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An Engineering Approach to Mathematical Modelling

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Abstract*: This paper presents mathematical modeling as an engineering tool for fixing solutions to certain engineering problems and shows that there exist a relationship between variables.Data analysis of experimental results to obtain a valid relationship between variables to establish a dependable conclusion is admitted as a major challenge that may hinder the solutions to some scientific problems. In this review, the term, model is regarded as an imitation of something on a smaller scale, for example, a physical model. There is a mathematical model that stands as a mathematical representation of a set of relationships between variables or parameters. The act of constructing or fashioning a model of something or finding a relationship between variables is called modeling. In practice, a relationship can be found between two or more variables. For instance, intake versus chloride concentration, Water Related Diseases (WRD) against Total Coliform Bacteria (TCB), weight versus height of students, or temperature against specific heat of a body. When simultaneous measurements are taken on two or more variables, the next approach is to plot the data on a scatter diagram to obtain a rough estimate of the relationship (if any) between them. In principle, mathematical modeling can take different forms but this paper emphasized statistical methods of least squares (Regression and Correlation method). The specific objective of this research work is to express this relationship between variables in mathematical form by obtaining an equation connecting them in form of a model, get the model verified and give a detailed procedure on model calibration for such a model to be used for future prediction.*

Keywords*: Variables, Mathematical modelling, Model calibration, verification and prediction*

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1.0 Introduction

In everyday life, the word model may be taken as an imitation of something on a smaller scale. This may translates to an example of a physical model (Beard, 2015, 2016; Chatfield, 2013). There also exist a mathematical model that stands as a mathematical representation of a set of relationships between variables or parameters. The act of constructing or fashioning a model of something or finding a relationship between variables is called modeling. Because of limited resources, undue waste of resources and time should be voided through bthe design of suitable models. The process of mathematical modeling and prediction facilitate check on how effectively limited field data are put to use in decision making. The trend in modeling involves the correlation of existing records (data), establishment of the relationship between variables through a mathematical equations, the calibration of such equations by the assignment of values to associated constants, and the adoption of such equation(s) for forecasting or in the prediction of expected or unexpected events. Prediction is a vital tool in the making of useful decision-through the examination of various responses obtained from changes in controlled variables (Chow, 2015,2017).

Although mathematical models can assume different forms, in this revew, emphasis is concentrated on the statistical methods of least squares. That is the regression and correlation method. These models are handy and come in small packages in many personal computers (Pc) or can be bought and installed into the Pc as soft wares. Mathematical modeling may be taken as the mathematical representation of physical systems, for instance, surface and/or groundwater system(s) known as governing equations and the solution of such equations by analytical or numerical methods via a computer application.

Mathematical models have wide applications in virtually all fields of human endeavors; for instance in engineering, medicine, business planning, warfare, demography and space technology. Many models in operations research or engineering are constructed to aid decision-making. There are different classes of models available in the literature and their names are tied to the method of derivation examples of such models are as presented next (Davies, 2014,2017).

The modelling of water quality parameter is significant because of the importance of water and the increasing concerned about water pollution (Eddy *et al*., 2004)

2.0 Theoretical models

Theoretical model may be taken as an exposition of the abstract principles of a subject (in science or art) which is in contrast to the practical aspects. The theoretical model is based on theories rather than data. In theoretical modeling, changes as important features of a problem are incorporated in the stages of modeling. . Most of the basic scientific equations in use are based on the definition or fundamental of mass (i) conservation of mass principles (first principle); namely (ii) conservation momentum (iii) Conservation of energy (Fisher. and Tippett, 2018).

$$\frac{dE_t}{dt} = \frac{d\mathrm{H}}{dt} + \frac{dw}{dt} \quad (1)$$

In which E_t denotes the total energy of the system, H represents heat added; and W is the work done on the system. Based on the existing theoretical models, several additional models have been derived. The quality of the resultant model can be improved upon if more assumptions or features are incorporated into the modeling process (Dodge, 2016)

2.1. Empirical models

An empirical model is essentially the same as the experimental model except that in an empirical model, the data may not necessarily come from experiments (Foster, 2017). For example, a researcher may wish to investigate the relationship between rainfall amount, duration and intensity, such a relationship may be expressed according to equation 2 form with intensity inversely proportional to duration ;

$$i = \frac{A}{B+t} \quad (2)$$

Or in a non-linear form (power-law) of the form

$$i = aR^b \quad (3)$$

where i is the intensity of the is rainfall in mm/h , t is the duration in minutes (hours), a and b are the rainfall regional constants, R is the return period or frequency (year) while A and B are regional constants.

Empirical modeling involves the use of data from field measurements or data from questionnaires rather than data from laboratory experiments. The field data are observations that have regular occurrences such as temperature, rainfall, wind speed, hours of sunshine and so on. Empirical models are data-driven (Cochran and Cox, 2016; Cox, 2015):.

2.2 *Experimental models*

Occasions may occur in which a modeler wishes to establish the relationship between certain variables. Through information obtained from experiment instead of those that are based on certain assumptions (Blom, 2014). Data obtained from the experiment may be useful for curve fitting within some ranges. The model constructed based on data collected from experiments is called an experimental model. For example, in a laboratory experiment reported by Momoh (2004), the effect of paper waste on biogas production from a fixed amount of water hyacinth and cow dung was carried out at room temperature for over 60 days. The relationship between the total solids concentration y(% in 250 ml of water) and paper wastes x (% of total solids) in each digester showed a polynomial model of the form.

$$y = 3.8448 + 0.0615x - 0.0011x^2 + 3E - 05x^3 \quad (4)$$

2.3 *Derived model*

Models that are obtained from existing models (theoretical models) by way of expansion, substitution, or manipulation of some terms are called derived models. For instance, the conservation of momentum or Newton's second law of motion states that the change of momentum per unit time on a body is equal to the resultant of all external forces acting on the body. As shown in the following equations (equations 5 to 7)

$$\sum F = \frac{d(mv)}{dt} \quad (5)$$

$$\sum F = m\frac{dv}{dt} + v\frac{dm}{dt} \quad (6)$$

$$\sum F = m\frac{dv}{dt}(\,if\ m = constant) \quad (7)$$

where $\sum F$ is the net force acting on a body, m is the mass of the body, *v* is the velocity in m/s, t is the time in second and $\frac{dv}{dt}$ is the acceleration or the rate of change of momentum with time. Equation 6 can be used to derive the terminal velocity of a free-falling body near the earth's surface. The net force is taken to comprise of two opposing forces, namely downward pull of gravity F_D and the upward air resistance, Fu. The substitution of these parameters to equation 7 yields equation 8 as follows:

$$F_D + F_U = m\frac{dv}{dt} \quad (8)$$

However, F_D and F_U in equation 7can be evaluated as F_D = mg and F_U = Kv (K is the constant that is the drag velocity, Km/s) , which upon substitution generates equation 9

$$m\frac{dv}{dt} = mg - kv \quad (9)$$

The division of equation 9 by the mass of the falling body (m), then we have;

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

(10)Equation 10 is a derived model that can be solved to obtain the velocity of a free-falling body at any time (Benson, 2015)

3.0 Linear regression modeling of variables

The water quality data in Table 1 was used to fit a linear regression model and estimate the concentration of nitrate when TDS is 470.0 mg/l. The estimated value was compared with the actual value.

Table 1:Concentration of nitrate and total dissolved solid

TDS(mg/l)	64.0	81.5	750.0	680.0	530.0	470.0	280.0
Nitrate(mg/l)	0.138	0.42	38.75	35.85	32.70	30.90	23.5

Table 2:: Solution to the problem

TDS, x	**Nitrate, y**	x^2	xy
64.0	0.138	4,096.0	8.832
81.5	0.420	6,642.25	34.23
750.0	38.75	562,500.0	29,062.5
680.0	35.85	462,400.0	24,378.0
530.0	32.70	280,900.0	17,331.0
470.0	30.90	220,900.0	14,523.0
280.0	23.50	78,400.0	6,580.0
2,855.5	**162.258**	**1,615,838.25**	**91,917.562**

3.1 Evaluation of the Coefficients

$$a_1 = \frac{n\sum xy - \sum y \sum x}{n\sum x^2 - (\sum x)^2} \quad (11)$$

$$= \frac{7(91{,}917.562) - (162.258)(2{,}855.5)}{7(1{,}615{,}838.25) - (2{,}855.5)^2}$$

$$= 0.05705$$

$$a_0 = \frac{\sum y - a_1 \sum x}{n} \quad (12)$$

$$= \frac{162.258 - (0.05705)(2{,}855.5)}{7} = -0.0912$$

The linear regression model of the above variables can be written as:

$$y = -0.0912 + 0.05705x$$

3.2 Verification of the model

Evaluating Nitrate for TDS of 420mg/l.$y\ (Nitrate) = -0.0912 + 0.05705x$
$= -0.0912 + (0.05705 \times 420)$
$= 26.722mg/l.$

Comparing the estimated and actual, there is a difference in values. Thus, there is a need for the variables to remodel in order to get an exact value or a value that is very close to the actual value.

3.3 Quadratic regression model

Consider a polynomial of the form given in equation

$$y = a_0 + a_1x + a_2x^2 \quad (13)$$

Given the n set of measurements, the squares estimates of a_0, a_1 and a_2 can be obtained in a similar way to that for linear regression. The sum of the squared deviations of the observed values of y from the predicted values is given by;

$$s = \sum(y - a_0 + a_1x + a_2x^2)^2 \quad (14)$$

Equation 14 can be minimized by setting its partial derivatives for a_0, a_1 and a_2 equal to zero, to obtain the following three simultaneous Equations;

$$\sum y = a_0 n + a_1 \sum x + a_2 \sum x^2 \quad (15)$$

$$\sum x^2 y = a_0 \sum x + a_1 \sum x^2 + a_2 \sum x^3 \quad (16)$$

$$\sum xy = a_0 \sum x^2 + a_1 \sum x^3 + a_2 \sum x^4 \quad (17)$$

Equation 17 is called the normal equations for the least square parabola and when solved gives a solution to the values of a_0, a_1 and a_2.

Consider the water quality parameters used in modeling in the regression model above. Using the above data compute;

(i) A quadratic regression model of the same data.

(ii) An estimate of the nitrate concentration for a known TDS value of 470 mg/l and compare the estimate with the actual value. The solution is presented in Table 3 which indicate, x and y variables to represent TDS and nitrate respectively.

Table 3: Parameters associated with equations 15 to 17

TDS x	**Nitrate** y	x^2	x^3	x^4	xy	x^2y
64.0	0.138	4.096.0	262,144.0	16,777,216.0	8.832	565.248
81.5	0.420	6,642.25	541,343.38	44,119,485.1	34.23	2,789.75
750.0	38.75	562,500.0	421,875,000	3.164×10^{11}	29,062.5	21,796,875
680.0	35.85	462,400.0	314,432,000	2.138×10^{11}	24.378.0	16,577,040
530.0	32.70	280,900.0	148,877,000	7.890×10^{10}	17,331.0	9,185,430
470.0	30.90	220,900.0	103,823,000	4.8797×10^{10}	14.523.0	6,825,810
280.0	23.50	78,400.0	21,952,000	6.1466×10^{9}	6,580.0	1,842,400
∑2,855.5	**∑162.26**	**∑1,615,838.25**	**∑1,011,762,487**	**∑6.6418×10^{11}**	**∑91,917.562**	**∑56,230,910.0**

Direct substitution from the Table into Equation 17 gives the following equations:

$162.26 = 7a_0 + 2855.5a_1 + 1{,}615{,}838.3a_2$

$91{,}917.56 = 2855.5a_0 + 1{,}615{,}838.3a_1 + 1{,}011{,}762{,}487a_2$

$56{,}230.91 = 1{,}615{,}838.3a_0 + 1{,}011{,}762{,}487a_1 + 6.641\times10^{11}a_2$

The division of the first, second and third factors in equation 17 by the factors; 7, 2855.5 and 1615838.3 and putting it into matrix format yields.

$$\begin{bmatrix}1 & 407.97 & 230{,}834.04\\ 1 & 565.07 & 354{,}320.60\\ 1 & 626.15 & 410{,}994.10\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix} = \begin{bmatrix}23.18\\ 32.19\\ 34.80\end{bmatrix}$$

By matrix triangulation (Gauss Elimination Technique), we have;

$$\begin{bmatrix}1 & 407.97 & 230{,}834.04\\ 0 & 157.14 & 123{,}486.60\\ 0 & 0 & 410{,}994.10\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix} = \begin{bmatrix}23.18\\ 9.01\\ -0.8256\end{bmatrix}$$

The solution to the unknown coefficients are obtained as follows:

$$a_2 = \frac{-0.825464}{9{,}586.34} = -8.6127\times10^{-5}$$

$$9.0 = 159a_1 + 123{,}486.6(-7.16712\times10^{-5})$$

$$a_1 = \frac{19.6455}{157.94} = 0.1244$$

In a similar manner $a_0 = -7.6855$

Thus, the result yields a quadratic mathematical model of the form;

$$y = -7.6855 + 0.1244x - 8.6127\times10^{-5}x^2$$

where y in the above Equation represents the concentration of nitrate and x representing the total dissolved solids (TDS) in each case.

3.4 Modification verification

This model can be verified using the same data in Table 1.3 and Nitrate concentration for a TDS value of 470mg/l. can be estimated and compared with the actual. Once the result gives the actual value within the set of variables, it does mean that the model has been verified.

$$y = -7.6855 + 0.1244x - 8.6127\times10^{-5}x^2$$

$$= -7.6855 + 0.1244(470) - 8.6127\times10^{-5}(470)^2 = 30.005mg/l$$

4.0 Conclusion

This result shows great improvement relative to the linear regression model earlier used on the same variables and has gone to verify this model.

A verified model is thus ready for calibration and calibrated model can be used for future prediction. The mathematical model presented above can be used for future prediction of the concentration of Nitrate once the solution is known.

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Conflct of Interest

The authors declared no conflict of Interest