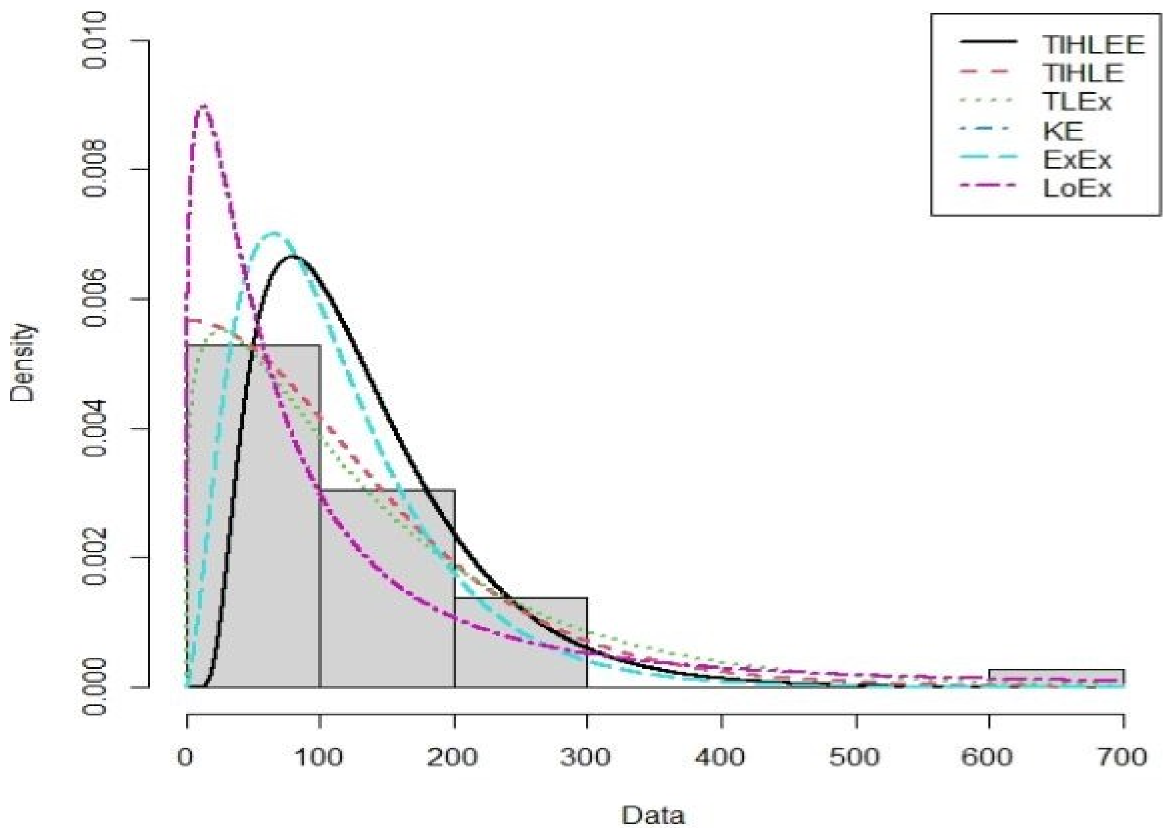

Fig. 5: Fitted pdfs for the TIHLEtE, TIHLE, TLEx, KE, ExEx, and LoEx distributions to the data set 1

Table 1: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	α	λ	θ	LL	AIC
TIHLEtE	0.6143	0.0189	0.5265	-207.4909	420.9818
TIHLE	0.0033	3.4313		-208.8547	421.7094
ToLE	1.2301		0.0040	-210.0476	424.0952
KE	0.0711	0.0538	0.1297	-213.8326	433.6653
ExEx	0.0154		2.7179	-227.4619	459.9237
LoEx	0.0137	1.2440		-212.0221	428.0442

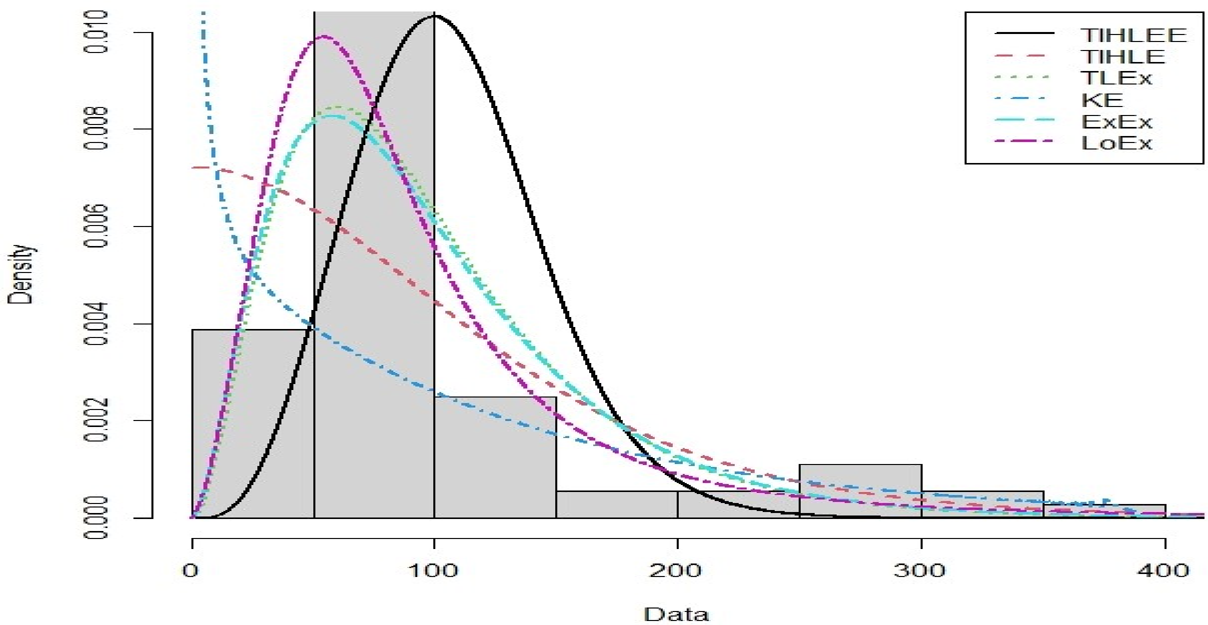

Fig. 6: Fitted pdfs for the TIHLEtE, TIHLE, TLEx, KE, ExEx, and LoEx distributions to the data set 2

Table 1 presents the results of the Maximum Likelihood Estimation of the parameters of the proposed distribution and the five comparator distributions. Based on the goodness of fit measure, the proposed distribution reported the minimum AIC value, though followed closely by the T1HLE. The visual inspection of the fit presented in Figure 5, also confirms the superiority of the proposed distribution amongst its comparators. Thus the proposed distribution 'best fit' COVID 19 data set amongst the range of distributions considered.

The second data set shown below represents the survival times (in days) of seventy-two (72) guinea pigs infected with virulent tubercle bacilli. It has been previously used by Bjerkedal (1960) and Shanker *et.al*., (2015):
12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83,84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

Data set 2

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	α	λ	θ	LL	AIC
TIHLEtE	3.558	0.3341	0.0541	-393.7824	793.5648
TIHLE	0.0029	4.8756		-400.0553	804.1106
ToLE	0.5000		0.0032	-423.6331	851.2662
KE	0.0291	0.0874	0.0936	-427.4184	860.8369
ExEx	0.0059		0.4756	-424.8795	853.759
LoEx	0.0128	0.8691		-429.0641	862.1282

Table 2 shows the results of the Maximum Likelihood Estimation of the parameters of the T1HLEtE distribution and the five comparator distributions. Based on the goodness of fit statistic AIC, the new distribution reported the minimum AIC value suggesting that the distribution is the 'best fit' to the survival times of the guinea pigs infected with tubercle bacilli. The visual inspection of the fit presented in Figure 6, also reaffirms the superiority of the new distribution amongst its comparators. Thus, using the survival function of the new distribution, one can obtain survival probabilities for different values of x in order to provide crucial summary information from the analyzed survival data.

8. 0 Conclusion

A new family of continuous distributions called the Type I Half Logistic Exponentiated-G (TIHLEt-G) family is proposed and studied. Some of the statistical aspects of the proposed family, such as explicit quantile function expressions, ordinary and incomplete moments, generating function, survivor functions, and order statistics are investigated. Some of the new family's sub-models were discussed. The method of maximum likelihood is used to estimate the model parameters. In comparison to well-known models, two real data sets are evaluated to highlight the importance and flexibility of one of the sub model. The findings reveal that the new model appears to be superior to the existing models considered and, therefore, provides new distribution to model data in many applications.

9.0 References

Adepoju, K. A. & Chukwu, O. I. (2015). Maximum Likelihood Estimation of the Kumaraswamy Exponential Distribution with Applications, *Journal of Modern Applied Statistical Methods*, 14, 1, pp. 208-214.

Afify, A. Z., Altun , E., Alizadeh, M. , Ozel, G. & Hamedani, G.G.(2017). The Odd Exponentiated Half-Logistic-G Family: Properties, Characterizations and Applications. *Chilean Journal of Statistics*, 8, 2, pp. 65-91.

Ahmad, Z., Elgarhy, M., & Hamedani, G. G. (2018). A new Weibull-X family of distributions: properties, characterizations and applications. *Journal of Statistical Distributions and Applications*, 5:5.

Alizadeh, M., Cordeiro, G. M., de Brito, E. & Demtrio, C. G. B. (2015a). The beta Marshall-Olkin family of distributions. *Journal of Statistical Distributions and Applications*, 2, 4, pp. 1-18.

Alizadeh, M., Emadi, M., Doostparast, M., Cordeiro, G. M., Ortega, E. M. M. & Pescim, R. R. (2015b). A new family of distributions: The Kumaraswamy odd log-logistic properties and applications. *Hacettepa Journal of Mathematics and Statistics*, 44, pp. 1491-1512.

Almarashia, Abdullah M., Elgarhyb, M., Elsehetryc, Mamhoud M., Golam Kibriad, B. M. & Algarnia, Ali, (2019). A new extension of exponential distribution with statistical properties and applications, *Journal of Nonlinear Sciences and Application*, 135145.

Al-Marzouki, S., Jamal, F., Chesneau, C., and Elgarhy, M., (2020). Topp-Leone Odd Frechet Generated Family of Distributions with Applications to COVID-19 Data Sets, *Computer Modeling in Engineering and Sciences*, 125, 1, pp. ,437-458

Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. & Shahbaz, M. Q. (2016). Topp-Leone family of distributions: Some properties and application. *Pak.j.stat.oper.res*, 12, 3, pp. 443-451.

Alzaatreh, A., Lee, C. & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, pp. 63-79.

Alzaghal, A., Lee, C. & Famoye, F. (2013). Exponentiated T-X family of distributions with some applications. *International Journal of Probability and Statistics*, 2, 3149. https://doi.org/10.5539/ijsp.v2n3p31

Balakrishnan, N. (1985). Order statistics from the half logistic distribution. *Journal of Statistical Computation and Simulation,* 20, pp. 287-309.

Oluvede, B., Chipepa, F. & Wanduku, D. (2021). The odd Weibull-Topp-Leone-G power series family of distributions: model, properties, and applications. *Journal of Nonlinear Sciences and Applications*, 268286, doi.org/10.22436/jnsa.014.04.06

Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with deferent doses of virulent tubercle bacilli, *American Journal of Hygiene*, 72: pp. 130-148.

Cordeiro, G. M., Ortega, E. M. & Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, 347, 8, pp. 1399-1429.

Cordeiro, G. M. & de Castro, M. (2011). A new family of generalized distribution. *Journal of Statistical Computations and Simulation*, 81, pp.883-898.

Cordeiro, G. M., Alizadeh, M. & Ortega, E. M. M. (2014a). The exponentiated half-logistic family of distributions: Properties and applications. *Journal of Probability and Statistics*,. doi.org/10.1155/2014/864396.

Cordeiro, G. M., Alizadeh, M. & Marinho, E. P. R. D. (2015). The Type I half-logistic family of distributions. *Journal of Statistical Computation and Simulation*, 86, pp. 707-728.

Cordeiro, G. M., Ortega, E. M., Popovic, B. V. & Pescim, R. R. (2014b). The Lomax generator of distributions: Properties, mini cation process and regression model. *Applied Mathematics and Computation*, 247, pp. 465-486.

Cordeiro, G. M., Pescim, R. R., Demetrio, C. G. B., Ortega, E. M. M. & Nadarajah, S. (2012). The new class of Kummer beta generalized distributions. *Statistics and Operations Research Transactions*, 36, pp. 153-180.

David, H. A. (1970). *Order statistics*, *Second edition*. Wiley, New York.

Dias, C.R. B., Cordeiro, G. M., Alizadeh, M., Marinho, P. R. D. & Cołlho, H. F. C. (2016). Exponentiated Marshall-Olkin family of distributions. *Journal of Statistical Distributions and Applications*, 3, 15, doi.org/10.1186/s40488-016-0051-2

Eugene, N., Lee, C. & Famoye, F. (2002). The beta-normal distribution and its applications. Communications in Statistics theory and Methods, 31, pp. 497- 512.

Falgore, J.Y. & Doguwa, S.I. (2020a). Kumaraswamy-Odd Rayleigh- G Family of Distributions with Applications. *Open Journal of Statistics*, 10, pp. 719-734.

Falgore, J.Y. & Doguwa, S.I. (2020b). Inverse Lomax-Exponentiated G (IL-EG) Family of Distributions: Properties and Applications. *Asian Journal of Probability and Statistics*. 9, 4, pp. 48-64.

Falgore, J.Y. & Doguwa, S.I. (2020c). Inverse Lomax G Family of Distributions with applications to Breaking Strength Data. *Asian Journal of Probability and Statistics,* 8, 2, pp. 49-60.

Gupta, R.C., Gupta, P.L. & Gupta, R.D. (1998). Modeling failure time data by Lehmann alternatives. *Commun. Stat. Theory Methods*, 27, pp. 887-904.

Gupta, R. D. & Kundu, D. (1999). Generalized exponential distributions, *Australian and New Zealand Journal of Statistics*, 41, pp. 173-188.

Ibrahim S, Doguwa S.I, Isah A & Haruna J. M.(2020b). The Topp Leone Kumaraswamy-G Family of Distributions with Applications to Cancer Disease. *Journal of Biostatistics and Epidemiology*, 6, 1, pp. 37-48

Ibrahim, S., Doguwa, S.I., Audu, I. & Muhammad, J. H. (2020a). On the Topp Leone exponentiated-G Family of Distributions: Properties and Applications.

Asian Journal of Probability and Statistics, 7, pp. 1-15.

Jamal, F., Elgarhy, M., Nasir, M., Ozel, G. & Khan, M. N. (2018). Generalized inverted Kumaraswamy generated family of distributions: Theory and applications, https://hal.archives ouvertes.fr/hal-01907258.

Johnson, N. L., Kotz, S. & Balakrishnan, N. (1994). *Continuous univariate distributions (Volume* 1, *2nd edition)*. John Wiley and Sons, Newyork.

Marshall, A.W. & Olkin, I. (1997). A new methods for adding a parameter to a family of distributions with application to the Exponential and Weibull families. *Biometrika*, 84, pp. 641-652.

Nasir, M. A., Tahir, M. H., Jamal, F. & Ozel. G. (2017). New Generalized Burr Family of Distributions for the Lifetime Data, *Journal of Statistics Applications and Probability*, 6, 2, pp. 401-417.

Oguntunde, P. E., Khaleel, M. A., Adejumo, A.O. & Okagbue, H. I., (2018). A study of an extension of the exponential distribution using logistic-x family of distributions, *International Journal of Engineering and Technology*, 7, 4, pp. 5467-5471.

Ristic, M. M. & Balakrishnan, N. (2011). The gamma-exponentiated exponential distribution. *Journal of Statistical Computation and Simulation*, 82, pp. 1191-1206.

Sanusi, A., Doguwa, S., Audu, I. & Baraya, Y. (2020a). Burr X Exponential-G Family of Distributions: Properties and Application. *Asian Journal of Probability and Statistics*, 7, pp. 58-75.

Sanusi, A., Doguwa, S., Audu, I. & Baraya, Y. (2020b). Topp Leone Exponential G Family of Distributions: Properties and Application. *Journal of Science and Technology Research*, 2, 4, pp. 83 -96.

Shanker, R., Fesshaye, H. & Selvaraj, S. (2015). On modelling lifetimes data using exponential and lindley distributions, *Biometric and Biostatistics International Journal*, 2, 5, pp. 1-9.

Silva, R., Silva, F.G., Ramos, M., Cordeiro, G., Marinho, P.& De Andrade, A. N. T. (2019). The Exponentiated Kumaraswamy-G Class: General Properties and Application. *Revista Colombiana de Estadstica,* 42, 1, pp. 1-33.

Silva, R.B., Bourguignon, M. & Cordeiro, G.M. (2014). The Weibull-G family of probability distributions. *Journal of Data Science*, 12, pp. 53-68.

Torabi, H. & Montazari, N. H. (2014). The logistic-uniform distribution and its application. *Communications in Statistics Simulation and Computation*, 43, 25512569, doi.org/10.1080/03610918.2012.737491

Usman, A. , Doguwa, S. I. S., Alhaji, B. B. & Imam, A. T.(2020). A New Generalized Weibull-Odd Frechet Family of Distributions: Statistical Properties and Applications. *Asian Journal of Probability and Statistics*, 9, 3, pp. 25-43.

Sengweni, W., Oluvede, B., & Makubate, B. (2021). The exponentiated half-logistic odd lindley-G family of distributions with applications Journal *of Nonlinear Sciences and Applications*, 287309, doi:10.22436/jnsa.014.05.01.

Yousof, H. M., Afy, A. Z., Nadarajah, S., Hamedani, G. & Aryal, G. R. (2018). The Marshall-Olkin generalized-G family of distributions with Applications. *Statistica*, 78, 3, pp. 273-295.

Zografos, K. & Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, 6, pp. 344- 362.

Conflict of Interest

The authors declared no conflict of interest.