Communication in Physical Sciences, 2021, 7(4): 398-410

On Flexibility of Inverse Lomax-Lindley distribution

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Received:07 September 2021/Accepted 30 November 2021/Published online:25 November 2021

Abstract: *In this work, a new extension of the Inverse Lomax family of distribution called Inverse Lomax Lindley (IL-L) distribution is proposed. Different properties of the new distribution are derived including moments, moment generating function, Renyi entropy, Shanon entropy, and order statistics. The performances of the maximum likelihood estimates of the parameters of the Inverse Lomax-Lindley distribution were evaluated through a simulation study. Application of the IL-L distribution to two real-life data sets has proved its flexibility.*

Keywords: *Entropy Inverse Lomax-Lindley distribution Inverse Lomax Rayleigh Monte Carlo Simulation Inverse Lomax Inverse Lomax-G Family*

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1.0 Introduction

Nowadays, modeling real data by generalized distributions remains consistent. Many generalized distributions were developed and applied in several areas. Nevertheless, many important issues still exist concerning real data which are not solved by existing methods. Lindley has proposed an alternative data model which is a mixture of exponential and gamma distributions with forms of non-monotonic hazard intensity. The Lindley distribution is one of the most important ones to study Stress-strength reliability Modeling. Ghitany *et al.* (2008) documented some of its structural properties and proved to be better suited for other types of data than for the exponential distribution.

There are many extended forms of the Lindley distribution in the statistical literature. For instance, Zero-Truncated Poisson–Lindley by Ghitany *et. al.* (2018), Generalized Lindley by Zakerzadeh and Dolati (2009), Negative Binomial Lindley by Zamani and Ismail (2010), Generalized Poisson Lindley Mahmoudi and Zakerzadeh (2010), Weighted Lindley by Ghitany *et. al.* (2011), Generalized Lindley Nadarajah et. al (2011), Marshall-Olkin extended Lindley by Ghitany *et. al.* (2012), Extended Lindley by Bakouch *et. al.* (2012), Power Lindley by Ghitany et. al. (2013) , Transmuted Quasi Lindley by Elbatal and Elgarhy (2013), Quasi-Lindley by Shanker and Mishra (2013), New Weighted Lindley by Asgharzadeh (2016), Transmuted two-parameter Lindley by Al-khazaleh *et. al.* (2016), New Lindley by Abd El-Monsef

(2016), Gamma Lindley by Nedjar and Zeghdoudi (2016), Inverse Power Lindley by Barco *et. al.* (2017), Truncated two-parameter Lindley by Aryuyuen (2018), Weibull Lindley by Asgharzadeh *et. al.* (2018), Alpha Power Transformed Power Lindley by Hassan *et. al.* (2019), and Weibull Marshall–Olkin Lindley by Afify *et. al.* (2020), among others.

Falgore and Doguwa (2020) proposed the Inverse Lomax-G (IL-G) family based on the T-X generator by Alzaatreh *et. al.* (2013) for any baseline G distribution with parameter vector ζ . The cumulative density function (cdf) and probability density function (pdf) of IL-G based on Falgore and Doguwa (2020) are given as

$$F(x;\vartheta)=\left(1+\frac{\beta\bar{G}(x;\upsilon)}{Gx;\upsilon)}\right)^{-\theta};\qquad x>0,\theta,\beta,\upsilon>0 \tag{1}$$

and

$$f(x;\vartheta)=\frac{\beta\theta g(x;\upsilon)}{[G(x;\upsilon)]^{2}}\left(1+\frac{\beta\bar{G}(x;\upsilon)}{Gx;\upsilon)}\right)^{-(1+\theta)}\qquad x>0,\theta,\beta,\upsilon>0 \tag{2}$$

where $\vartheta=(\theta,\beta,\upsilon)^{T}$, $\bar{G}(x;\upsilon)=1-G(x;\upsilon)$ and also θ and β are the two additional parameters that are added to make the baseline distribution much more flexible. Moreover, Falgore *et. al.* (2021) has extended this family with Rayleigh distribution.

In this paper, we propose a new model called the Inverse Lomax-Lindley (IL-L) distribution. The cdf and pdf of Lindley distribution are

$$G(x;\alpha)=1-\frac{1+\alpha(1+x)}{1+\alpha}\exp\left\{-\alpha x\right\}\qquad x>0,\alpha>0 \tag{3}$$

and

$$g(x;\alpha)=\frac{\alpha^{2}(1+x)}{1+\alpha}\exp\left\{-\alpha x\right\}\qquad x>0,\alpha>0 \tag{4}$$

The objective of this paper is to propose a new distribution called the Inverse Lomax-Lindley distribution which has the capacity of providing a more robust compound probability distribution when modeling real-life data set. This new distribution adds two additional parameters to the baseline (Lindley) distribution.

This article is structured as follows: The definition of IL-L was presented in Sec. 2. In Sec 3, some of the properties of the IL-L model were presented. A Monte Carlo simulation study was presented in Sec. 4. A maximum likelihood method of estimation of the IL-L distribution is presented in Sec. 5. In Sec. 6, the application of the Inverse Lomax Lindley (IL-L) distribution and its competitors are discussed. While Sec. 7 concludes the paper.

2.0 The Inverse Lomax Lindley (IL-L) distribution

By setting the Lindley cdf given in equation (3) back in the equation (1), we have the cdf of IL-L distribution as

$$F_{IL-L}(x;\vartheta)=\left\{1+\beta\frac{\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\left\{-\alpha x\right\}}{1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\left\{-\alpha x\right\}}\right\}^{-\theta}\qquad x>0,\theta,\beta,\alpha>0 \tag{5}$$

and the corresponding pdf, hazard function, and survival function given in equations (6), (7), and (8) respectively as

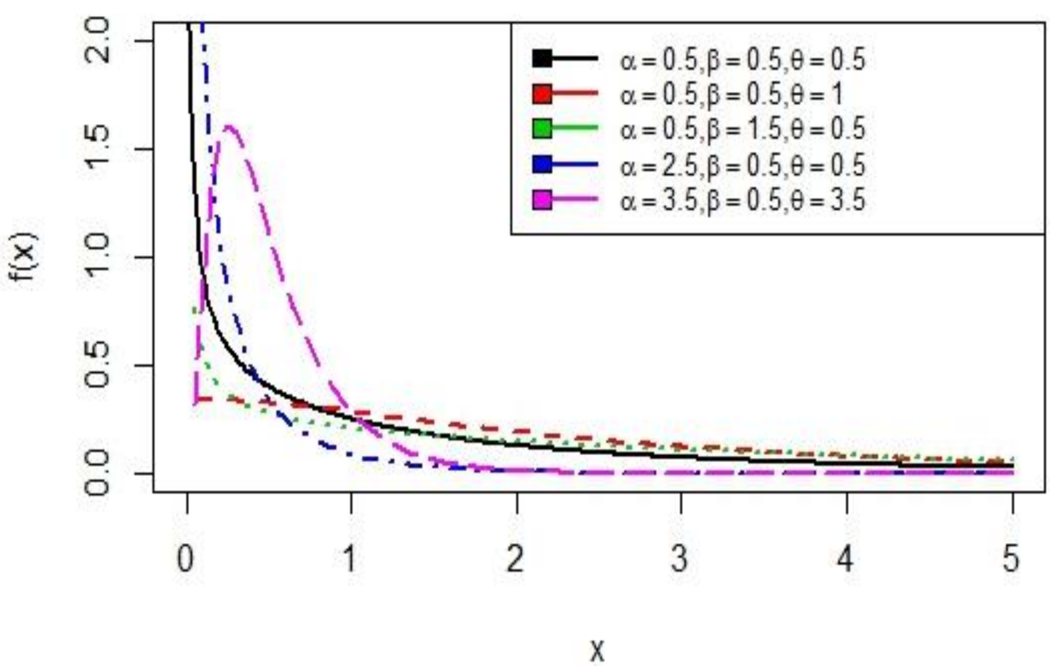
$$f_{IL-L}(x;\vartheta)=\frac{\alpha^{2}\beta\theta(1+x)\exp\{-\alpha x\}}{(1+\alpha)\left(1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}\right)^{2}}\left\{1+\beta\frac{\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}{1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}\right\}^{-(\theta+1)}\quad x>0,\theta,\beta,\alpha>0 \tag{6}$$

$$h_{IL-L}(x;\vartheta)=\frac{\alpha^{2}\beta\theta(1+x)\exp\{-\alpha x\}}{(1+\alpha)\left(1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}\right)^{2}\left[1-\left\{1+\beta\frac{\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}{1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}\right\}^{-\theta}\right]}\left\{1+\beta\frac{\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}{1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}\right\}^{-(\theta+1)}$$
$$x>0,\theta,\beta,\alpha>0 \tag{7}$$

$$S_{IL-L}(x;\vartheta)=1-\left\{1+\beta\frac{\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}{1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}\right\}^{-\theta}\qquad x>0,\theta,\beta,\alpha>0 \tag{8}$$

where β is a scale,while θ and α are shape parameters, respectively.

(a)



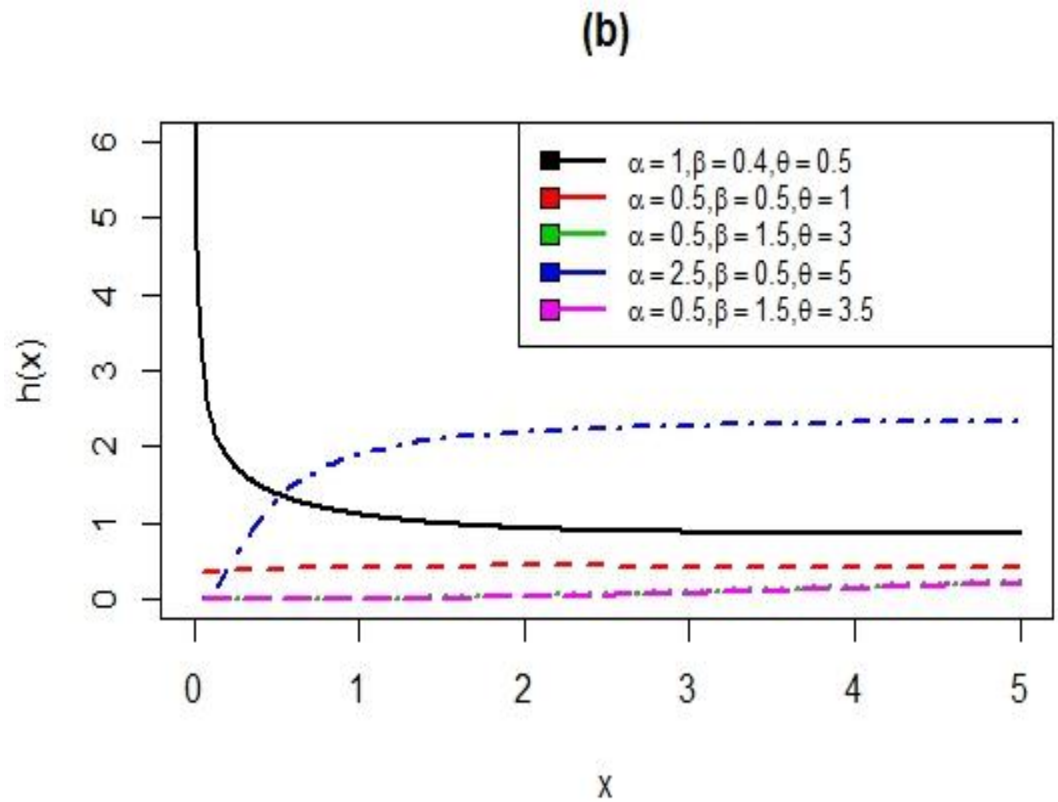


Fig. 1: pdf and hazard Plots of IL-L distribution with various parameter values

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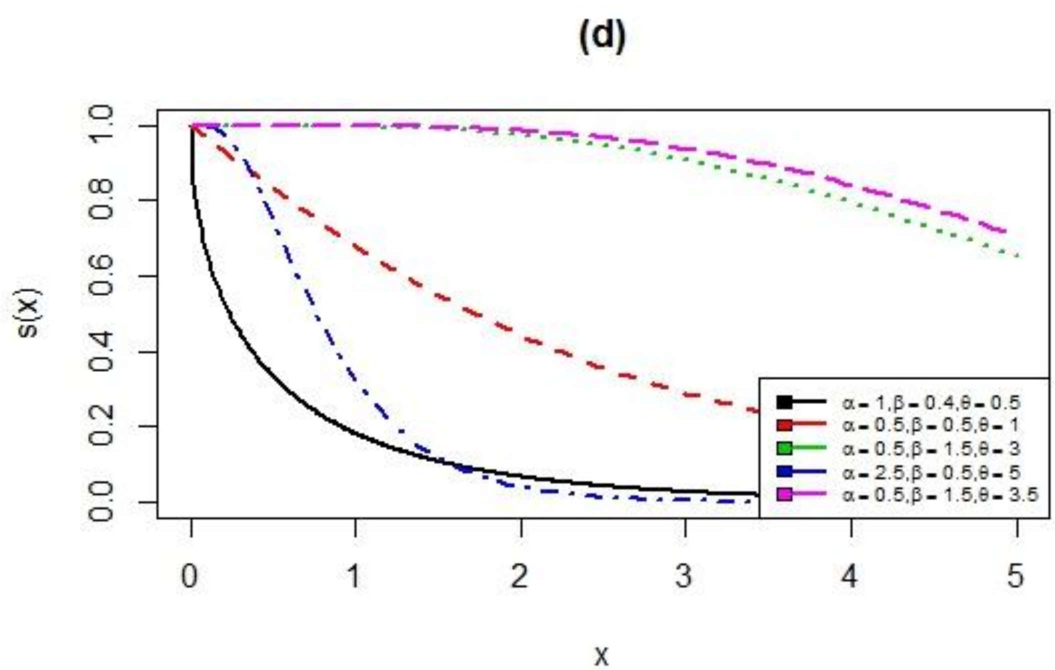

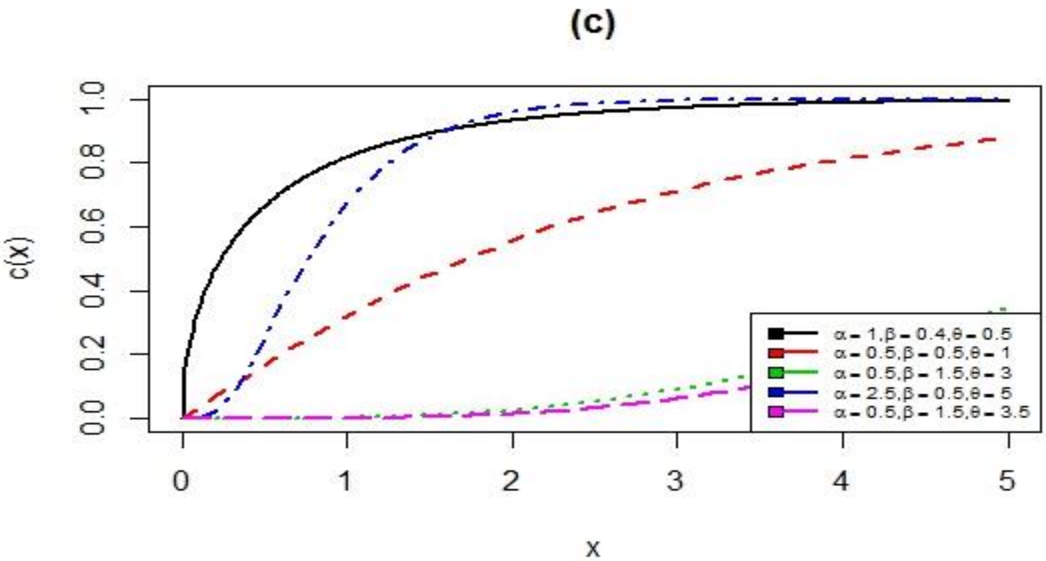

Fig. 2: cdf and survival functions Plots of IL-L distribution with various parameter values

2.1 Validity of the cdf and pdf of the IL-L distribution

The cdf in the equation (5) is valid.

3.0 Properties of IL-L distribution

3.1 Quantile Function

Theorem If $0 < u < 1$ and a random variable X follows the $IL-L(\theta,\beta,\alpha)$, then the quantile function can be given as

$$x = \frac{1}{\alpha}\left\{-W_{-1}\left[-\frac{k(1+\alpha)e^{-(1+\alpha)}}{(1+k)}\right]-\alpha-1\right\} \tag{9}$$

Where $k = \dfrac{u^{-\frac{1}{\theta}}-1}{\beta}$, and W_{-1} in the negative branch of Lambert W function.

Proof

Let the cdf in the equation (5) be equal to u, $u \in (0,1)$. Then,

$$\frac{\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}{1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}} = \frac{u^{-\frac{1}{\theta}}-1}{\beta}$$

this can be simplified as

$$\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\} = \frac{\left(\frac{u^{-\frac{1}{\theta}}-1}{\beta}\right)}{\left(1+\frac{u^{-\frac{1}{\theta}}-1}{\beta}\right)}$$

Let k= $\dfrac{u^{-\frac{1}{\theta}}-1}{\beta}$, by rearranging we have

$$\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}=\frac{k}{1+k} \tag{10}$$

Multiply both sides of the equation (10) by $-(1+\alpha)\exp\{-(1+\alpha)\}$ This reduce to

$$(1+\alpha+\alpha x)\exp-\{1+\alpha+\alpha x\}=\frac{-k(1+\alpha)\exp\{-(1+\alpha)\}}{1+k}$$

The Lambert W function implies that if $Xe^{X}=Y$ $\quad X=W(Y)$ i.e X is the Lambert W function of the real argument Y. In the same manner, we can say that $-(1+\alpha+\alpha x)$ is the Lambert W function of

$$\frac{-k(1+\alpha)\exp\{-(1+\alpha)\}}{1+k}\text{, i.e }-(1+\alpha+\alpha x)=W\left(\frac{-k(1+\alpha)\exp\{-(1+\alpha)\}}{1+k}\right) \tag{11}$$

Since $(1+\alpha+\alpha x)>0$ then, the W function specified above is the negative branch. Therefore, we may write the equation (11) as

$$-(1+\alpha+\alpha x)=W_{-1}\left(\frac{-k(1+\alpha)\exp\{-(1+\alpha)\}}{1+k}\right) \tag{12}$$

To solve for x, let $Y=\left(\dfrac{-k(1+\alpha)\exp\{-(1+\alpha)\}}{1+k}\right)$. Then $(1+\alpha+\alpha x)=-W_{-1}(Y)$. Finally,

$$x=\frac{1}{\alpha}\left\{-W_{-1}\left[-\frac{k(1+\alpha)e^{-(1+\alpha)}}{(1+k)}\right]-\alpha-1\right\}.$$

Table 1: A sample values for the quantile function of IL-L distribution for $\alpha,\beta,$ and θ.

u	0.5,0.5,0.5	0.5,1,2	1,0.5,3	0.5,0.5,2.5
.1	0.0299	1.6267	0.6359	1.2839
.2	0.1191	2.3381	0.9072	1.8355
.3	0.2671	2.9598	1.1554	2.3367
.4	0.4771	3.5695	1.4085	2.846
.5	0.7598	4.2095	1.6836	3.3989
.6	1.1397	4.9217	1.9996	4.0341
.7	1.6695	5.7691	2.3869	4.8132
.8	2.4752	6.8806	2.9097	5.8658
.9	3.9687	8.6535	3.7661	7.5931

Table 1 provides the various sample quantiles for the IL-L distribution at various parameter values for four different combinations.

3.2 Moments

Suppose a random variable X follows $IL-L(\alpha,\beta,\theta)$, then the $r^{th}(\mu_r)$ raw moments for the IL-L distribution are computed from the following:

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f_{IL-L}(\alpha,\beta,\theta)dx$$

$$= \frac{\alpha^2\beta\theta}{(1+\alpha)}\int_0^\infty \frac{x^r(1+x)\exp\{-\alpha x\}}{\left(1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}\right)^2}\sum_{i=0}^{\theta+1}\beta^i(-1)^i\binom{i-\theta}{i}\left[\frac{\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}{1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}\right]^i dx \qquad (13)$$

after some simplifications and algebra, we have

$$\mu_r' = L_{i,j,k,l}\left[\frac{\Gamma(r+l+1)}{(\alpha+\alpha i+\alpha j)^{r+l+1}}+\frac{\Gamma(r+l+2)}{(\alpha+\alpha i+\alpha j)^{r+l+2}}\right] \qquad (14)$$

where

$$L_{i,j,k,l} = \frac{\sum_{i=0}^{\theta+1}\sum_{k,l=0}^{\infty}(-1)^i\alpha^2\beta^{i+1}\theta\binom{i-\theta}{i}\frac{\Gamma(i-j+1)\Gamma(k+1)}{k!l!\Gamma(i-j+1-k)\Gamma(k+1-l)}}{\sum_{j=0}^{\infty}(-1)^j\binom{2+i}{j}(1+\alpha)^{i-j}}$$

Based on the r^{th} moments given in equation (14), some statistical measures can be obtained by setting $r = 1,2,3,4\ldots$ as presented in Table 2. The moment generating function (mgf) of the IL-L distribution can be presented using Taylor's series expansion as

$$M_x(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mu_r'. \qquad (15)$$

Table 2: Moments of IL-L (α,β,θ) distribution

μ_r'	0.5,0.5,0.5	0.5,1,2	1,0.5,3
μ_1'	1.5012	4.7778	1.9961
μ_2'	6.2256	31.3333	5.7669
μ_3'	41.6637	262	22.1944
μ_4'	379.572	2674.667	107.4806
sd	1.9929	2.9165	1.3351
cv	1.3275	0.6104	0.6689
cs	6.6359	1.5632	2.2467
ck	13.5355	26.1309	17.6834

Table 2 shows the sample moments for the IL-L distribution at different parameter values. The standard deviation (sd), coefficient of variation (cv), coefficient of skewness (cs), and coefficient of kurtosis (ck) are given by:

$$sd = \sqrt{\mu_2' - \mu^2},$$

$$cv = \sqrt{\frac{\mu_2^{'}}{\mu^2} - 1},$$

$$cs = \frac{\mu_3^{'} - 3\mu_2^{'}\mu + 2\mu^3}{(\mu_2^{'} - \mu^2)^{\frac{3}{2}}},$$

and

$$ck = \frac{\mu_4^{'} - 4\mu_3^{'}\mu + 6\mu^2\mu_2^{'} - 3\mu^4}{(\mu_2^{'} - \mu^2)^2}.$$

3.3 Entropy

Entropy measures the average volume of information received by identifying the effect of a random trial. The Rényi entropy for the IL-L distribution is given as

$$E_{i,j,k,m} = \frac{\Gamma(k+m+1)}{(\alpha i + \alpha j + \alpha\Omega)^{k+m+1}} \qquad (16)$$

3.4 Order Statistics

The distribution of order statistics for the IL-L distribution is given as

$$K_{i,j,k,l} = (1+x)(1+\alpha(1+x))^{k-l}\exp\{-\alpha(k+l+1)x\} \qquad (17)$$

where

$$K_{i,j,k,l} = \frac{\sum_{i=0}^{v-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j\theta+\theta+j+1}(-1)^{i+j+k}\binom{v-1}{i}\binom{n-v+i}{j}\binom{-j\theta-j-\theta+k-2}{k}\alpha^2\beta^{1+k}\theta}{(1+\alpha)^{1+k+l}\sum_{l=0}^{\infty}(-1)^l\binom{2+k}{l}}$$

4.0 Monte Carlo Simulation Studies

Monte Carlo simulation is used to check the performance of the MLEs of the parameters for the IL-L distribution. The results of all simulations are obtained from 1000 replications using R-Project. In each replication, a random sample of size n is obtained from X follows $IL-L(\alpha,\beta,\theta)$. The random number generation for the IL-L distribution is performed by the inversion method using the quantile function defined earlier. Three different combinations of true parameter values in the first row of Table 3 are used for the data generating processes. Table 3 lists the MSEs with their corresponding biases (in parentheses) obtained from five different sample sizes. Three cases were considered. These are; case1 $(\alpha = \beta = \theta = 0.5)$, case2 $(\alpha = 0.5, \beta = 1.5, \theta = 0.5)$, and case3 $(\alpha = 2.5, \beta = 0.5, \theta = 0.5)$.

Table 3: Simulation Results for the three cases

Parameters	Case1	Case2	Case3
		n=100	
$\hat{\alpha}$	0.5523	0.5207	2.6413
$\hat{\beta}$	0.7286	1.9951	0.6519
	0.5107	0.5127	0.5139
$MSE_{\hat{\alpha}}(Bias)$	0.0332 (0.0523)	0.0156 (0.0207)	1.1146 (0.1413)
$MSE_{\hat{\beta}}(Bias)$	0.3999 (0.2286)	2.5601 (0.4951)	0.2919 (0.1519)

$MSE_{\hat{\theta}}(Bias)$	0.0072 (0.0107)	0.0077 (0.0127)	0.0076 (0.0139)
	n=200		
$\hat{\alpha}$	0.5251	0.5054	2.5173
$\hat{\beta}$	0.6092	1.7222	0.5575
$\hat{\theta}$	0.5043	0.5058	0.5065
$MSE_{\hat{\alpha}}(Bias)$	0.0168 (0.0251)	0.0101 (0.0054)	0.6617 (0.0173)
$MSE_{\hat{\beta}}(Bias)$	0.1333 (0.1092)	0.8921 (0.2222)	0.0998 (0.0575)
$MSE_{\hat{\theta}}(Bias)$	0.0034 (0.0043)	0.0037 (0.0058)	0.0038 (0.0065)
	n=500		
$\hat{\alpha}$	0.5114	0.4974	2.4343
$\hat{\beta}$	0.5517	1.5689	0.5161
$\hat{\theta}$	0.5011	0.5026	0.5031
$MSE_{\hat{\alpha}}(Bias)$	0.0064 (0.0114)	0.0059(-0.0025)	0.4527(-0.0656)
$MSE_{\hat{\beta}}(Bias)$	0.0609 (0.0517)	0.2769 (0.0689)	0.0569 (0.0161)
$MSE_{\hat{\theta}}(Bias)$	0.0013 (0.0011)	0.0014 (0.0026)	0.0016 (0.0031)
	n=1000		
$\hat{\alpha}$	0.5051	0.4976	2.4061
$\hat{\beta}$	0.5244	1.5348	0.498
$\hat{\theta}$	0.5007	0.5014	0.5023
$MSE_{\hat{\alpha}}(Bias)$	0.0036 (0.0051)	0.0036(-0.0023)	0.3786(-0.0938)
$MSE_{\hat{\beta}}(Bias)$	0.0214 (0.0244)	0.1330 (0.0348)	0.0347(-0.0019)
$MSE_{\hat{\theta}}(Bias)$	0.0006 (0.0007)	0.0006 (0.0014)	0.0007 (0.0023)

5.0 Estimation

Here, the maximum likelihood method for estimating the parameters of the IL-L is considered. The log-likelihood for the IL-L distribution is given as:

$$ll = n\log\left(\frac{\alpha^2\beta\theta}{(1+\alpha)}\right)+\sum_{i=1}^{n}\log(1+x_i)-\alpha\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}\log\left(1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}\right)$$

$$-(\theta+1)\sum_{i=1}^{n}\log\left\{1+\beta\frac{\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}{1-\left(\frac{1+\alpha(1+x)}{1+\alpha}\right)\exp\{-\alpha x\}}\right\} \tag{18}$$

To get the estimates of the parameters, we differentiate the equation (18) partially concerning each parameter and equate to zero as follows:

$$\frac{\partial ll}{\partial \alpha}=\frac{2n}{\alpha}-\frac{n}{1+\alpha}-\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}\frac{p_i}{1-p_i}\left(x_i-q_i+\frac{1}{1+\alpha}\right)$$
$$-\beta\left(\theta+1\right)\sum_{i=1}^{n}\left[\frac{p_i\left(1-p_i\right)\left(q_i-\frac{1}{1+\alpha}-x_i\right)-p_i^2\left(x_i-q_i+\frac{1}{1+\alpha}\right)}{\left(1-p_i\right)\left(1-p_i+\beta p_i\right)}\right] \tag{19}$$

$$\frac{\partial ll}{\partial \beta}=\frac{n}{\beta}-\left(\theta+1\right)\sum_{i=1}^{n}\left(\frac{p_i}{1-p_i+\beta p_i}\right) \tag{20}$$

$$\frac{\partial ll}{\partial \theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\log\left(\frac{1-p_i+\beta p_i}{1-p_i}\right) \tag{21}$$

where $q_i=\frac{\left(1+x_i\right)}{1+\alpha\left(1+x_i\right)}$ *and* $p_i=\left(\frac{1+\alpha\left(1+x_i\right)}{1+\alpha}\right)e^{-\alpha x_i}$.

6.0 Applications

To investigate the advantage of the proposed distribution, we consider two real data sets. The first dataset represents uncensored data corresponding to remission times (in months) of a sample of bladder cancer patients studied in Falgore *et. al.* (2019) and Merovci (2013). The second data set is on the failure times of aircraft windshields studied Kharazmi *et al.* (2020). The parameter estimates are obtained using the maximum likelihood method. Then, we present comparison criteria values: Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The smaller are the values of these statistics, the better is the fit to the data. Table 4 shows that IL-L distribution is a good fit and can be the considered best model for this data set. The estimates for Lindley and Transmuted Lindley distributions are as reported in Merovci (2013). . We provide more information from a histogram of the data given in Figure 3 with fitted lines which are the best three models: IL-L, exponential and gamma. Table 5 also shows that IL-L distribution is a good fit, barely better than the Weibull model for this data set. We provide a graphic visualization using a histogram of the data in Figure 3 with fitted lines based on the best three models: IL-L, Weibull, and gamma.

Table 4: The MLEâ€™s, AIC, and BIC of parameters for the first data set

Distribution	Estimated parameter			-LL	AIC	BIC
IL-L	0.07368	0.09458	1.10115	409.447	824.895	833.451
Trans. Lindley	0.156	0.617	-	415.15	834.31	840.01
Lindley	0.196	-	-	419.52	841.06	843.91
Weibull	1.04773	9.56	-	414.087	832.174	837.878
Gamma	1.17253	0.12521	-	413.368	830.736	836.44
Lognormal	1.75345	1.07308	-	415.094	834.189	839.893
Exponential	0.10677	-	-	414.342	830.684	833.536

Table 5: The MLEâ€™s, AIC, and BIC of parameters for the second data set

Distribution	Estimated parameter			-LL	AIC	BIC
IL-L	1.67955	28.0723	0.94983	129.311	264.623	271.95
Weibull	2.39321	2.86813		31.288	266.577	271.462
Gamma	3.52837	1.37691		38.395	280.791	285.676
Lognormal	0.79269	0.68363		\55.659	315.318	320.204
Exponential	0.39023	-		64.988	331.975	334.418

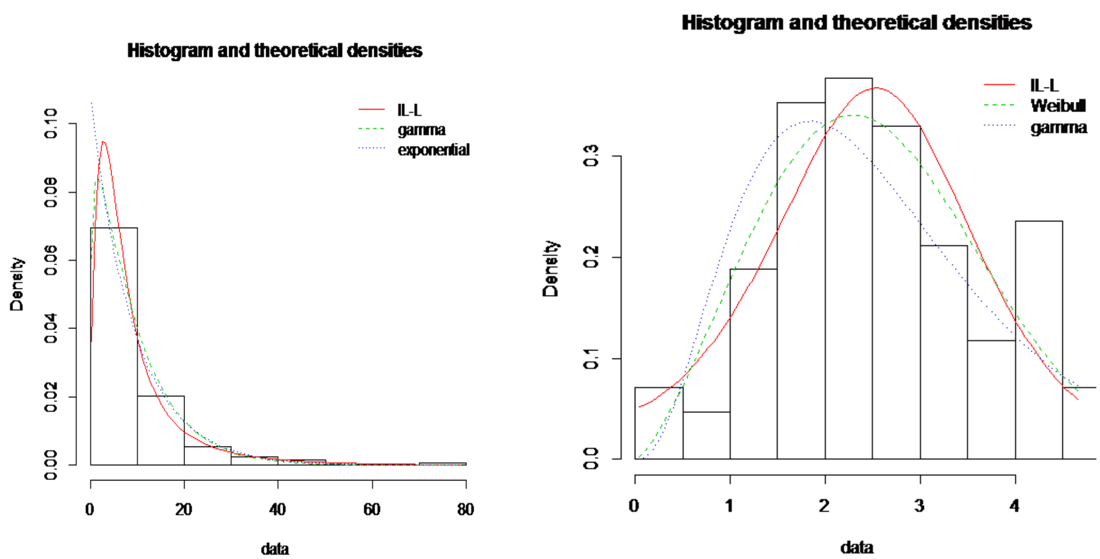

Figure 3: Fitted densities of IL-L, Gamma and Exponential distributions by the left based on the first data set and with Weibull and Gamma distributions-by the right based on the second data.

7.0 Conclusion

A new extension of Lindley distribution is proposed. The flexibility of the proposed distribution was demonstrated as follows: by varying the shape parameter θ , the distribution takes many shapes as shown in Figures 1 and 2, the estimates of the parameters exhibit good properties as shown in the simulation table, and finally, the applicability of the proposed distribution as shown in the last two Tables, has shown the superiority of the IL-L distribution. However, Figure 3 demonstrated the fitness of the IL-L distribution as it fits the two data sets well.

8.0` Acknowledgment

The authors of the manuscript appreciate the efforts of the reviewers and the Editor(s) for improving the quality of the manuscript.

9.0 Author's contribution

The first author initiates the first draft and wrote sections one and two, the second author did sections four and six, the third author wrote section three, and finally, the fourth author wrote section five.

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Conflict of Interest

The authors declared no conflict of interest.