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Chaos Synchronization Based on Linear and Adaptive Controls: Theory and Experiment

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Abstract: *In this paper, we report on the theoretical and experimental investigation of chaotic synchronization using of a single variable linear feedback and adaptive controllers. Based on the Lyapunov stability theory, theoretical approaches to the design of controls are presented, and the results are validated numerically and by employing electronic circuit experiments. We used two typical oscillators, namely, the Lorenz and Sprott chaotic systems to demonstrate our results; while off-the-shelf components on breadboard were used to experimentally implement the proposed single variable controllers. We specifically show that synchronization of two chaotic systems can be experimentally realized when the strength of the feedback exceeds a theoretically determined threshold.*

***Keywords**: Chaos, Synchronization, Linear feedback controller, Adaptive controller, single variable*

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1.0 Introduction

Synchronization is a form of collective behaviour that occur due to specific coupling or by deliberate introduction of forcing on two or more oscillators whose dynamics can be periodic or chaotic, such that their dynamics can complimentarily be adjusted, and in the course of time a common dynamics is achieved. The occurrence of chaos synchronization proposed by Pecora and Carroll in (1990) is in particular very fascinating and with wide applications in nearly all disciplines ranging from chemical reactions to biological systems; as well as in power converters, communication and networking systems, surveillance and control systems, secured at a processing (Aguilar-Lpez *et al*., 2014; Filali *et al*., 2014; Bhatnagar and Wu, 2015; Choi *et al*, 2017 ; Ren *et al.*, 2013) to name but a few. These wide varieties of applications have triggered enormous research advances on various effective synchronization approaches-which can be broadly classified as linear and non linear feedback controls proposed during the last three decades (Niu *et. al.*, 2014; Ping and Fei, 2011; Hu and Xu, 2008; Hua and Xua, 2008; Salarieh and Alasty, 2009; Guo *et al.*, 2009; Vincent and Guo, 2009; Onma *et al.*, 2016; Almatroud *et al.*, 2016; Liu *et al.*, 2010; Ricardo and Rafael, 2008; Yao *et al.*, 2014; Siddique and Rehan, 2016; Hua *et al.*, 2016; Abd *et al.*, 2017; Liu *et al.*, 2010; Ojo *et al.*, 2014). Prominent among these, we make mention of the linear state feedback method (Niu *et. al.*, 2014; Ping and Fei 2011; Hamed *et al.*, 2018), adaptive control method (Hu and Xu, 2008; Hua and Xua, 2008; Salarieh and Alasty 2009; Guo *et al.*, 2009; Vincent and Guo, 2009; Onma *et al.*, 2016; Almatroud *et al.*, 2016; Mahmoud and Abood, 2017; Liu *et al.*, 2018), observer control method (Ricardo and Rafael, 2008; Yao *et al.*, 2014; Siddique and Rehan, 2016; Hua *et al.*, 2016; Abd *et al.*, 2017), fuzzy control method (Liu *et al.*, 2010; Liu and Zheng, 2009; Hanene *et al.*, 2020), backstepping method (Ojo *et al.*, 2014; Shaohua *et al*., 2020). Among the aforementioned control strategies and many more available today for stabilization and synchronization of chaotic systems, theoretical analysis have shown that the linear state feedback (Niu *et. al.*, 2014; Ping and Fei, 2011; Hamed *et al.*, 2018; Wang and Wang, 2011) and adaptive control (Hu and Xu, 2008; Salarieh and Alasty, 2009; Guo *et al.*, 2009; Vincent and Guo, 2009; Onma *et al.*, 2016; Wang and Wang, 2011; Pallov and Sharma, 2020) are highly promising in terms of realization of simple control inputs with potential experimental applications as well as low energy cost requirement. Consequently, they have recently received renewed interest, for instance in the synchronization of Lorenz-Stenflo systems (Wang and Wang, 2011; Pallov and Sharma, 2020; Yang, 2014), Chen-Lee system (Liu and Gua, 2017; Yaping *et al.*, 2020) and a four-dimensional power system model (Shaohua *et al*., 2020). Since the beginning of the studies on chaos synchronization, the design of coupling scheme that will ensure stable synchronization was considered as a crucial research question which to-date has remained fundamentally relevant (Wang and Wang, 2011; Yang, 2014; Stefanski *et al.*, 2009; Olusola *et al.*, 2010). In one of our previous papers, we employed a combination of Lyapunov direct and Linear Matrix Inequality (LMI) methods to determine the threshold coupling for the onset of stable synchronous behaviour of unidirectionally and linearly coupled parametrically excited pendula (Olusola *et al.*, 2010). More importantly, many existing reports on non linear controls and in particular adaptive control techniques have been purely focused on theoretical analysis, without recourse to experimental implementation, to the best of our

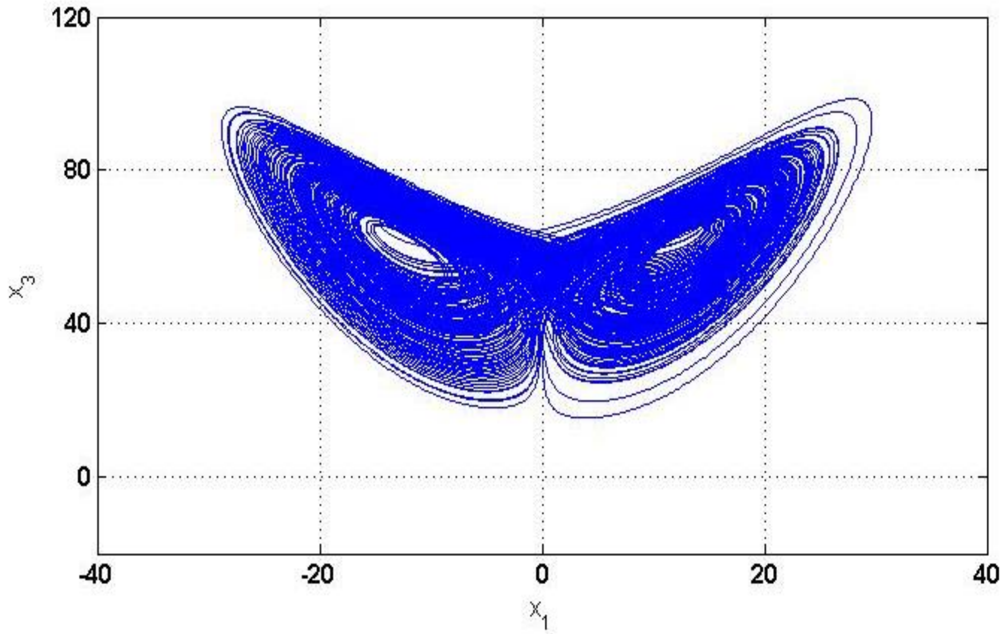

Fig. 1. (Colour online) Phase portrait of the chaotic attractor of Lorenz system with parameters a = 28, b =$\frac{8}{3}$ and c = 10

knowledge (Wang and Wang, 2011; Yang, 2014; Liu and Gua, 2017; Ni *et al.*, 2017; Chunhua *et al.*, 2020). Emphasis have been placed on experimental realizations of chaos synchronization schemes as being key to many real life chaos synchronization-based applications and concerted research efforts have been devoted to implementing various chaos synchronization schemes (Hua *et al.*, 2016; Perlikowski *et al.*, 2008; Arellano *et al.*, 2013; Mart'inez *et al.*, 2014; Ahmed *et al.*, 2017; Egunjobi *et al.*, 2018).Recently, we reported experimental evidence for synchronization via cyclic coupling (Egunjobi *et al.*, 2018). In the present paper, we provide experimental evidence for chaotic synchronization from the perspective of single variable linear state feedback control and adaptive control.

We first employ the linear state feedback approach and adaptive control technique respectively to design simple, efficient, and experimentally realizable controllers for synchronization of nonlinear oscillators using Lorenz and Sprott systems as classical oscillators. Then, the effectiveness and feasibility of the designed controllers are demonstrated numerically and validated with an electronic experiment using off-the-shelve components on the breadboard. To the best of our knowledge, synchronization of chaotic oscillators via a single variable approach has not been experimentally implemented and reported in the literature. The rest of the paper is organized as follows: In sections 2 and 3, we present analytic and numerical evidences of synchronization in identical Lorenz and Sprott oscillators respectively based on single variable adaptive, while section 4 is focused on experimental implementation. The paper is concluded in section 5.

2. 0 Synchronization of Lorenz System

2.1 Design of linear feedback controller

Based on the linear state error feedback approach, we consider the drive Lorenz system (Lorenz, 1964) as:

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1)\\ \dot{x}_2 &= bx_1 - x_2 - x_1x_3\\ \dot{x}_3 &= x_1x_2 - cx_3\end{aligned} \tag{1}$$

where a = 28, b = $\frac{8}{3}$ and c = 10 are control parameters that make the system exhibits chaotic behaviour as depicted in Fig. 1.

The controlled slave system is given as:

$$\begin{aligned}\dot{y}_1 &= a(y_2 - y_1) + u_1\\ \dot{y}_2 &= by_1 - y_2 - y_1y_3 + u_2\\ \dot{y}_3 &= y_1y_2 - cy_3 + u_3\end{aligned} \tag{2}$$

where $u_1, u_2 and\, u_3$ are control functions to be designed. The system error function is defined as;

$$e_1 = y_1 - x_1,\ \ e_2 = y_2 - x_2,\ \ e_3 = \ y_3 - x_3 \tag{3}$$

Differentiating equation (3) with respect t time, the error dynamics is

$$\dot{e}_1 = \dot{y}_1 - \dot{x}_1,\ \ \dot{e}_2 = \dot{y}_2 - \dot{x}_2,\ \ \dot{e}_3 = \ \dot{y}_3 - \dot{x}_3 \tag{4}$$

Using equation (1) and equation (2) in equation (4), we obtain:

$$\begin{aligned}\dot{e}_1 &= a(e_2 - e_1) + u_1\\ \dot{e}_2 &= be_1 - e_2 + x_1x_3 - y_1y_3 + u_2\\ \dot{e}_3 &= -ce_3 + y_1y_2 - x_1x_2 + u_3\end{aligned} \tag{5}$$

Theorem 1

For drive system (1) and response system (2), if we choose the controller as:

$$u_1 = -ke_1, u_2 = 0, u_3 = 0 \tag{6}$$

where k is a feedback plus satisfying the condition

$$k > \frac{(a+b-x_3)^2c + x_2^2 - 4ac}{4c} \tag{7}$$

then, the response (2) and drive (1) systems can be fully synchronized withthe proposed controller (6).

Proof

Suppose the Lyapunov function for system (5) is given as:

$$V_1 = \frac{1}{2}(x_1^2 + e_2^2 + e_3^2) \tag{8}$$

We can express the time derivative of V_1 along the trajectories of systemgiven by equation (5) as:

$$\begin{aligned}\dot{V}_1 &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3\\ &= -(a + k)e_1^2 - e_2^2 - ce_3^2 + (a + b - x_3)e_1e_2 + e_1e_3x_2\\ &= -e^TPe\end{aligned} \tag{9}$$

where $[e_1, e_2, e_3]^T$ and

$$P = \begin{pmatrix} a+k & \frac{-(a+b-x_3)}{2} & -\frac{x_2}{2} \\ \frac{-(a+b-x_3)}{2} & 1 & 0 \\ -\frac{x_2}{2} & 0 & c \end{pmatrix} \quad (10)$$

If the feedback and the update *k* satisfies the following conditions;

$$n_1 > 0, n_2 > 0 \; and \; n_3 > 0 \quad (11)$$

where

$$n_1 = a + k$$
$$n_2 = n_1 - \frac{(a+b-x_3)^2}{4} \quad (12)$$
$$n_3 = -\frac{x_2^2}{4} + c(a+k) - \frac{c(a+b-x_3)^2}{4}$$

then, $\dot{V}_1 \leq 0$ can be obtained. Substituting equation (12) into Equation (11) and after some algebraic manipulations, we obtain that there must be a constant *k* that satisfies equation (7) and $\dot{V}_1 \geq 0$ such that $> \frac{c(a+b-x_3)^2 + x_2^2 - 4ac}{4c}$. From Lyapunov stability theory, it is clear that the error dynamical system (4) is stable at the origin (0, 0, 0) asymptotically. This implies that the drive-response system (1) and (2) with linear feedback control (6) can be synchronized. The proof is thus completed.

2.2 Adaptive control design

Theorem 2

Considering the drive system (1) and response system (2), if the adaptive controller function is chosen such that:

$$u_1 = -ke_1, \; u_2 = -(a+c)e_1, \; u_3 = 0 \quad (13)$$

and the feedback gain *k* is updated by the following law;

$$\dot{k} = \alpha e_1^2 \quad (14)$$

where $\alpha$is a positive constant, then the systems, equations 1 and 2 with the controller (13) and equation (14) are in synchronized state.

Proof

The error dynamics of the drive-response and adaptive controlled Lorenz systems can be written as;

$$\dot{e}_1 = a(e_2 - e_1) - ke_1$$
$$\dot{e}_2 = (b - a - c)e_1 - e_2 + x_1x_3 - y_1y_3 \quad (15)$$
$$\dot{e}_3 = -ce_3 + (y_1e_2 - x_2e_1).$$

By choosing a Lyapunov function (*V*) for equation (15) as:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) + \frac{1}{2\alpha}(k - \bar{k})^2, \quad (16)$$

where $\bar{k}$ *is the estimate of* k;

the time derivative of *V* along the solution of (15) is given as:

$$\dot{V}_1 = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 - (k - \bar{k})e_1^2, \quad \leq -(a + \bar{k})e_1^2 - e_2^2 - ce_3^2 + (a - b - x_3)e_1e_2 + e_1e_3x_2 \quad = -e^TPe \quad (17)$$

where $[|e_1|, |e_2|, |e_3|]^T$ and

$$P = \begin{pmatrix} a+k & \frac{(x_3-b+c)}{2} & -\frac{x_2}{2} \\ \frac{(x_3-b+c)}{2} & 1 & 0 \\ -\frac{x_2}{2} & 0 & c \end{pmatrix} \tag{18}$$

If the feedback plus k satisfies the condition:

$$n_1 > 0, n_2 > 0 \; and \; n_3 > 0 \tag{19}$$

where

$$\begin{aligned} n_1 &= a + \bar{k}, \\ n_2 &= n_1 - \frac{(x_3-b+c)^2}{4} \\ n_3 &= \frac{x_2^2}{4} + c\left[\left(a+\bar{k}\right) - \frac{(x_3-b+c)^2}{4}\right], \end{aligned} \tag{20}$$

then we can show that $\dot{V}_1 \leq 0$ is negative definite. Substituting equation (20) into Equation (19) and after some algebraic manipulations, one readily obtains $\bar{k} > \frac{c(x_3-b+c)^2+x_2^2-4ac}{4c}$. Implying that there must be a constant $\bar{k}$ that satisfies equation (14) and $\dot{V}_1 \leq 0$and according to Lyapunov stability theory, the errordynamical system (15) is asymptotically stable at the origin (0, 0, 0). That is,the drive system (1) and response system (2) with the linear feedback control Equation 13 can be synchronized. The proof is thus completed.

2.3 Numerical simulation

In order to verify the effectiveness and feasibility of the single variable nonlinear controller (6) and adaptive controller (13) for the Lorenz system, the fourth-order Runge Kutta (ode45) routine was employed with initial conditions [$x_1(0)$ = 0.3, $x_2(0)$ = 0.2, $x_3(0)$ = −0.2] for drive system and [$y_1(0)$ = 0.03, $y_2(0)$ = 0.02, $y_3(0)$ = −0.02] for response system. The parameters were fixed as follows: *a = 10, b = 28 and c = 8/3* to ensure that the system is in a chaotic dynamics state. We numerically solved drive system (1) and response system (2) with controllers $u_i(t)$, as defined in equation (6) and equation (13), respectively. The error dynamics variables are chaotic before controller activation. However, on the activation of linear state feedback controller (6) coupled with adaptive controller (13) as shown in Figs. 2 and 3, respectively, the error dynamics converge to the stable equilibrium point (0, 0, 0) as time $t \to \infty$, implying that system (1) and system (2) are in the state of synchronization.

3.0 Synchronization of Identical Sprott Systems

In this section, we propose simple nonlinear controllers for synchronizing identical Sprott systems [Sprott, 2000] We take the drive system as

$$\begin{aligned} \dot{x}_1 &= a x_2 \\ \dot{x}_2 &= x_1 + x_3 \\ \dot{x}_3 &= x_1 + x_2^2 - x_3 \end{aligned} \tag{21}$$

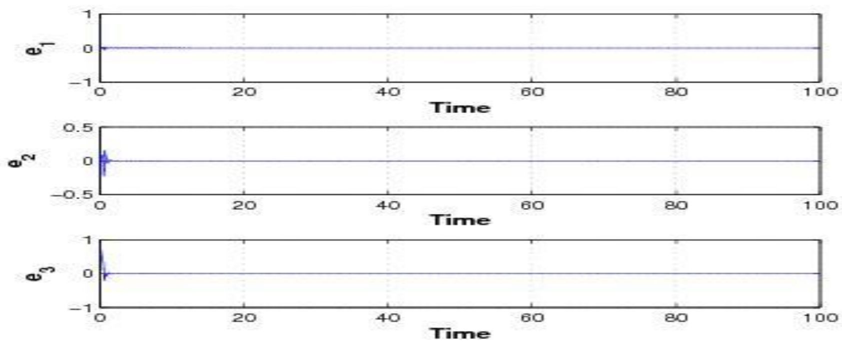

Fig. 2. (Colour online) Time response curve for synchronization errors of two identical Lorenz systems with the linear state feedback controller in (6) activated at*t* ≥ *20s*

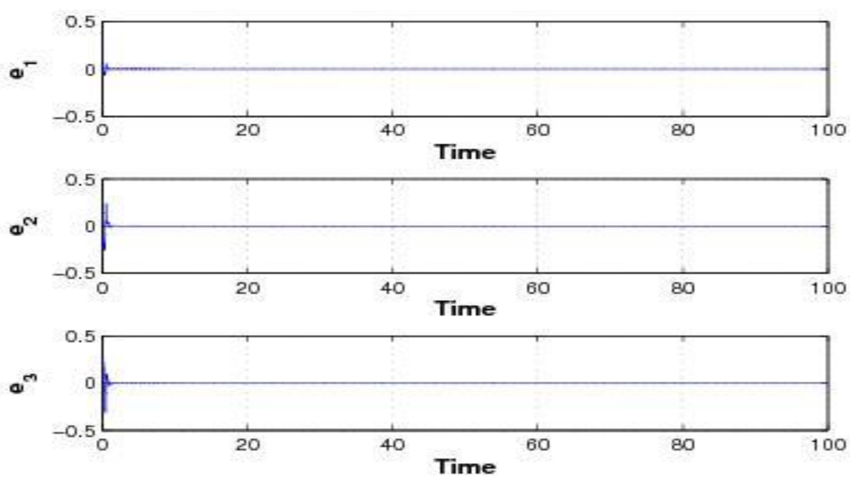

Fig. 3. (Colour online) Time response curve for synchronization errors of two identical Lorenz systems with the adaptive controller in (13) activated at *t* ≥*20s*

where a is the control parameter and when $a = 0.2$, the system exhibits chaotic behaviour as shown in Fig. 4.

The controlled slave system is given as:

$$\dot{y}_1 = ay_2 + u_1$$
$$\dot{y}_2 = y_1 + y_3 + u_2 \qquad (22)$$
$$\dot{y}_3 = y_1 + y_2^2 - y_3 + u_3$$

where $u_1, u_2 and\ u_3$ are control functions to be designed. The system error function is defined as;

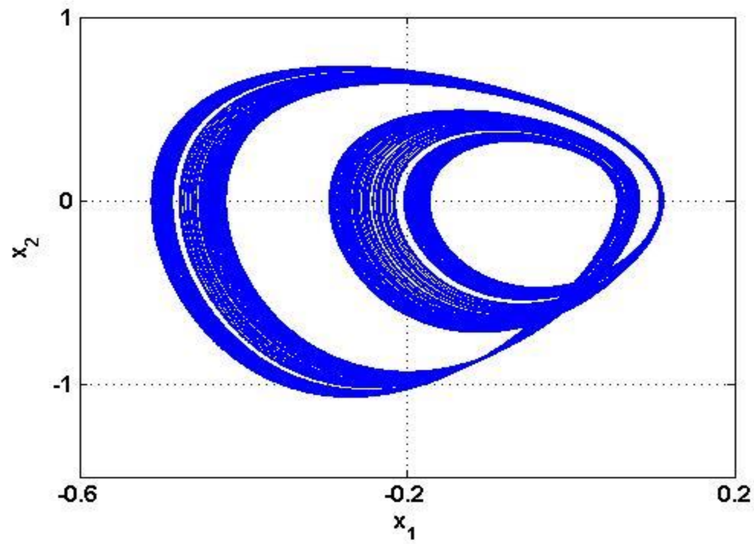

Fig. 4. (Colour online) Phase portrait of the chaotic attractor of Sprott system withparameter *a = 0.2*.

$e_1 = y_1 - x_1,\ \ e_2 = y_2 - x_2,\ \ e_3 = \ y_3 - x_3$ (23)

The time derivatives of the error functions which is the error dynamical system is obtained using equation (21) and equation (22) in equation (23); and is given as

$$\dot{e}_1 = ae_2 + u_1$$
$$\dot{e}_2 = e_1 + e_3 + u_2 \quad (24)$$
$$\dot{e}_3 = e_1 + x_2e_2 - y_2e_2 - e_3 + u_3$$

3.1 Linear state feedback control for synchronization of sprott system

Theorem 3

For drive system (21) and response system (22), if we choose the controllers $u_i(t)$ as follows:

$u_1 = -ke_1, u_2 = 0, u_3 = 0,$ (25)

where k is the feedback plus, satisfying the condition; k > 0, then the response system (22) associated with the proposed controller (25) and the drive system(21) can be synchronized.

Proof

Let the Lyapunov function for the error system (24) be given as:

$$V_1 = \frac{1}{2}(x_1^2 + e_2^2 + e_3^2) \quad (26)$$

The time derivative of V1 along the trajectories of error dynamics is given as;

$$\dot{V}_1 = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3$$
$$= -(a+k)e_1^2 - e_2^2 - ce_3^2 + (a+b-x_3)e_1e_2 + e_1e_3x_2 \quad (27)$$
$$= -e^TPe$$

where $e = [e_1, e_2, e_3]^T$ and

$$P = \begin{pmatrix} k & -\frac{(a+1)}{2} & -\frac{1}{2} \\ \frac{(a+1)}{2} & 1 & -\frac{(1+x_2+y_2)}{2} \\ -\frac{1}{2} & -\frac{(1+x_2+y_2)}{2} & 1 \end{pmatrix} \quad (28)$$

If the feedback plus $k$satisfies the condition;

$$n_1 > 0, n_2 > 0 \; and \; n_3 > 0 \quad (29)$$

where

$$n_1 = k,$$
$$n_2 = -(a+1)^2 \quad (30)$$
$$n_3 = \left[-\frac{k\left(x^2+y^2+1\right)^2+k(a+1)^2+(a+1)(x^2+y^2+1)}{4}\right]$$

then $\dot{V}_1 \le 0$ is negative definite can be obtained. According to the theory of Lyapunov stability, the error dynamical system (24) is asymptotically stable. That is, the drive system (21) and response system (22) with the feedback control (25) can be synchronized. The proof is completed.

3.2 Adaptive control for synchronization of Sprott system

Theorem 4

For drive system (21) and response system (22), if we choose the adaptive controller as;

$$u_1 = -ke_1, u_2 = -ae_1, u_3 = 0 \quad (31)$$

and the feedback plus k is updated by the law:

$$\dot{k} = \beta e_1^2 \quad (32)$$

where β is a positive constant, then the two Sprott systems of (21) and (22) satisfying equation (31) and equation (32) are in a state of synchronization.

Proof

The error system of the two Sprott systems under adaptive control is as follows;

$$\dot{e}_1 = ae_2 - ke_1$$
$$\dot{e}_2 = (1-a)e_1 + e_3, \quad (33)$$
$$\dot{e}_3 = e_1 + x_2e_2 + y_2e_2 - e_3).$$

We choose a Lyapunov function (V) as:

$$V = \frac{1}{2}(e_1^2+e_2^2+e_3^2) + \frac{1}{2\beta}(k-\bar{k})^2, \quad (34)$$

where $\bar{k}$ *is the estimate of* k; The time derivative of V along the solution of (33) is given as:

$$\dot{V}_1 = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 - \left(k-\bar{k}\right)e_1^2,$$
$$\le -\bar{k}e_1^2 - e_3^2 + e_1e_2 + e_1e_3 + (1+x_2+y_2)e_2e_3 \quad (35)$$
$$= -e^TPe$$

where $[|e_1|, |e_2|, |e_3|]^T$and

$$P = \begin{pmatrix} \bar{k} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{(x_2+y_2+1)}{2} \\ -\frac{1}{2} & -\frac{(x_2+y_2+1)}{2} & 1 \end{pmatrix} \quad (36)$$

If the feedback plus k satisfies the condition:

$$n_1 > 0, n_2 > 0 \; and \; n_3 > 0 \quad (37)$$

where

$$n_1 = \bar{k},$$
$$n_2 = -\frac{1}{4} \quad (38)$$

$n_3 = -\bar{k}\frac{(x^2+y^2+1)^2}{4} - \frac{1}{4}(1 + (x^2 + y^2 + 1))$

then $\dot{V}_1 \leq 0$ can be obtained. Substituting (38) into (37) and computing the inequalities, there must be a constant $\bar{k}$, that satisfies (32) and $\dot{V}_1 \leq 0$.

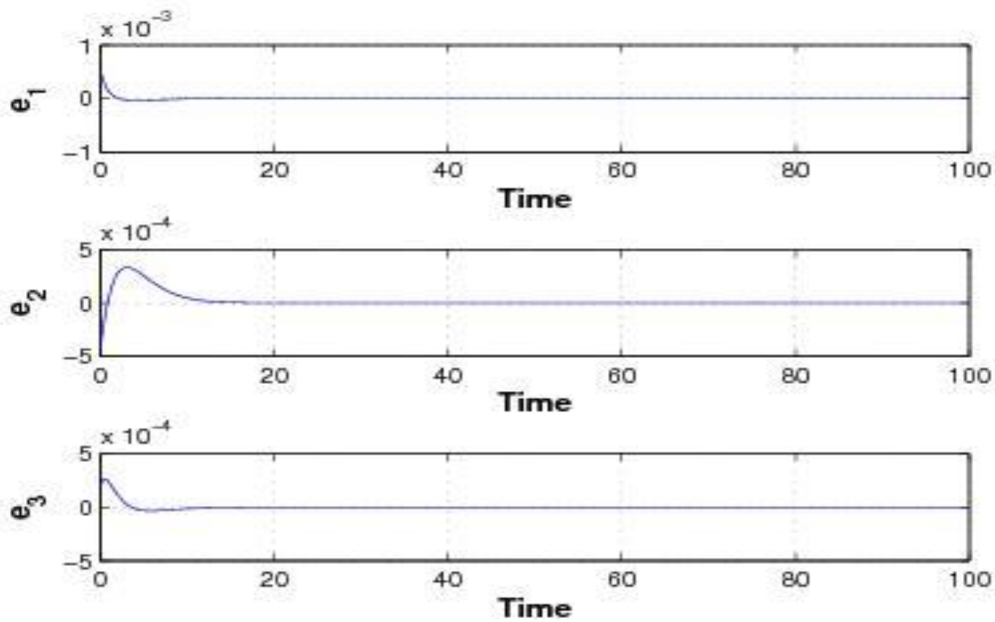

Fig. 5: (Colour online) Time response curve for synchronization errors of two identical Sprott systems with the linear state feedback controllers activated at *t* ≥ *20s*.

According to the theory of Lyapunov stability, the error dynamical system (33) is asymptotically stable. i.e., the drive (21) and response (22) systems with the adaptive controller (31) can be synchronized. The proof is thus completed.

3.3 Numerical simulation

Following the numerical procedure in section 3, we simulated the Sprott system [Sprott] using the parameter, a = 0.2 and the initial conditions of the drive system (21) and response system (22) chosen as [x_1 (0) = 0.0001, x_2(0) = 0.0001, x_3 (0) = 0.0001], and [y_1 (0) = 0.001, y_2 (0) = 0.001, y_3 (0) = 0.001], respectively. The synchronization errors between systems (21) and (22) are shown in Figs. 5 and 6. From Fig. 5, we can see that the errors $e_1(t)$, $e_2(t)$and $e_3(t)$ have been stabilized to the equilibrium point (0, 0, 0) after the linear state feedback controller (25) is activated. Thus, system (21) and system (22) are in a state of synchronization. Furthermore, the initial conditions are taken as [x_1 (0) = 0.0001, x_2 (0) = 0.0002, x_3 (0) = −0.0001], and [y_1(0) = −0.001, y_2 (0) = 0.002, e_1(0) = 0.003]; and by switching on the adaptive control given by Eq (31) with β = 1, we show in Fig. 6 the global asymptotic convergence of the error dynamics $e_1(t)$, $e_2(t)$ and $e_3(t)$ as $t \to \infty$implying that system (21) and system (22) achieve global synchronization.

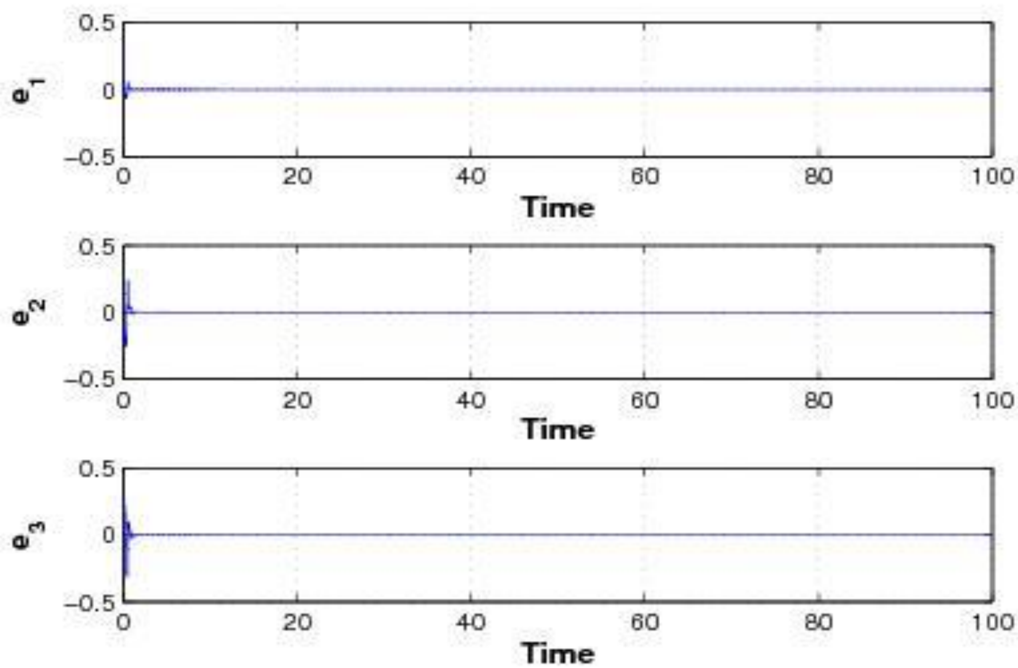

Fig. 6. (Colour online) Time response curve for synchronization errors of two identical Sprott systems with the adaptive controllers activated at $t \geq 20s$.

4.0 Synchronization: Electronic Implementation

In this section, we present the experimental evidence of synchronization in the Lorenz system and Sprott systems based on linear state feedback controllers and adaptive controllers (6), (31), (13) and (21) respectively. The electronic circuit realization of the coupled systems was done using microcontroller-based rogramming as illustrated in Fig. 7. The exadecimal form of the system code was saved in the AVR micro-controller (ATMEGA 328P-PU)EPROM memory using a micro-controller programmer (Arduino). The generated pulse width modulated (PWM) output is fed into a low pass filter to obtain continuous output waveform observed in a digital oscilloscope (Yokogawa DL9140, 5GS/s, 1GHz). The frequency of the PWM output may be varied to ensure its compatibility with the bandwidth of the oscilloscope. The timer of the microcontroller is used to vary the frequency. In Fig. 8, we display the oscilloscope traces of the phase portrait showing the chaotic attractors for = 60.3k and R = 49.9k of the (a) Sprott oscillator and (b) Lorenzsystem respectively. Notice that Figs. 8(a & b) reproduces the exact chaotic structures obtained from numerical simulations as shown in Figs. 1 and 4, respectively.

We begin the experimental verification of the single variable scheme using off-the-shelve components on a breadboard by implementing drive-response Lorenzsystems as paradigmatic oscillators, connected through a coupling resistance,R_k.We first identified the critical coupling R_{kc} = 0.05k.When $R_k$is below the critical value (say R_k = 0.01k <R_{kc} = 0.05k) as displayed in Fig. 9(a), complete synchronization could not be achieved between the two Lorenz systems.

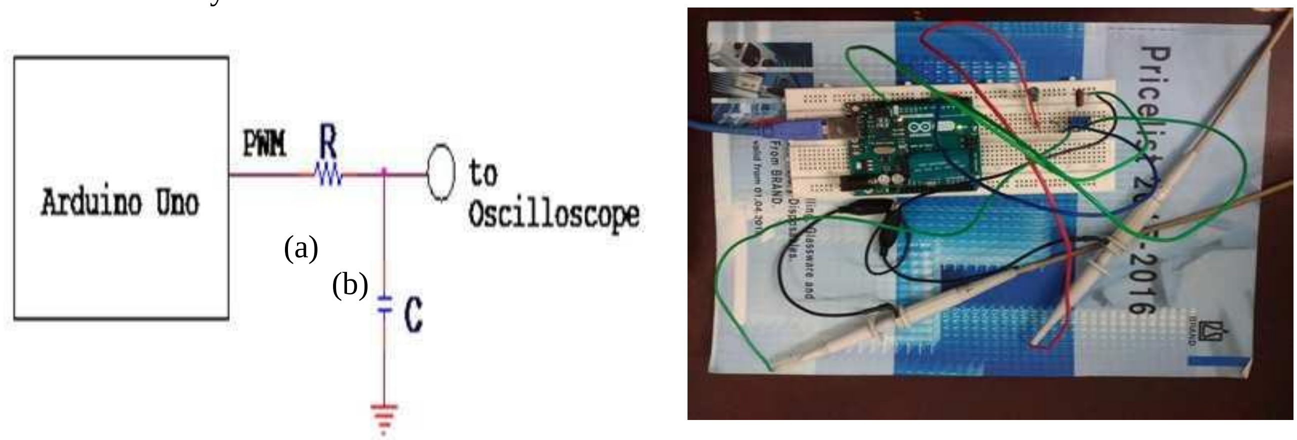

Fig. 7: (a)Schematic diagram for experimental implementation using Adruino UNO hardware (microcontroller), and (b) Experimental setup using the Adruino UNO hardware (microcontroller) to implement the differential equations.

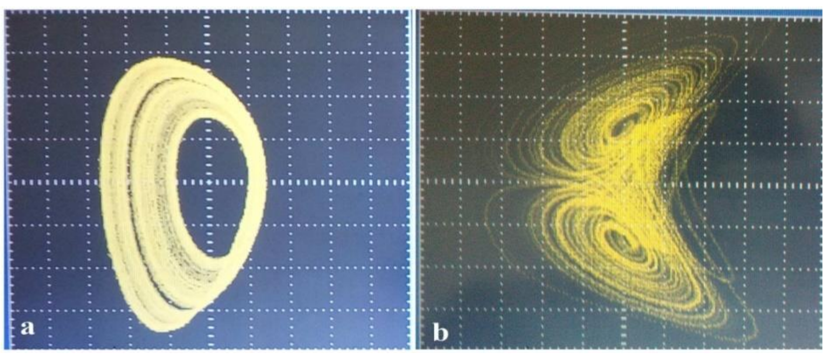

Fig. 8. (Color online) Oscilloscope traces of chaotic attractor of a: (a) single Sprott system for R = 60.3k, and; (b) single Lorenz system for R = 49.9k.

In Fig. 9(b) however, we show the result of oscilloscope traces for $x_1 vs\, x_2$ for $R_k = 0.10k > R_{kc} = 0.05k$, in which the two nearly identical Lorenz systems evolve into the synchronous state at coupling resistance. Remarkably, our result provides experimental validation for the synchronization of chaotic systems based on a single variable controller.Remarkably, previous reports on the application of single variable controllers (Wang and Wang, 2011) focused on the theoretical derivation and numerical simulation. We have, in the present study showed that by using off-the-shelve components a breadboard, synchronization via a single variable controller of two nearly identical Lorenz systems can evolve into synchronization at a coupling above the threshold value.

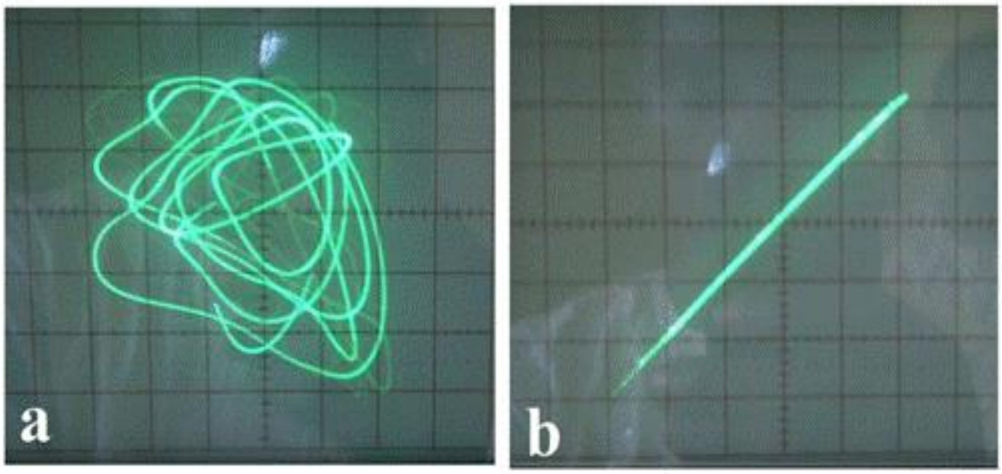

Fig. 9. Oscilloscope traces for two Lorenz systems interacting via a single variable controller $x_1 vs\ x_2$ plots for (a) $R_k = 0.01k > R_{kc} = 0.05k$ - no synchronization, (b) $R_k = 0.10k > R_{kc} = 0.05k$ - synchronization.

Furthermore, experimental verification of the feasibility and effectiveness of the adaptive control approach is provided. Here, we found a threshold resistance value R_k as R_{kc} = 0.68k below which synchronization would not be attained. In Fig. 10(a), the result of oscilloscope traces for$x_1 verse\ x_2$ are shown for R_k= 0.50k <R_{kc} = 0.68k. Clearly, the two nearly identical Lorenz systems could not attain a synchronous state at coupling resistance below the threshold value. Under adaptive controls (13) however, the two Lorenz systems achieve complete synchronization as the resistance, $R_k$progressively take on values greater than the critical value R_{kc} = 0.68k as displayed in Fig. 10. Notably, the present results also show that a drive-response chaotic oscillator interacting via adaptive controller (13) can evolve into complete synchronization when the strength of interaction is above the critical value. Thus, this experimental result validates the feasibility of the adaptive control scheme presented.

As a second example, we used two Sprott circuits for our experiment and apply a single variable controller given by Equation (25) by connecting a coupling resistance R_k between two nodes of the Sprott circuits that establish a connection via $x_1 \rightarrow x_2$and then varying the coupling parameter, R_k, to verify the synchronization of two nearly identical Sprott systems.

The values of the components were chosen as close as possible since, in practice, two oscillators cannot be strictly identical. In this case, the critical resistance is $R_{kc} = 0.10$. An almost complete synchronization was achieved for $R_k = 0.15k > R_{kc}$ as depicted in Fig. 11(b), where we show the oscilloscope traces of $x_1 vs\ x_2$ evolving into a synchronous state as the coupling strength exceeds a critical value. In Fig. 11(a),$R_k = 0.07k < R_{kc}$, and the drive-response Sprott system could not attain a synchronous state. Thus, validating the theoretical results in the case of the single variable feedback approach for synchronization of two Sprott systems

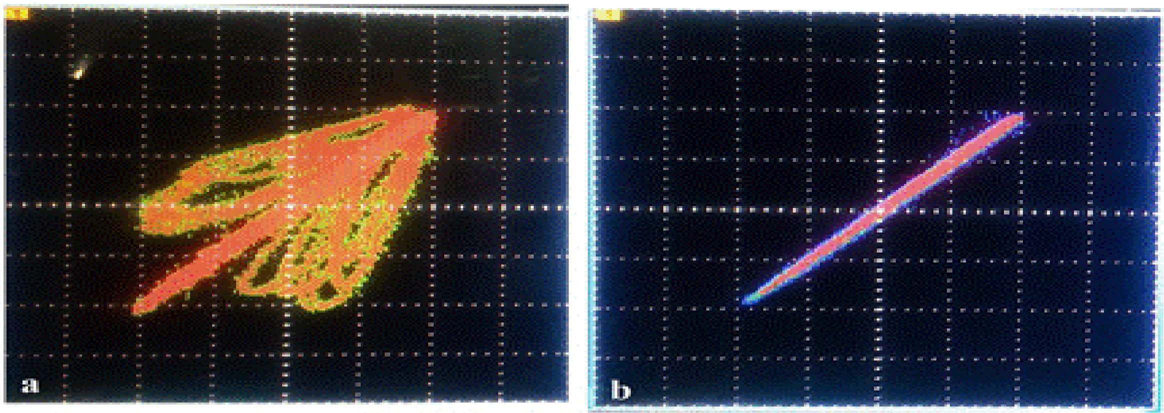

Fig. 10: Oscilloscope traces for two Sprott systems interacting via adaptive controller $x_1 vs\ x_2$ plots for (a) $R_k = 0.05k < R_{kc} = 0.68k$ - no synchronization, (b)$R_k = 0.85k > R_{kc} = 0.68k$ - synchronization.

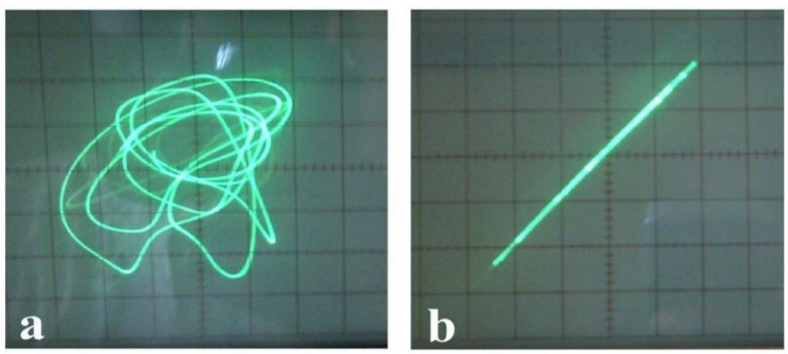

Fig. 11: Oscilloscope pictures for two Sprott systems interacting via single variable controller $x_1 vs\ x_2$ plots for (a)$R_k < R_{kc}$ (b) $R_k > R_{kc}$.

Finally, we give experimental results for two Sprott circuits using an adaptive controller in equation 31. This was achieved by connecting resistanceR_k between two nodes of the Sprott circuits connection via $x_1 \rightarrow x_2$ and then varying the coupling parameter,R_k to examine synchronization behaviour. Similar to the drive-response Lorenz system, an almost complete synchronization was achieved when the strength of the interaction is greater than a threshold value,R_{kc} = 2.5k, as depicted in Fig. 12(b) for R_k = 2.96k, where we display the oscilloscope traces of $x_1 vs\ x_2$evolving into the synchronous state as the coupling strength exceeds a critical value. Below threshold coupling resistance values, for instance)R_k = 2.20k, the coupled system could not attain a synchronization state as shown in Fig. 12(a). Thus, this second experimental result on adaptive control for two Sprott systems gives further experimental validation of the feasibility of adaptive control schemes.

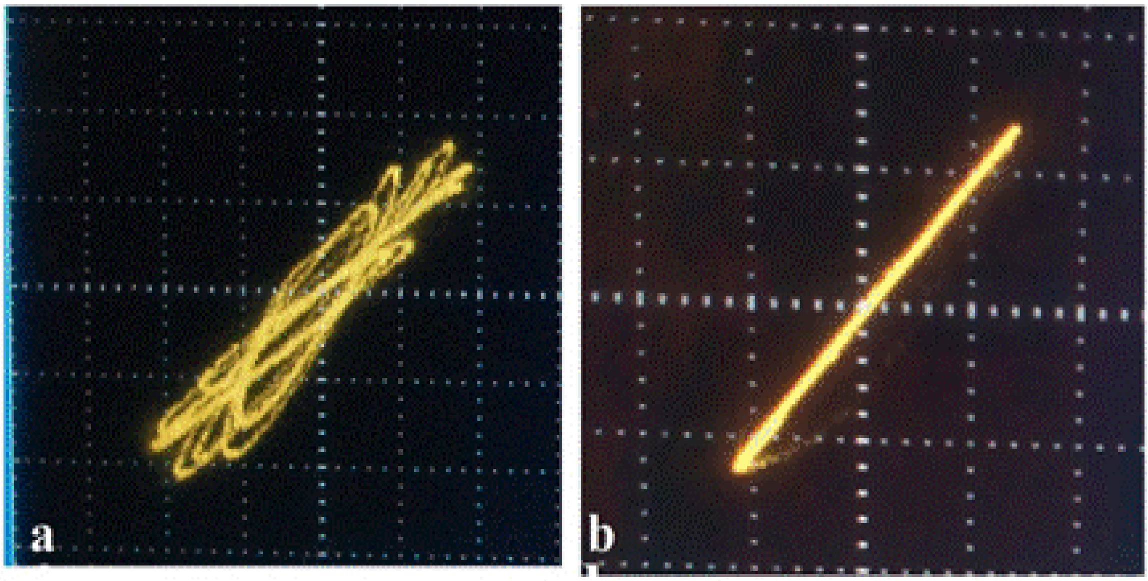

Fig. 12. Oscilloscope pictures for two Sprott systems interacting via adaptive controller $x_1 vs\ x_2$ plots for (a)R_k = 2.20k <R_{kc}, (b) R_k = 2.96k >R_{kc}.

5.0 Conclusion

Synchronization of nonlinear oscillators using single variable linear feedback and adaptive control techniques were examined in this paper. Using Lorenz and Sprott systems as typical chaotic oscillators, simple and efficient controllers were designed based on Lyapunov stability theory. First, the performance and feasibility of the designed controllers were verified by means of numerical simulation and we showed that the two systems studied attained synchronization state asymptotically as the controllers were activated. Furthermore, by using off-the-shelve components on the breadboard, we identified the critical/threshold coupling resistances for each scheme and illustrated the transition from asynchronous behaviour to stable synchronization as the coupling is varied beyond the critical value. We remark that previous results on single variable nonlinear and adaptive controls were more concerned with theoretical designs as well as numerical simulations (See for instance Refs. [[Stefanski et al, 2004; Olusola et al. 2010]). Consequently, the present study has advanced the existing results by providing experimental validation of theoretical results. The derived results can be generalized to all chaotic and hyperchaotic systems which exhibit more complex dynamics and therefore ensure security in information transmission.

6.0 References

Abd, M. H., Tahir, F. R., Al-Suhail, G. A. & Pham, V.-T. (2017). An adaptive observer synchronization using chaotic time-delay system for secure communication. *Nonlinear Dynamics,* 90, pp. 2583-2598.

Aguilar-Lpez, R., Martnez-Guerra, R. & Perez-Pinacho, C. (2014). Nonlinear observer for synchronization of chaotic systems with application to secure data transmission. Er. *Phys. J. Spec. Top*. 223, pp. 1541-1548, Ahmed, H., Ushirobira, R. & Efimov, D. (2017): Experimental study of the robust global synchronization of Brockett oscillators. The European Physical Journal Special Topics 226(15), pp. 3199-3210.

Almatroud Othmana, A., Noorania, M. S. M. & Mossa Al–Sawalha, M. (2016): Adaptive dual synchronization of chaotic and hyperchaotic systems with fully uncertain parameters, Optik – International Journal for Light and Electron Optics 12, 19, pp. 7852–7864 (2016).

Arellano-Delgado, A., Lo´pez-Guti´errez, R. M., Cruz-Hern´andez, C., Posadas-Castillo, C. Cardoza-Avendan˜o, L., & Serrano-Guerrero, H. (2013): Experimental network synchronization via plastic optical fiber. *Optical Fiber Technology* 19, 2, pp. pp. 93–108.

Bhatnagar, G. & Wu, Q. M. J. (2015). A novel chaos based secure transmission of biometric data. *Neurocomputing*,147, pp. 444-455.

Hamed T, Saeid S & Hamidreza T. (2018). Chaos synchronization and parameter identification of a finance chaotic system with unknown parameters, a linear feedback controller. *Alexandria Engineering Journal,* 57, 3, pp. 1519-1524.

Choi, H. & Lee, J. (2017). Principles, Applications, and Challenges of Synchronization in Nature for Future Mobile Communication Systems. Hindawi, *Mobile Information Systems,* https://doi.org/10.1155/2017/8932631

Cheng, C., Gao, F. Xu, J., Wang, Y. & Yuan, T. (2020). Adaptive Control Design for Arneodo Chaotic System with Uncertain Parameters and Input Saturation. Hindawi *Advances in Mathematical Physics,* https://doi.org/10.11 55/2020/3285414

Egunjobi, A. I., Olusola, O. I., Njah, A. N., Saha, S. & Dana, S. K. (2018). Experimental evidence of chaos synchronization via cyclic coupling. Commun *Nonlinear Science and Numerical Simulations,* 56, pp. 588-595.

Filali, R. L., Benrejeb, M. & Borne, P. (2014). On observer based secure communication design using discrete-time hyperchaotics systems. *Communication in Nonlinear Science Numerical Simulation*, 19, 5, pp. 1424–1432.

Guo, R., Vincent, U. E. & Idowu, B. A. (2009): Synchronization of Chaos in RCL- shunted Josephson Junction using a simple adaptive control, Physica Scr. 79, 035801.

Medhaffar, H., Feki, M. & Derbe, N. (2020). Adaptive fuzzy control for the stabilisation of chaotic systems. International *Journal of Automation and Control*,14, 2, pp.115 – 137.

Hu, M. F., & Xu, Z. Y. (2008): Adaptive feedback controller for projective synchronization. *Nonlinear Anal., Real World Appl*., 9, 3, 1253-1260.

Hua, C., Li, J., Yang, Y. & Guan, X. (2016). Extended-state-observer-based finite-time synchronization control design of teleoperation with experimental validation. *Nonlinear Dynamics*, 85, pp. 317-331.

Hua, M. F. & Xua, Z. Y. (2008): Adaptive feedback controller for projective synchronization. Nonlinear Analysis: *Real World Applications* ,9, pp. 1253 - 1260.

Liu L. & Guo, R. (2017). Control problems of Chen-Lee system by adaptive control method. *Nonlinear Dynamics,*., 87, pp. 503-510.

Liu, X., Zhai, D., Dong, J. & Zhang, Q. (2018). A Adaptive fault-tolerant control with prescribed performance for switched nonlinear pure-feedback systems. *Journal of the Franklin Institute* 355, pp. 273-290.

Liu, Y.J., Wang, W., Tong, S.C., & Liu, Y.S. (2010). Robust adaptive tracking control for nonlinear systems based on bounds of fuzzy approximation parameters. *IEEE Trans. Syst. Man Cybern., Part A, Syst. Hum*. 40, 1, pp. 170-184.

Liu, Y.J. & Zheng, Y.Q. (2009): Adaptive robust fuzzy control for a class of uncertain chaotic systems. *Nonlinear Dynamics*,. 57(3), pp. 431-439.

Lorenz, E.N. (1964). Deterministic nonperiodic flow. *Journal of Atmospheric Science*, 20, pp. 130–141.

Mahmoud, E.E., & Abood, F.S (2017): A novel sort of adaptive complex synchronizations of two indistinguishable chaotic complex nonlinear models with uncertain parameters and its applications in secure communications. *Results in Physics*, 7 pp. 4174-4182.

Machuc, J. L. (2014): Experimental Synchronization by Means of Observers. *Journal of Applied Research and Technology*, 12, 1, pp. 52–62.

Ni, J. K., Liu, L., Liu, C. X., Hu, X. O. & Li. A. A. (2017). Chaos suppression for a four-dimensional fundamental power system model using adaptive feedback control. *Trans. Inst. Measurement and Control*, 39, 2, pp. 194-207.

Niu, H., Ma, S. & Fan, T., Chen, C. & He, P. (2014). Linear state feedback stabilization of unified hyperchaotic systems. *World Journal of Modelling and Simulation* 10, 1, pp. 34–48.

Ojo, K.S. Njah, A. N., Ogunjo, S.T. & Olusola, O.I. (2014). Reduced Order Functional Projective Synchronization of three Josephine junctions using backstepping technique. *Nonlinear Dynamics and Systems Theory*, 14, 12, pp. 110–133.

Olusola, O.I., U.E. Vincent, & Njah, A.N. (2010). Multistability and basin crisis in synchronized parametrically driven oscillators. *Nonlinear Dynamics*, 62, pp. 717-727.

Onma, O.S. Olusola, O.I. & Njah, A.N. (2016). Control and synchronization of chaotic and hyperchaotic Lorenz systems via extended adaptive control techniques, Far East *Journal of Dynamical systems*, 28, 1, pp. 1 – 32.

Pallov Anand & B. B. Sharma (2020). *Synchronization of Chain Network of 4-D Lorenz-Stenflo Systems using Contraction based Backstepping*. 2020 2nd International Conference on Innovative Mechanisms for Industry Applications (ICIMIA)

Pecora, L. M. & Caroll,T. L. (1990). Synchronization of Chaotic Systems., Chaos, 25, 097611, https://doi.org/10.1063/1.4917383

Perlikowski, P., Jagiello, B., Stefanski, A. & Kapitaniak, T. (2008). Experimental observation of ragged synchronizability. Phys. Rev. E 78, 017203.

Ping He, & Fei Tan. (2011): Linear State Feedback Stabilization for Controlled Chaotic Systems. *International Journal of Nonlinear Science* 123, pp. 373–384.

Ren, H. P., Baptista, M. S., & Grebogi, C. (2013). Wireless communication with chaos, *Physics Letters and Review*, 3:110, 18, doi: 10.1103/PhysRevLett.110.184101.

Ricardo, A. L. & Rafael, M.. G. (2008). Synchronization of a class of chaotic signals via robust observer design. *Chaos Solitons Fractals* 37, 2, pp. 581-587.

Salarieh, H. & Alasty, A. (2009): Adaptive synchronization of two chaotic systems with stochastic unknown parameters. *Communication in Nonlinear Science and Numerical Simulation*,. 14, 2, pp. 508-519.

Shaohua Luo, Frank L. Lewis, Yongduan Song & Kyriakos G. Vamvoudakis (2020). Adaptive backstepping optimal control of a fractional-order chaotic magnetic-field electromechanical transducer. *Nonlinear Dynamics*, 100, pp. 523–540.

Siddique, M. & Rehan, M. (2016): A concept of coupled chaotic synchronous observers for nonlinear and adaptive observers-based chaos synchronization. *Nonlinear Dynamics*, 84(4), pp. 2251-2272.

Sprott J.C. (2000): A New Class of Chaotic Circuit. Phys. Letts. A 266, 19–23.

Stefanski, A., Wojewoda, J. & Kapitaniak, T. (2004): Simple estimation of synchronization threshold in ensembles of diffusively coupled chaotic systems. *Phys. Rev. E* 70, 026217.

Vincent, U.E. & Guo, R. (2009): A simple adaptive control for full and reduced–order synchronization of uncertain time-varying chaotic systems, *Communication in Nonlinear Science and Numerical Simulation*, 14, pp. 3925–3932.

Wang, X.-Q.& Wang, Y.-Q. (2011): Adaptive control for synchronization of a four-dimensional chaotic system via a single variable. *Nonlinear Dynamics*, 65, pp. 311-316.

Yang, Chi-Ching (2014). Adaptive Single Input Control for Synchronization of a 4D LorenzStenflo Chaotic System. *Arabian Journal of Science and Engineering*, 39, pp. 2413-2426.

Yao, J.Y., Jiao, Z.X. & Ma, D.W. (2014). Adaptive robust control of DC motors with extended state observer. *IEEE Trans. Ind. Electron.*, 61, 7, pp. 3630-3637.

Yaping Hu, Yaru Zhang, & Rongwei Guo (2020). *Coexistence of anti-phase and complete synchronization in the Chen-Lee system. Conference: 2020* 39th Chinese Control Conference (CCC).

Conflict of Interest

The authors declared no conflict of interest. All authors took part in analyzing the results, proofreading and effecting all corrections. All authors read and approved the final manuscript