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Mathematical Modelling of an Investor's Wealth with different Stochastic Volatility Models

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***Abstract**: This paper investigates the application of various stochastic volatility models in determining optimal investment strategies in the stock market. The study explores the geometric Brownian motion (GBM), constant elasticity of variance (CEV), modified CEV (M-CEV), and Heston volatility models. Each model offers a unique perspective on volatility dynamics and option pricing. The research formulates the Hamilton-Jacobi-Bellman (HJB) equations for each model and employs the Legendre transformation method to convert them into linear partial differential equations (PDEs). The quadratic utility function is utilized to derive optimal portfolio distributions under each model. Numerical simulations are conducted to analyze the impact of market parameters such as appreciation rate, volatility, interest rate, elasticity parameter, tax, and investor's wealth on the optimal portfolio distribution. The results indicate that optimal investment strategies vary significantly based on market conditions and investor preferences. Overall, this study provides valuable insights into the dynamic nature of financial markets and offers practical guidance for portfolio optimization and risk management strategies.*

***Keywords**: Optimal portfolio distribution, stochastic volatility, Ito's lemma, Hamilton Jacobi Bellman equation, financial market.*

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1.0 Introduction

The stock market (SM) is an essential part of the financial market, which makes it possible for the buying and selling of equity representing ownership in publicly traded companies. SM pricing has to do with determining fair values for individual stocks, which is often determined by the forces of fundamental analysis, demand and supply, market sentiment and other factors. Accurate pricing is of utmost importance for investors seeking to make the right investment decisions, and risk management and ultimately make optimal returns on investment. Stochastic volatility is a model that recognizes that the volatility of the underlying asset is not constant but rather follows a random process over time (Merton, 1976). Traditional models that assume constant volatility may fail to capture the dynamic nature of financial markets and lead to inaccurate pricing. Stochastic volatility models provide a more realistic representation of volatility dynamics, allowing for time-varying volatility levels and capturing empirical observations such as volatility clustering, mean reversion, and the presence of jumps.Various types of stochastic volatility models have been developed to better capture the dynamics of volatility in financial markets. These models include:

The GBM model, proposed by (Black & Scholes, 1973), forms the foundation of modern option pricing theory. It assumes that stock prices follow a log-normal distribution and that volatility is constant over time. While GBM itself is a foundational model for understanding asset price dynamics, Merton's work laid the groundwork for connecting it with the idea of stochastic volatility.(Merton, 1976) introduced the concept of stochastic volatility by incorporating a stochastic process for volatility into the framework of GBM. He recognized that the constant volatility assumption of the Black-Scholes model was not fully representative of real-world market behaviour, leading him to explore the idea of allowing volatility to vary over time. Merton's incorporation of stochastic volatility into the GBM framework was a significant step toward modelling more realistic price dynamics in financial markets.(Agbam & Azubuike, 2021; Hamzah *et al*, 2021) investigated the prediction of stock market prices using the GBM Model and the results show that in the simulation, there are some actual stock prices located outside trajectory realization that may be from the GBM model. Thus, the model did not predict accurately the price behavior of some of the listed stocks. (Muhamad & Ani, 2022) investigated forecasting the crude oil price in Malaysia using the GBM model and the result shows that the log return of crude oil prices is normally distributed, and GBM models were suitable to forecast crude oil prices but for short-term prediction. (Akpanibah& Catherine, 2023) investigated the application of the GBM model to FM and they showed that the model is appropriate for forecasting SM pricing on the floor of Nigeria stock exchange at least for one month. However, the introduction of other stochastic volatility models was driven by the limitations of the GBM model and the need to better capture the complexities of real-world financial markets.

.The introduction of the CEV model to the concept of stochastic volatility can be attributed to (Cox, 1975). He introduced the CEV model as an extension of the GBM model. The CEV model addresses one of the limitations of the GBM model - the assumption of constant volatility. Cox recognized that while constant volatility was unrealistic, a power-law relationship between the asset's volatility and its price level could offer a more flexible way to model time-varying volatility.

(Adeleke*et al*, 2019) examine the option pricing implications of the CEV model in the Nigerian stock market. The authors find evidence supporting the applicability of the CEV model, suggesting that it can effectively capture the dynamics of option prices in this market. (Aremo*et al*, 2019) did an empirical study of the CEV model using data from the NSE. They studied the relationship between stock returns and volatility, as captured by the CEV model. Their results show that the CEV model gives a reasonable representation of stock returns in the SM. Ogunjo & Adegbaju (2020) studied SV modelling of stock returns under the CEV model in the NSE; they investigated the dynamics of stock returns and volatility, taking into consideration the proposition of the CEV model. They showed that the model can effectively capture the volatility dynamics in the Nigerian SM.

Many other authors such as (Elliott & Wagalath (2018); Wang & Zhu (2018), Akpanibah & Oghenero, 2018; Li & Zhou (2018); Osu *et al* (2018); Yang & Zhang (2019); Tikhomirov & Nikitina (2019); Cui & Xu (2019); (Ballotta*et al*, 2019); Li & Lee (2019); Chan & McAleer (2020) Wang & Meng (2019); Chan & McAleer (2020); Hu & Huang (2020); Liu & Liu (2020); Akpanibah & Ini, 2020; Akpanibah & Ini, 2021; Li & Cheng (2020); Song & Liang, 2021) studied the CEV model under different conditions.

The M-CEV model is an extension of the CEV model and is aimed at redefining the CEV model by incorporating mean-reverting properties into the volatility process. This addition allows the model to better capture both short-term fluctuations and long-term trends in volatility, making it more suitable for certain market conditions. Some authors have used the M-CEV model to model their problem. This includes but is not limited to (Madalena *et al*, 2019) investigated option pricing under the M-CEV model with stochastic interest rates. Their study provided insights into the impact of interest rate fluctuations on option prices, enhancing the understanding of pricing dynamics in the stock market.(Liu, *et al*, 2020) extended the work of (Madalena *et al*, 2019) by studying options pricing using the M-CEV model with stochastic interest rates. Their study further explored the implications of interest rate uncertainty on option pricing, Olorunfemi & Oyewunmi (2018) studied an option pricing model under the MCEV process. Adedoyin & Babajide (2019) investigated the parameters of the MCEV model using implicit and explicit methods. Hence addressed the challenges of parameter estimation and gave possible solutions for accurate modeling. (Jianjun *et al*, 2018) explored option pricing and risk measurement under the M-CEV model with stochastic interest rates. They gave clear insight into the understanding of risk management in option pricing.

Adedoyin & Babajide (2019) extended their research to analyze option pricing and Greeks under the MCEV model with stochastic volatility. (Adedoyin & Babajide (2020), Akpanibah & Ini (2021), Amadi *et al*, 2022) further expanded their research by examining portfolio optimization under the MCEV model with stochastic volatility. Their study offered insights into optimal portfolio allocation strategies, considering the dynamic nature of volatility in the stock market.

The Heston Volatility Model (HVM) was introduced by Steven L. Heston. He developed this model as an innovative solution to address the limitations of previous stochastic volatility models and to better capture the complex dynamics of option prices and volatility observed in financial markets.

Alina*et al*, (2018) gave a total review of the HVM Model with applications. They discussed the mathematical formulation of the HVM model and its capability to capture the dynamics of asset prices and volatility and proceeded to solve the HVM using numerical methods and discuss its limitations and possible areas for improvement. Aliyu (2018) compared the pricing of European options using the Heston Model and the Black-Scholes Model. Their study highlights the advantages of the Heston Model in capturing the volatility smile observed in the SM. Chen & Zhang (2018) propose a closed-form solution for the HVM with stochastic interest rates. Tiwari *et al*, (2019) extend the HVM by incorporating time-dependent parameters. The authors discussed the effect of some parameters on option pricing and explored the Greeks of European options under this extended model. Kim & Lee (2019) investigate the impact of transaction costs and liquidity on option pricing under the Heston Model. They pointed out the advantages of introducing transaction costs and liquidity effects in option pricing models and risk management strategies.

Also, several other authors such as (Sheng & Rong, 2014; Akpanibah & Samaila, 2021; Akpanibah et al, 2020) used the HVM to model the price of risky assets in portfolio optimization and proceed to obtain optimal distribution strategies under different assumptions.

Hence in this paper, we investigate different volatility models available in the financial market such as GBM, CEV, M-CEV and the Heston Volatility model as it plays a crucial role in determining the fraction of an investor's wealth to be invested in different assets at any given time. Also, to determine the optimal control problem for a given portfolio and also obtain the OPD under quadratic utility function and discuss the relationships between the OPD and other parameters.

2.0 Formulation of Hjb Equations

In this section, we introduce four volatility models used in modelling the stock market prices of any risky asset. Our interest here is to determine the wealth function of an investor with a portfolio comprising of one risk-free asset and a risky asset whose price process follows the geometric Brownian motion, Heston volatility, constant elasticity of variance and modified constant elasticity of variance models. Furthermore, we will obtain an optimization problem in the form Hamiton Jacobi Bellman equation which is a non-linear PDE

2.1 *Geometric Brownian Motion(GBM)*

$$\frac{dA}{A} = rdt \quad A(0) = 1 \tag{1}$$

$$\frac{ds}{s} = \mu dt + \sigma dw \tag{2}$$

where μ is the appreciation rate of the risky asset and r is the interest rate of the risk-free asset, w(t) is the standard Brownian motion defined on a complete probability space (Ω, f, p).

Let $l,$ be the amount an investor is allowed to invest in the risky asset.

$$dV = lV\frac{ds}{s} + V(1-l)\frac{dA}{A} + Cdt \tag{3}$$

Substituting (1) and (2) into (3)

$$dV = \left[V\left(l(\mu - r) + r\right) + C\right]dt + lV\sigma dw \tag{4}$$

The value function (Hamilton Jacobi Bellman equation)

Next, we defined the value function of the investor's utility at the expiration of his or her investment as follows

$$\mathrm{Q}(t,V) = Max\left[E(u,T)\right] \tag{5}$$

From Ito's lemma Taylor series expansion, and maximum principle can obtain the (HJB) equation as follows

$$dQ = Q_t\,dt + Q_v\,dV + \frac{1}{2}Q_{VV}(dV)^2$$

$$dQ = Q_t\,dt + Q_V\left[\left[V\left(l(\mu - r) + r\right) + C\right]\right] + \frac{1}{2}Q_{vv}l^2V^2\sigma^2$$

According to Ito's lemma $dwdt = dtdt = 0, (dw)^2 = dt$

$$\begin{cases} \dfrac{dQ}{dt \to 0} = Q_t + Q_V\left[\left[V\left(l(\mu - r) + r\right) + C\right]\right] + \dfrac{1}{2}Q_{vv}l^2V^2\sigma^2 = 0 \\ \mathrm{Q}(T,V) = U(V) \end{cases} \tag{6}$$

where $Q_{t,}$ Q_V *and* Q_{VV} are partial derivatives of the first and second order w.r.t time, and wealth.

Differentiating (6) with respect to $l,$ we have

$$l = -\left(\frac{(\mu - r)Q_V}{V\sigma^2 Q_{VV}}\right) \tag{7}$$

Substituting (7) into (6)

$$Q_t - Q_V\left(\frac{(\mu - r)}{\sigma^2 Q_{VV}}\right)(\mu - r) - \frac{1}{2}\left(\frac{\left(Q_v(\mu - r)\right)^2}{V\sigma^2 Q_{VV}}\right) \tag{8}$$

Hence the investor's wealth and the Hamilton Jacobi-Bellman Equation under the Geometric Brownian Motion are given in (4) and (8).

2.2 *Modified Constant Elasticity of Variance (MCEV)*

$$\frac{ds}{s} = (\eta + k\xi^2 s_t^{2\lambda})dt + \xi s_t^{\lambda} dw \tag{9}$$

Where η is the appreciation rate of the risky asset and r is the interest rate of the risk-free asset, w(t) is the standard Brownian motion defined on a complete probability space (Ω, f, p). Let l, be the fraction of an investor's wealth V(t) allowed to be invested in the risky asset.
Substituting (1) and (9) into (3), we have

$$dV = \left[V(l(\eta + k\xi^2 S_t^{2\lambda} - r) + r) + C\right]dt + lV\xi S_t^{\lambda} dw \tag{10}$$

The value function (Hamilton Jacobi Bellman equation)
Next, we defined the value function of the investor's utility at the expiration of his or her investment as follows

$$Q(t, S, V) = Max\left[E\left(u, T\right)\right] \tag{11}$$

From Ito's lemma Taylor series expansion, and maximum principle can obtain the (HJB) equation as follows

$$dQ = Q_t dt + Q_S dS + Q_V dV + \frac{1}{2}Q_{SS}(dS)^2 + \frac{1}{2}Q_{VV}(dV)^2 + Q_{SV}(dS)(dV) \tag{12}$$

Substituting (9) and (10) into (12), we have

$$\begin{bmatrix} Q_t dt + Q_s\left[\left(\eta s + K\xi^2 S^{2\lambda+1}\right)\right] \\ +Q_v\left[\left[V(l(\eta + K\xi^2 S_t^{2\lambda} - r) + r) + C\right]\right] \\ +\frac{1}{2}Q_{ss}\xi^2 S^{2\lambda+1} + \frac{1}{2}Q_{vv}l^2V^2\xi^2S^{2\lambda} + Q_{sv}lV\xi^2S^{2\lambda+1} \end{bmatrix} = 0 \tag{13}$$

Where $Q_{t,}\ Q_{S,} Q_V, Q_{SS,}\ Q_{VV}\ and\ Q_{SV}$ are partial derivatives of the first and second order concerning time (t), risky asset's price (s), and wealth (V). The boundary condition is given as $Q(S, T, V) = U(V)$
Differentiating (13) with respect to l, we have

$$l = \frac{-\left[Q_V\left(\eta + K\xi^2 S^{2\lambda} - r\right) + Q_{SV}\xi^2 S^{2\lambda+1}\right]}{V\xi^2 Q_{VV}} \tag{14}$$

Substituting (14) into (13), we have

$$Q_t + Q_s(\eta S + K\xi S^{2\lambda+1}) - Q_V\left[\frac{\left[Q_V\left(\eta + K\xi^2 S^{2\lambda} - r\right) + Q_{SV}\xi^2 S^{2\lambda+1}\right]}{\xi^{2\lambda}Q_{VV}}\right](\eta + K\xi^2 S^{2\lambda} - r) + r) + C) +$$

$$\frac{1}{2}Q_{SS}\xi^2 S^{2\lambda+2} - \frac{1}{2}\left[\left[\frac{\left[Q_V\left(\eta + K\xi^2 S^{2\lambda} - r\right) + Q_{SV}\xi^2 S^{2\lambda+1}\right]^2}{\xi^{2\lambda}Q_{VV}}\right]\right] - \left[\frac{\left[Q_V\left(\eta + K\xi^2 S^{2\lambda} - r\right) + Q_{SV}\xi^2 S^{2\lambda+1}\right]}{\xi^{2\lambda}Q_{VV}}\right]\xi^2 S^{2\lambda+1} = 0$$

This is the Hamilton Jacobi-Bellman Equation for the MCEV

2.3 Heston Volatility Model

$$\begin{cases} \dfrac{d\mathcal{S}(t)}{\mathcal{S}(t)} = \left((r + \mathfrak{H}k)dt + \sqrt{k}dW_{s}\right), \\ \mathcal{S}(0) = s_0 \end{cases} \tag{15}$$

$$\begin{cases} dk(t) = \mathfrak{T}(\mathfrak{b}\varphi - k)dt + \sigma\sqrt{k}dW_c, \\ k(0) = k_0 \end{cases} \tag{16}$$

where $r > 0$ is the interest rate of the risk-free asset, $\sqrt{k}$ is the volatility of the stock market price, $\mathfrak{b}$ is the long-term price variance, $\mathfrak{T}$ is the rate of reversion to the long-term price variance, σ is the volatility of the volatility and $\mathfrak{H}$ is the expected appreciation rate of the stock market price.
Let l, be the proportion of an investor's wealth allowed to be invested in the risky asset.

Substituting (1), (15) and (16) into (3)

$$\mathrm{dV(t)} = \begin{bmatrix} \{\mathrm{V(t)}\left(l\,(\mathrm{t})(\mathfrak{H} - r)\mathrm{k} + r\right) + \mathrm{C}\}\mathrm{dt} + \left(\mathrm{V(t)}\,l\,(\mathrm{t})\sqrt{\mathrm{k}}\right)\mathrm{dW}_{s} \\ \mathrm{V(0)} = \mathrm{v}_0 \end{bmatrix} \tag{17}$$

The value function (Hamilton Jacobi Bellman equation)

Next, we defined the value function of the investor's utility at the expiration of his or her investment as follows

$$\mathrm{Q}(t,\mathrm{k},V) = Max\left[E\left(u,T\right)\right] \tag{18}$$

From Ito's lemma Taylor series expansion, and maximum principle can obtain the (HJB) equation as follows

$$dQ = Q_t dt + Q_S dS + Q_V dV + \frac{1}{2}Q_{SS}(dS)^2 + \frac{1}{2}Q_{VV}(dV)^2 + Q_{SV}(dS)(dV)$$

The HJB equation is given thus by the application of Ito's lemma

$$\begin{cases} \sup\limits_{l} \left\{ \begin{matrix} \mathrm{Q_t} + \mathrm{Q_v}\left[\mathrm{v}\left(l\,\mathrm{k}(\mathfrak{H} - r) + r\right) + \mathrm{C}\right] \\ + \mathfrak{T}(\mathrm{b} - \mathrm{k})\mathrm{Q_k} + \frac{1}{2}\left(\mathrm{v}\,l\,(\mathrm{t})\sqrt{\mathrm{k}}\right)^2 \mathrm{Q_{vv}} \\ + \frac{1}{2}\sigma^2 \mathrm{k}\mathrm{Q_{kk}} + \left(\sigma \mathrm{v}\mathrm{k}\,l\,\rho\right)\mathrm{Q_{vk}} \end{matrix} \right\} = 0 \\ \mathrm{Q}(\mathrm{k},T,V) = U(V) \end{cases} \tag{19}$$

Differentiating (19) with respect to l, what we have

$$l = \frac{-\left[Q_V\left(KL(t)\right) + Q_{SV}L\right]}{VLQ_{VV}} \tag{20}$$

Substituting (20) into (19), we have

$$dQ = Q_t + Q_s(rS + SKL(t)) - Q_V\left[\frac{\left[Q_V\left(KL(t)\right) + Q_{SV}L\right]}{VLQ_{VV}}\right](KL(t) + r) + C) +$$

$$\frac{1}{2}Q_{SS}L - \frac{1}{2}\left[\left[\frac{\left[Q_V\left(KL(t)\right) + Q_{SV}L\right]^2}{L}\right]\right] - \left[\frac{\left[Q_V\left(KL(t)\right) + Q_{SV}L\right]}{\xi^{2\lambda}Q_{VV}}\right]Q_{SV} \tag{21}$$

2.4 Constant Elasticity of Variance (CEV)

$$\frac{ds}{s} = \mu dt + \sigma S^{\beta} dw \tag{22}$$

Where μ is the appreciation rate of the risky asset and r is the interest rate of the risk-free asset, w(t) is the standard Brownian motion defined on a complete probability space (Ω, f, p).

Let α be the tax on the invested fund, then substituting (1) and (22) into (3), we have

$$dV = \left[V\left(l\left(\mu - r\right) + \left(r - \alpha\right)\right) + C\right]dt + lV\sigma S^{\beta} dw \tag{23}$$

Next, we defined the value function of the investor's utility at the expiration of his or her investment as follows

$$H(t,S,V) = Max\ E\left[U\left(T\right)\right] \tag{24}$$

From Ito's lemma Taylor series expansion, and maximum principle can obtain the (HJB) equation as follows

$$dQ = Q_t dt + Q_S dS + Q_V dV + \frac{1}{2}Q_{SS}(dS)^2 + \frac{1}{2}Q_{VV}(dV)^2 + Q_{SV}(dS)(dV) \tag{25}$$

Substituting (1), (22) and (23) into (25), we have

$$dQ = Q_t + Q_s \mu S + Q_V \left[V\left(l\left(\mu - r \right) + \left(r - \alpha \right) \right) + C \right] + \frac{1}{2} Q_{SS} \sigma^2 S^{2\beta+2} + Sup \left\{ \begin{array}{l} l^2 (\frac{1}{2} \sigma^2 S^{2\beta} Q_{VV}) \\ +l \left[\begin{array}{l} \sigma^2 S^{2\beta+2} Q_{sv} \\ +(\mu - r) Q_V \end{array} \right] \end{array} \right\} = 0 \qquad (26)$$

Where $Q_{t,}$ $Q_{S,}Q_V$, $Q_{SS,}$ Q_{VV} *and* Q_{SV} are partial derivatives of the first and second order concerning time (t), risky asset's price (s), and wealth (V). The boundary condition is given as $Q(S,T,V) = U(V)$

Differentiating (26) with respect to l, we have

$$l = -\frac{\left[Q_V \left(\mu - r \right) + \sigma^2 S^{2\beta+1} Q_{SV} \right]}{V \sigma^2 S^{2\beta} Q_{VV}} \qquad (27)$$

Substituting (27) into (26), we have

$$\left\{ \begin{array}{l} Q_t + \mu s Q_s + \left(r - \alpha \right) V Q_V + \frac{1}{2} \sigma^2 S^{2\beta+2} Q_{SS} - \left(\frac{((\mu - r) Q_V + \sigma^2 S^{2\beta+1} Q_{SV})^2}{2\sigma^2 S^{2\beta+1} Q_{VV}} \right) = 0 \\ V(T,S,V) = U(V) \end{array} \right\} \qquad (28)$$

Our interest is to solve for Q in (28) and substitute it in (27) to obtain the optimal investment strategy. Since (28) is a non-linear PDE, we will use the Legendre transformation method in (Li et al, 2013, Gao 2009) to transform it into a linear PDE.

3.0 Methods

The Legendre transform is one of the dynamic programming approaches used in solving optimization problems in the field of mathematical finance. It is used to convert a non-linear PDE into a linear PDE. By using the Legendre transform, complex equations can be reformulated and its solution can be determined by solving a system of linear equations. The dual theory, on the other hand, is based on the idea that the value of the system can be expressed as the maximum expected reward of a portfolio of non-anticipative assets.

Theorem1: Let $f: R^n \to R$ be a convex function for $z > 0$,defined the Legendre transform $L(z) = \max_v \left\{ f(v) - zv \right\}$ where $L(z)$ is the Legendre dual of $f(v)$.(Jonson and Sircar 2002).

Since $f(x)$ is convex, from theorem 1, we can define the Legendre transform as follows

$$g(t,z) = \inf \left\{ \begin{array}{l} v \mid Q(t,v) \geq \\ zv + \hat{Q}(t,v) \end{array} \right\} 0 < t < T \qquad (29)$$

where $\hat{Q}$ is the dual of Q and $z > 0$ is the dual variable of Q .The value of Q where this optimum is attained is denoted by $g(t,z)$, so that

The functions g and $\hat{Q}$ are closely related and can be referred to the dual of Q. These functions are related as follows

$$\hat{Q}(t,z) = Q(t,g) - zg$$

Where

$$g(t,z) = v, Q_v = z, g = -\hat{Q}_z \qquad (30)$$

At terminal time, we denote

$$\hat{U}(z) = \sup \left\{ U(v) - zx \mid 0 < v < \infty \right\}$$

And

$$G(z) = \sup \left\{ v \mid U(v) \geq zv + \hat{U}(z) \right\}$$

As a result

$$G(z) = \left(U^1 \right)^{-1} (z) \qquad (31)$$

where G is the inverse of the marginal utility U and note that

At terminal time T , we can define

$$\mathrm{g}(T,\mathrm{v}) = \inf_{v>0}\left\{ \mathrm{v} \mid U(\mathrm{v}) \geq zv + \hat{Q}(t,z) \right\} \text{and} \hat{V}(t,z) = \sup_{x>0}\left\{ U(x) - zx \right\}$$

So that

$$\mathrm{g}(T,z) = \left(U^{1}\right)^{-1}(\mathrm{z}) \tag{32}$$

From the definition of Legendre transform,

$$Q_V = z \tag{33}$$

$$\mathrm{Q}(t,s,z) = Q(t,s,g) - zg \tag{34}$$

$$g(t,s,\mathrm{z}) = v \tag{35}$$

Differentiating (34), we have

$$Q_t = \hat{Q}_t, \mathrm{Q}_s = \hat{Q}_s, \mathrm{Q}_{ss} = \hat{Q}_{ss} - \frac{\hat{Q}_{sz}^2}{\hat{Q}_{sz}}, \mathrm{Q}_{sv} = -\frac{\hat{Q}_{sz}}{\hat{Q}_{zz}}, Q_{vv} = \frac{-1}{\hat{Q}_{zz}} \tag{36}$$

4.0 Results and Discussion

In this section, we will be solving for the value function $Q(\mathrm{t},\mathrm{v},\mathrm{s})$ from the HJB equation in (28) and the solution obtained will be used to obtain our investment strategies under exponential and logarithmic utility functions. To achieve this, we apply the Legendre transformation method to reduce the non-linear PDE in (28) to a linear PDE.

Substituting (36) into (27) and (28), we have

$$\hat{Q}_t + \mu s \hat{Q}_s + rv\mathrm{z} + \frac{1}{2}\sigma^2 s^{2\beta+2}\hat{Q}_{ss} + \frac{(\mu-r)^2}{2\sigma^2 s^{2\beta}} z^2 \hat{Q}_{zz} - z\left(\mu - r\right)s\hat{Q}_{sz} = 0 \tag{37}$$

$$l^*(t) = \frac{(\mu-r)z + \sigma^2 s^{2\beta+1}\left(-\frac{\hat{Q}_{sz}}{\hat{Q}_{zz}}\right)}{\frac{-1}{\hat{Q}_{zz}\sigma^2 s^{2\beta}}} \tag{38}$$

Differentiating (37) and (38) with respect to z and using (30), we have

$$g_t + \mu s g_s + \left(r - \alpha\right)(g + zg_z) + \frac{1}{2}\sigma^2 s^{2\beta+2} g_{ss} + \frac{(\mu-r)^2}{2\sigma^2 s^{2\beta}} z^2 g_z + \frac{(\mu-r)^2}{2\sigma^2 s^{2\beta}} z^2 g_{zz} - s\left(\mu - r\right)(g_s + zg_{sz}) = 0 \tag{39}$$

$$l^*(t) = \frac{\sigma^2 s^{2\beta+1} g_s - z(\mu-r)g_z}{\sigma^2 s^{2\beta}} \tag{40}$$

4.1 Optimal Portfolio Distribution under Quadratic Utility Function

In this section, the optimal value function and the optimal portfolio distribution for an investor will be solved for under the quadratic utility function by solving (39) for the value function and substituting the solution into (40) to obtain the optimal portfolio distribution of the investor.

From the work of Li et al, (2013), the quadratic utility function is given thus

$$\mathrm{U}(\mathrm{v}) = (v - c)^2, \tag{41}$$

where is the investor's wealth and c is a constant coefficient.
From (32) and (41), we have

$$g(T,s,z) = \frac{1}{2}z + c \tag{42}$$

Next, we form a solution of the form

$$g(t,s,z) = Zh(t,s) + a(t) \tag{43}$$

$$a(t) = c, h(T,s) = \frac{1}{2}$$

Differentiating (43), we have

$$g_t = zh_t + a_t, g_s = zg_s, g_{ss} = zh_{ss}, g_z = h, g_{zz} = 0, g_{sz} = h_s \tag{44}$$

Substituting (44) into (39), we have

$$h_t + (2r-\mu)sh_s + \frac{1}{2}\sigma^2 s^{2\beta+2} + \frac{(\mu-r)^2}{\sigma^2 s^{2\beta}}h - 2(r-\alpha)h + a_t - (r-\alpha)a(t) = 0 \tag{45}$$

Splitting (45), we have

$$h_t + (2r-\mu)sh_s + \tfrac{1}{2}\sigma^2 s^{2\beta+2} + \frac{(\mu-r)}{\sigma^2 s^{2\beta}} - 2(r-\alpha)h = 0$$
$$h(T,s) = \tfrac{1}{2} \tag{46}$$

$$a_t - (r-\alpha)a(t) = 0$$
$$a(t) = c \tag{47}$$

Solving (47), we have

$$a_t = (r-\alpha)a(t)$$

$$\frac{da}{dt} = (r-\alpha)a$$

$$\frac{da(t)}{a(t)} = (r-\alpha)dt$$

$$\ln a(t) = (r-\alpha)t + \ln M_1$$

$$\ln a(t) - \ln M_1 = (r-\alpha)t$$

$$\ln\left(\frac{a(t)}{M_1}\right) = (r-\alpha)t$$

$$\frac{a(t)}{M_1} = e^{(r-\alpha)t}$$

$$a(t) = M_1 e^{(r-\alpha)t} \tag{48}$$

$$a(t) = M_1 e^{(r-\alpha)t} = \text{c}$$

$$M_1 = \frac{c}{e^{(r-\alpha)T}} = ce^{-(r-\alpha)T} \tag{49}$$

Substitute (49) in (48), we have

$$a(t) = ce^{(r-\alpha)T}e^{(r-\alpha)T}$$

$$a(t) = ce^{(r-\alpha)(t-T)} \tag{50}$$

Next, we apply the power transformation method to (46), and assume a solution of the form

$$h(t,s) = n(t,x), x = s^{-2\beta}$$
$$n(T,x) = \frac{1}{2} \tag{51}$$

Differentiating (51), we have

$$h_t = n_t, h_s = -2\beta S^{-2\beta-1} n_x, h_{ss} = 2\beta(2\beta+1)S^{-2\beta-2} n_x + 4\beta^2 S^{-4\beta-2} n_{xx} \tag{52}$$

Substitute (52) into (46), we have

$$n_t + \beta\left[-2(2r-\mu)x + \sigma^2(2\beta+1)\right]n_x + 2\sigma^2\beta^2 x n_{xx} + \frac{(\mu-r)^2}{\sigma^2}xn - 2(r-\alpha)n = 0 \tag{53}$$

Lets

$$n(t,x) = K(t)e^{L(t)x}$$
$$K(T) = \frac{1}{2}, L(T) = 0 \tag{54}$$

Differentiating (54), we have

$$n_t = K_t e^{L(t)x} + K(t)L_t x e^{L(t)x}$$
$$n_x = KLe^{Lx}, n_{xx} = KL^2 e^{Lx} \tag{55}$$

Substituting (55) into (53), we have

$$\frac{K_t}{K} + \beta(2\beta+1)\sigma^2 L - 2(r-\alpha) + x\left[L_t - 2\beta(2r-\mu)L + 2\sigma^2\beta^2 L^2 + \frac{(\mu-r)^2}{\sigma^2}\right] = 0 \tag{56}$$

Splitting (56) into two equations, we have

$$\frac{K_t}{K} + \beta(2\beta+1)\sigma^2 L - 2\left(r-\alpha\right) = 0$$
$$K(T) = \tfrac{1}{2} \tag{57}$$

$$L_t - 2\beta(2r+\mu)L + 2\sigma^2\beta^2 L^2 + \frac{(\mu-r)^2}{\sigma^2} = 0$$
$$L(T) = 0 \tag{58}$$

If

$$-2\beta^2 = A, 2\beta(2r-\mu) = B, -(\mu-r)^2 = C$$

then (58) becomes

$$L_t - BL - A\sigma^2 L^2 - \frac{C}{\sigma^2} = 0, L(T) = 0$$

$$L_t = A\sigma^2 L^2 + BL + \frac{C}{\sigma^2}$$

$$\frac{dL}{dt} = A\sigma^2 L^2 + BL + \frac{C}{\sigma^2}$$

$$\frac{dL}{A\sigma^2 L^2 + BL + \frac{C}{\sigma^2}} = dt \tag{59}$$

If $B^2 - 4AC < 0$

Integrating (59) with respect to t,

Next,

$$u_1 = \frac{-B + \sqrt{B^2} - 4AC}{2A\sigma^2}, u_2 = \frac{-B - \sqrt{B^2} - 4AC}{2A\sigma^2}$$

Is the solution of $A\sigma^2L^2+BL+\dfrac{C}{\sigma^2}=0$

Then we can write (59) as

$$\frac{\left(\dfrac{1}{L(t)-u_1}-\dfrac{1}{L(t)-u_2}\right)}{A\sigma^2(u_1-u_2)}dL=dt \tag{60}$$

Integrating (60) with respect to t, we have

$$\frac{1}{A\sigma^2(u_1-u_2)}\int\frac{dL}{L(t)-u_1}-\frac{dL}{L(t)-u_2}=\int dt$$

$$\int\frac{dL}{L(t)-u_1}-\frac{dL}{L(t)-u_2}=A\sigma^2(u_1-u_2)\int dt+M_2$$

$$\ln(L(t)-u_1)-\ln(L(t)-u_2)=A\sigma^2(u_1-u_2)t+\ln M_2$$

$$\ln\frac{(L(t)-u_1)}{(L(t)-u_2)}=A\sigma^2(u_1-u_2)t+\ln M_2$$

$$\ln\left(\frac{L(t)-u_1}{L(t)-u_2}\right)-\ln M_2=A\sigma^2(u_1-u_2)t$$

$$\frac{L(t)-u_1}{L(t)-u_2}=e^{M_2A\sigma^2(u_1-u_2)t}$$

$$L(t)-u_1=(L(t)-u_2)e^{M_2}e^{A\sigma^2(u_1-u_2)t}$$

$$L(t)-u_1=(L(t)-u_2)M_3e^{A\sigma^2(u_1-u_2)t}$$

$$L(t)-u_1=L(t)e^{A\sigma^2(u_1-u_2)t}-u_2M_3e^{A\sigma^2(u_1-u_2)t}$$

$$L(t)-L(t)e^{A\sigma^2(u_1-u_2)t}=u_1-u_2M_3e^{A\sigma^2(u_1-u_2)t}$$

$$L(t)(1-e^{A\sigma^2(u_1-u_2)t})=u_1-u_2M_3e^{A\sigma^2(u_1-u_2)t}$$

$$L(t)=\frac{u_1-u_2M_3e^{A\sigma^2(u_1-u_2)t}}{1-e^{A\sigma^2(u_1-u_2)t}}$$

But L(T) = 0

$$0=\frac{u_1-u_2M_3e^{A\sigma^2(u_1-u_2)t}}{1-e^{A\sigma^2(u_1-u_2)t}}$$

$$M_3=\frac{u_1}{u_2e^{A\sigma^2(u_1-u_2)t}}$$

$$L(t)=\frac{u_1-u_2\left(\dfrac{u_1}{u_2e^{A\sigma^2(u_1-u_2)T}}\right)e^{A\sigma^2(u_1-u_2)t}}{1-\dfrac{u_1}{u_2e^{A\sigma^2(u_1-u_2)T}}e^{A\sigma^2(u_1-u_2)t}}$$

$$L(t)=\frac{u_1-u_1e^{A\sigma^2(u_1-u_2)(t-T)}}{1-\dfrac{u_1}{u_2}e^{A\sigma^2(u_1-u_2)(t-T)}} \tag{61}$$

Let $k_1 = \dfrac{-B+\sqrt{B^2}-4AC}{2A\sigma^2}$ ' $k_2 = \dfrac{-B-\sqrt{B^2}-4AC}{2A\sigma^2}$ and $G_t = \dfrac{k_1 - k_2 e^{2\beta^2(k_1-k_2)(t-T)}}{1-\dfrac{k_1}{k_2}e^{2\beta^2(k_1-k_2)(t-T)}}$ (62)

Then (62) becomes

$$L(t) = \frac{G(t)}{\sigma^2} \tag{63}$$

Substituting (63) into (57), and solving for K, we have

$$K(t) = \frac{1}{2}e^{\{(K_1\beta(2\beta+1)-2(r-\alpha)(T-t)\}}\left\{\frac{K_2-K_1}{K_2-K_1e^{2\beta^2(K_1-K_2)(t-T)}}\right\}\frac{2\beta+1}{2\beta}$$

Hence the optimal strategy for the investor under quadratic utility is given as

$$l^*(t) = \left(\frac{\sigma^2 S^{2\beta+1}zh_s - z(\mu-r)h}{\sigma^2 s^2\beta} = \frac{\sigma^2 S^{2\beta+1}(\dfrac{g-a(t)}{h})(-2\beta s^{-2\beta-1}n_x)}{\sigma^2 s^2\beta} - \frac{\dfrac{g-a(t)}{h}(\mu-r)h}{\sigma^2 s^2\beta}\right)$$

$$l^*(t) = \frac{g-a(t)}{h}\frac{\left[-2\beta G(t)-(\mu-r)\right]}{\sigma^2 s^2\beta} = \left(\frac{-2\beta G(t)}{\mu-r}-1\right) \tag{64}$$

Where $G_t = \dfrac{k_1 - k_2 e^{2\beta^2(k_1-k_2)(t-T)}}{1-\dfrac{k_1}{k_2}e^{2\beta^2(k_1-k_2)(t-T)}}$

$$k_1 = \frac{-B+\sqrt{B^2}-4AC}{2A\sigma^2} \text{ ' } k_2 = \frac{-B-\sqrt{B^2}-4AC}{2A\sigma^2}$$

$$A = -2\beta^2, B = 2\beta(2r-\mu), C = -(\mu-r)^2$$

4.2 Numerical Simulation and Discussion

In this section, we present some numerical simulations on the effect of some market parameters on the optimal portfolio distribution and discuss the graphs presented.

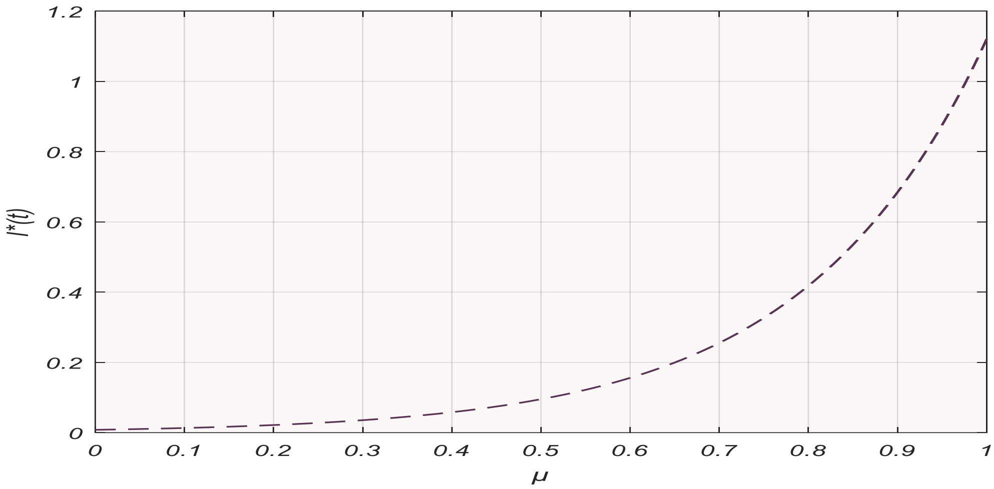

Fig. 1: Evolution of optimal portfolio distribution with appreciation rate of the risky asset

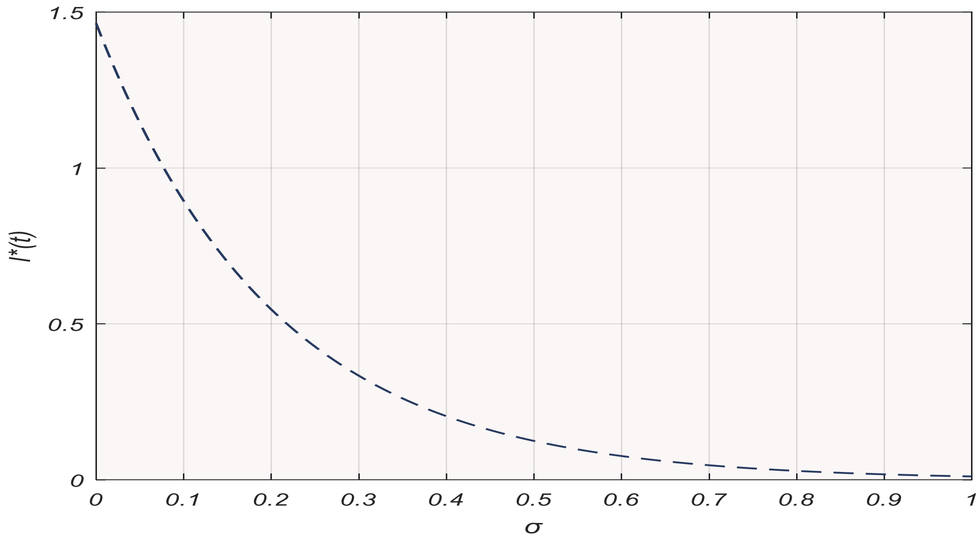

Fig. 2: Evolution of optimal portfolio distribution with instantaneous volatility of the risky asset

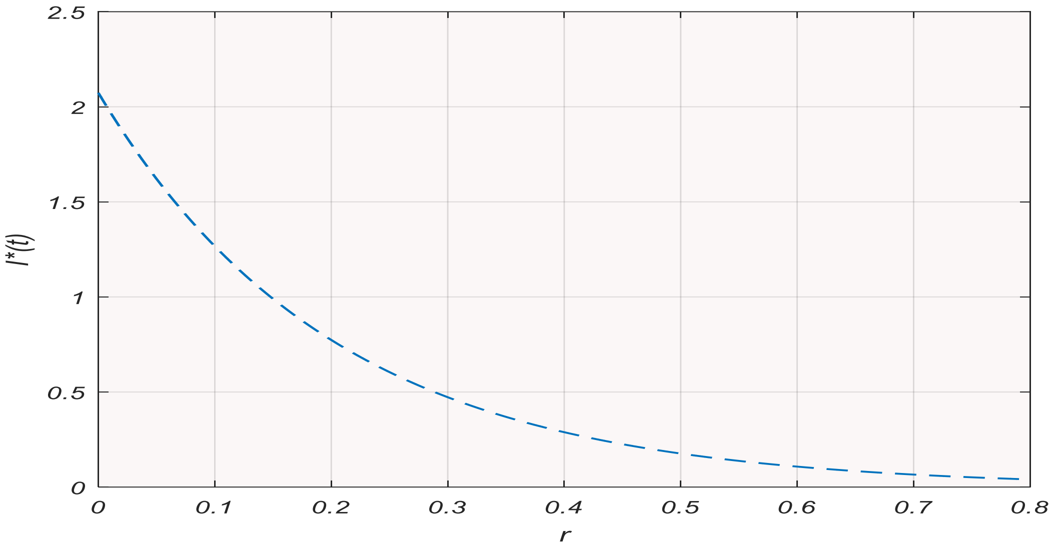

Fig. 3: Evolution of optimal portfolio distribution with predetermined interest rate

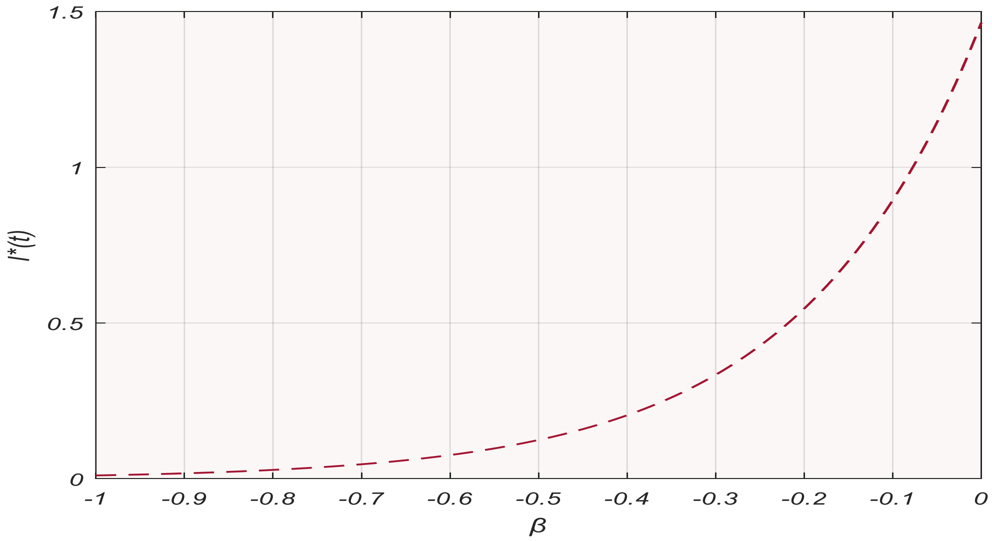

Fig. 4: Evolution of optimal portfolio distribution with elasticity parameter of the risky asset

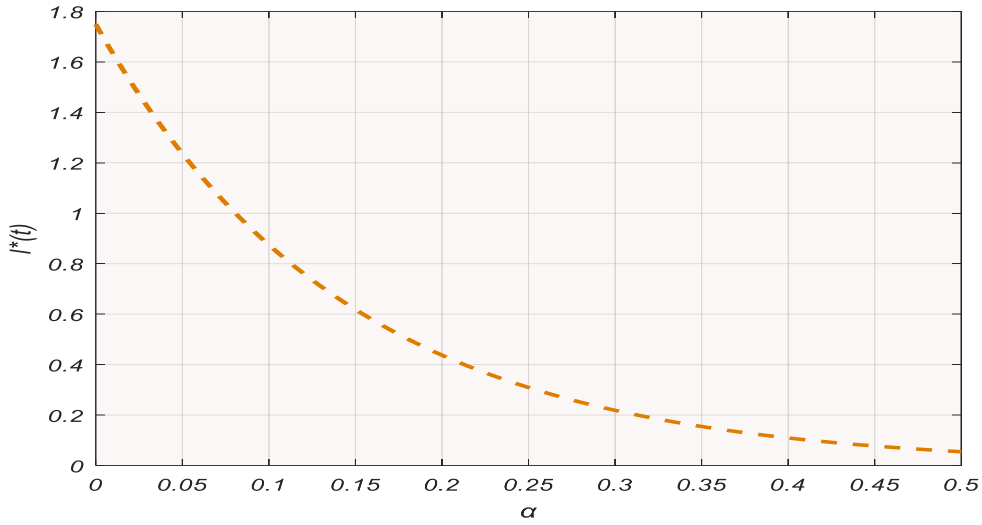

Fig. 5: Evolution of optimal portfolio distribution with tax on the risky asset

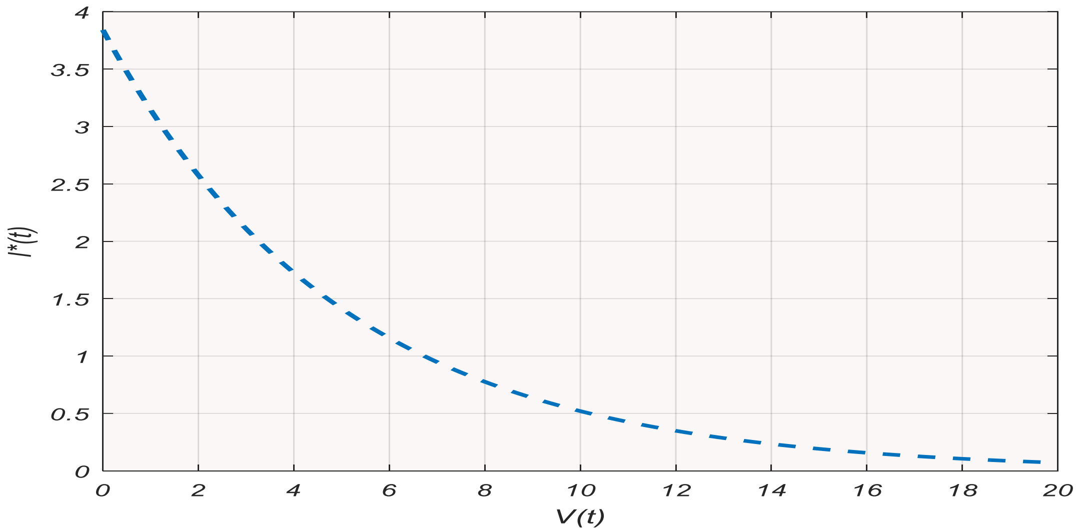

Fig. 6: Evolution of optimal portfolio distribution with wealth of the investor

In Fig. 1, the graph of the optimal portfolio distribution against the appreciation rate of the risky asset is presented; the graph shows that the optimal portfolio distribution is an increasing function of the appreciation rate of the risky asset. The graph implies that any asset with a higher appreciation rate will naturally be appealing and attractive to an investor; hence the investor may be willing to commit more of his resources to such an asset with the expectation of more returns and vice versa. In Fig. 2, the graph of optimal portfolio distribution against the instantaneous volatility is presented; the graph shows that the optimal portfolio distribution is inversely proportional to the instantaneous volatility of the risky asset. This implies that the higher the instantaneous volatility of the risky asset, the higher the risk involved in the investment in such asset. Hence this may create more fears in the mind of the investor toward investing in the risky asset, furthermore, reduce the proportion of the investor's wealth in risky asset. In Fig. 3, the graph of the optimal portfolio distribution with the predetermined interest rate is presented; the graph shows that the optimal portfolio distribution is a decreasing function of the predetermined interest rate. The graph implies that a member with large funds prefers to invest where there is a lesser risk since they may not want to lose what they have gathered already.

In Fig. 4, the graph of optimal portfolio distribution for the risky asset with elasticity parameter of the risky asset is presented; the graph shows that the proportion of the risky asset increases with an increase in the elasticity parameters. The implication of this graph shows that higher elasticity leads to a decrease in the instantaneous volatility of the risky asset and from Fig. 2, a decrease in instantaneous volatility leads to an increase in optimal portfolio distribution. In Fig. 5, the graph of optimal portfolio distribution for the risky asset with the tax on invested fund size is presented; the graph shows that optimal portfolio distribution for the risky asset decreases with an increase in tax on the invested fund. This graph implies that members may be discouraged from investing in highly taxed investments and vice versa. In Fig. 6, the graph of optimal portfolio distribution for the risky asset with the investor's wealth is presented. We observed that the optimal portfolio distribution is a decreasing function of the wealth function. This implies that an investor will invest more in risky assets if their wealth is small and vice versa.

5.0 Conclusion

In this study, we have examined various stochastic volatility models, including the geometric Brownian motion (GBM), constant elasticity of variance (CEV), modified CEV (M-CEV), and Heston volatility models. Through the formulation of Hamilton-Jacobi-Bellman (HJB) equations and the application of the Legendre transformation method, we have derived optimal portfolio distributions under each model, considering a quadratic utility function.

Our numerical simulations have provided valuable insights into the impact of market parameters on optimal portfolio distribution, including appreciation rate, volatility, interest rate, elasticity parameter, tax, and investor's wealth. The results highlight the dynamic nature of financial markets and the importance of considering various factors in portfolio optimization and risk management strategies. Based on the results obtained and the scope limitation of the study, the recommendation made are diversification, continuous monitoring of market conditions, utilization of advanced stochastic volatility models, and the need for further research to enhance investment decision-making processes.

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Availability of data and materials

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Authors' Contributions

The first author modelled and solve the problem, the second author carried out the numerical simulations and both authors analysed the results and proof read the paper.