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Application of Hulthén-Hellmann Potential (HHP) to Predict the Mass-Spectra of Heavy Mesons

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Abstract: *In this study, we adopt Hulthén plus Hellmann potentials as the quark-antiquark interaction potential for predicting the mass spectra of heavy mesons. The radial Schrödinger equation was solved using the series expansion method and the energy equation was obtained. The energy equation is applied for predicting the mass spectra of heavy mesons such as charmonium* $(c\bar{c})$ *and bottomonium* $(b\bar{b})$. *Four special cases were considered when some of the potential parameters were set to zero, resulting in Hellmann potential, Yukawa potential, Coulomb potential, and Hulthén potential, respectively. The present potential provides satisfying results in comparison with experimental data and work of other researchers.*

Keywords: *Hulthén potential; Hellmann potential; Schrödinger equation; Heavy mesons; Series expansion method*

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1.0 Introduction

The solution of the Schrödinger equation (SE) for a physical system in quantum mechanics is of great importance because the knowledge of Eigen energy and wave function contains all possible information about the physical properties of a system under study (Inyang *et al.,* 2021; Mutuk, 2018; Allosh *et al.,*2021; Edet and Okoi, 2019; Ikot,2020). The study of behavior of several physical problems in physics requires us to solve the non-relativistic or relativistic equation. A good description of many features of these problems can be obtained using non-relativistic models that is the quark-antiquark strong interaction is described by a phenomenological potential (Horchani *et al.,* 2021; Onate *et al.,*2021; Abu-shady *et al.,* 2021). Heavy mesons have provided extremely useful probes for the deconfined state of matter because the force between a heavy quark and anti-quark is weakened due to the presence of gluons which lead to the dissociation of it bound states (Vega and Flores, 2016). The heavy mesons and their interaction are well described by the SE (Prasanth *et al.,*2020; Ukewuihe *et al.,* 2021; Inyang *et al.,* 2021).

The solution of the spectral problem for the SE with spherically symmetric potentials is of major concern in describing the spectra of heavy mesons (Rani *et al.,*2018). Potential models offer a rather good description of the mass spectra of heavy mesons such as bottomonium, and charmonium (Mansour and Gamal,2018). In predicting the mass spectra of heavy mesons, confining-type potentials are generally used. The holding potential is the Cornell potential with two terms one of which is responsible for the Coulomb interaction of the quarks and the other corresponds to a confining term (Al-Oun *et al.,* 2015). In past, this type of potential has been studied by many researchers using different techniques (Al-Jamel, 2019; Abu-Shady,2016 ; Ciftci, and Kisoglu, 2018; Bayrak *et al.,*2006; Hall and Saad, 2015). The confining potentials may be in different forms depending upon the interaction of the particles within the system. The harmonic oscillator and the hydrogen atom are the two potentials in which solutions to the SE are found exactly. On the other hand, to obtain the approximate solutions, some techniques are employed. Example of such techniques include, asymptotic iteration method(AIM) (Inyang *et al*., 2021; Oyewumi, and Oluwadare, 2016), Laplace transformation method (Abu-Shady, and Khokha,2018;Abu-Shady *et al.*,2018), super symmetric quantum mechanics (SUSYQM) (Ikhdair, 2009;Abu-Shady and Ikot, 2019; Al-Jamel, 2019;Onate *et al.,*2021), the Nikiforov-Uvarov (NU) method (Ntibi *et al.,* 2020; Okoi *et al.,*2020; Edet *et al*., 2019;Inyang *et al.,* 2021;Edet *et al.,* 2020;Inyang *et al.,*2020;Inyang *et al.,* 2021; Edet *et al.,* 2020;Ekpo *et al*.,2020;William *et al.,*2020;Inyang *et al*.,2021; Okon *et al*.,2016;Abu-Shady *et al*.,2019; Inyang *et al*., Omugbe, 2020;Ita, and Ikeuba, 2013;Thompson *et al.,*2021;Abu-Shady, 2015;Akpan *et al*.,2021;Inyang *et al*.,2021;Inyang *et al*.,2021) ,the Nikiforov-Uvarov Functional Analysis (NUFA) method (Ikot *et al*., 2021; Rampho *et al.,* 2020), the series expansion method (SEM) (Inyang *et al.,* 2020; Ibekwe *et al.,*2020;Inyang *et al*.,2021;Abu-Shady and Fath-Allah, 2019;Inyang *et al*.,2021; Ibekwe *et al.,* 2021), the analytical exact iterative method(AEIM) (Khokha *et al.,* 2016),WKB approximation method (Omugbe *et al.,* 2020;Omugbe *et al.,*2021;Omugbe *et al.,*2022; Omugbe, 2020; Omugbe,2020;Hitler *et al*.,2017), the Exact quantization rule (EQR) (Qiang *et al*.,2008;Inyang *et al*.,2020) and others (Mutuk, 2019). The Hulthén potential, (1942) is a short-range potential that behaves like a Coulomb potential for small values of r and decreases exponentially for large values of r . It has been used in many branches of physics, such as nuclear and particle physics, atomic physics, solid-state physics, and chemical physics (Nwabuzor *et al.,*2021).

The Hellmann potential,(1935) which is a superposition of an attraction Coulomb potential and a Yukawa potential has been studied extensively by many authors in obtaining the energy of the bound state in atomic, nuclear, and particle physics (Rai, and Rathaud,2015; Nasser *et al.,* 2014; Ikhdair, and Falaye, 2013).

Recently, there has been great interest in combining two potentials in both the relativistic and non-relativistic regimes (William *et al*., 2020). The essence of combining two or more physical potential models is to have a wider range of applications. Hence, in the present work, we aim at solving the SE with the combination of Hulthén and Hellmann potential (HHP) analytically using the series expansion method and apply the results to predict the mass spectra of heavy mesons such as bottomonium and charmonium, in which the quarks are considered as spinless particles for easiness. The adopted potential is of the form (William *et al.,* 2020).

$$V(r) = -\frac{A_0 e^{-\phi r}}{1-e^{-\phi r}} - \frac{A_1}{r} + \frac{A_2 e^{-\phi r}}{r} \qquad (1)$$

where A_0, A_1 and A_2 are potential strength parameters and ϕ is the screening parameter.

The expansion of the exponential terms in Eq. (1) (up to order three, to model the potential to interact in the quark-antiquark system) yields,

$$V(r) = -\frac{\beta_0}{r} + \beta_1 r - \beta_2 r^2 + \beta_3 \qquad (2)$$

where

$$\left.\begin{aligned} -\beta_0 &= -A_1 + A_2 - \frac{A_0}{\phi},\ \ \beta_1 = \frac{A_2\phi^2}{2} - \frac{A_0\phi}{12} \\ \beta_2 &= \frac{A_2\phi^3}{6},\ \ \beta_3 = \frac{A_0}{2} - A_2\phi \end{aligned}\right\} \qquad (3)$$

2.0 Solutions of the Schrödinger equation with Hulthén plus Hellmann potential

We consider the radial SE of the form (Ibekwe *et al.,* 2021)

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-V(r)\right)-\frac{l(l+1)}{r^2}\right]R(r)=0 \tag{4}$$

Where is the angular quantum number taking the values 0,1,2,3,4…, μ is the reduced mass for the heavy mesons, r is the internuclear separation and E_{nl} denotes the energy eigenvalues of the system. The substitution of Eq. (2) into Eq. (4) gives

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+\left[\varepsilon+\frac{P}{r}-Qr+Sr^2-\frac{\mathrm{L(L+1)}}{r^2}\right]R(r)=0 \tag{5}$$

Where

$$\left.\begin{aligned}\varepsilon=\frac{2\mu}{\hbar^2}\left(E_{nl}-\beta_3\right),\mathrm{P}=\frac{2\mu\beta_0}{\hbar^2}\\ Q=\frac{2\mu\beta_1}{\hbar^2},\mathrm{S}=\frac{2\mu\beta_2}{\hbar^2}\end{aligned}\right\} \tag{6}$$

$$L\left(L+1\right)=l\left(l+1\right) \tag{7}$$

The simplification of Eq. (7), yields equation 8

$$L=-\frac{1}{2}+\frac{1}{2}\sqrt{\left(2l+1\right)^2} \tag{8}$$

The anzats wave function is defined as follows (Rani *et al.*,2018).

$$R(r)=e^{-\alpha r^2-\beta r}F(r) \tag{9}$$

where α and β are positive constants whose values are to be determined in terms of potential parameters. The differentiation of Eq. (9) generates equations. (10) and (11) as follows:

$$R'(\mathrm{r})=F'(\mathrm{r})\mathrm{e}^{-\alpha r^2-\beta r}+F(r)\left(-2\alpha r-\beta\right)e^{-\alpha r^2-\beta r} \tag{10}$$

$$\begin{aligned}R''(\mathrm{r})&=\mathrm{F}''(\mathrm{r})\mathrm{e}^{-\alpha r^2-\beta r}+F'(r)\left(-2\alpha r-\beta\right)e^{-\alpha r^2-\beta r}\\&+\left[\left(-2\alpha\right)+\left(-2\alpha r-\beta\right)\left(-2\alpha r-\beta\right)\right]F(r)e^{-\alpha r^2-\beta r}\end{aligned} \tag{11}$$

Upon the substitution of Eqs. (9), (10) and (11) into Eq. (5) and subsequent division by $e^{-\sigma p^2-\rho p}$, equation 12 is obtained:

$$F''(\mathrm{r})+\left[-4\alpha r-2\beta+\frac{2}{r}\right]F'(\mathrm{r})+\begin{bmatrix}\left(4\alpha^2+S\right)r^2+\left(4\alpha\beta-B\right)r\\+\left(P-2\beta\right)\frac{1}{r}-\frac{L(L+1)}{r^2}+\left(\varepsilon+\beta^2-6\alpha\right)\end{bmatrix}F(r)=0 \tag{12}$$

The function $F(r)$ is considered as a series of the form given by equation 13 (Rani *et al.*,2018).

$$F(r)=\sum_{n=0}^{\infty}a_n r^{2n+L} \tag{13}$$

The first and second derivatives of Eq. (13) gives,

$$F'(r)=\sum_{n=0}^{\infty}\left(2n+L\right)a_n r^{2n+L-1} \tag{14}$$

$$F''(r)=\sum_{n=0}^{\infty}\left(2n+L\right)\left(2n+L-1\right)a_n r^{2n+L-2} \tag{15}$$

By substituting Eqs. (13),(10)and (15) into Eq.(12) ,we get

$$\begin{aligned}&\sum_{n=0}^{\infty}(2n+L)(2n+L-1)a_n r^{2n+L-2}+\left[-4\alpha r-2\beta+\frac{2}{r}\right]\sum_{n=0}^{\infty}(2n+L)a_n r^{2n+L-1}\\&+\left[\left(4\alpha^2+S\right)r^2+\left(4\alpha\beta-B\right)r+\left(P-2\beta\right)\frac{1}{r}-\frac{L(L+1)}{r^2}+\left(\varepsilon+\beta^2-6\alpha\right)\right]\sum_{n=0}^{\infty}a_n r^{2n+L}=0\end{aligned} \tag{16}$$

Collecting powers of r in Eq. (16) we have

$$\sum_{n=0}^{\infty}a_n\left\{\begin{aligned}&\left[\left(2n+L\right)\left(2n+L-1\right)+2\left(2n+L\right)-L\left(L+1\right)\right]r^{2n+L-2}\\&+\left[-2\beta\left(2n+L\right)+\left(P-2\beta\right)\right]r^{2n+L-1}\\&+\left[-4\alpha\left(2n+L\right)+\varepsilon+\beta^2-6\alpha\right]r^{2n+L}\\&+\left[4\alpha\beta-Q\right]r^{2n+L+1}+\left[4\alpha^2+S\right]r^{2n+L+2}\end{aligned}\right\}=0 \tag{17}$$

Equation (17) is linearly independent implying that each of the terms is separately equal to Zero, noting that r is a non-zero function; therefore, it is the coefficient of r that is zero. With this, we obtain the relation for each of the terms.

$$\left(2n+L\right)\left(2n+L-1\right)+2\left(2n+L\right)-L\left(L+1\right)=0 \tag{18}$$

$$-2\beta\left(2n+L\right)+P-2\beta=0 \tag{19}$$

$$-4\alpha\left(2n+L\right)+\varepsilon+\beta^2-6\alpha=0 \tag{20}$$

$$4\alpha\beta-Q=0 \tag{21}$$

$$4\alpha^2+S=0 \tag{22}$$

From Eqs. (19) and (22) we have

$$\beta=\frac{P}{4n+2L+2} \tag{23}$$

$$\alpha=\frac{\sqrt{-S}}{2} \tag{24}$$

The energy equation can be obtained from equation 20 as follows:

$$\varepsilon=2\alpha\left(4n+2L+3\right)-\beta^2 \tag{25}$$

The substitution of Eqs. (3),(6), (8), (23) and (24) into Eq. (25) yields the energy eigenvalues of HHP as,

$$\begin{aligned}E_{nl}=&\sqrt{\frac{-\hbar^2A_2\phi^3}{12\mu}}\left(4n+2+\sqrt{\left(2l+1\right)^2}\right)\\&-\frac{2\mu}{\hbar^2}\left(A_1-A_2+\frac{A_0}{\phi}\right)^2\left(4n+1+\sqrt{\left(2l+1\right)^2}\right)^{-2}+\frac{A_0}{2}-A_2\phi\end{aligned} \tag{26}$$

Special cases

1. Setting $A_0=0$ in Eq. (27) we obtain energy equation for Hellmann potential

$$E_{nl}=\sqrt{\frac{-\hbar^{2}A_{2}\phi^{3}}{12\mu}}\left(4n+2+\sqrt{\left(2l+1\right)^{2}}\right)$$

$$-\frac{2\mu}{\hbar^{2}}\left(A_{1}-A_{2}+\frac{A_{0}}{\phi}\right)^{2}\left(4n+1+\sqrt{\left(2l+1\right)^{2}}\right)^{-2}-A_{2}\phi \qquad (27)$$

2. Setting $A_1 = A_2 = 0$ in Eq. (26) we obtain energy equation for Hulthén potential

$$E_{nl}=-\frac{2\mu}{\hbar^{2}}\frac{A_{0}^{2}}{\phi^{2}}\left(4n+1+\sqrt{\left(2l+1\right)^{2}}\right)^{-2}+\frac{A_{0}}{2} \qquad (28)$$

3. Setting $A_0 = A_2 = \phi = 0$ in Eq. (26) we obtain energy equation for Coulomb potential

$$E_{nl}=-\frac{2\mu A_{1}^{2}}{\hbar^{2}}\left(4n+1+\sqrt{\left(2l+1\right)^{2}}\right)^{-2} \qquad (29)$$

4. Setting $A_0 = A_1 = 0$ in Eq. (26) we obtain energy equation for Yukawa potential

$$E_{nl}=\sqrt{\frac{-\hbar^{2}A_{2}\phi^{3}}{12\mu}}\left(4n+2+\sqrt{\left(2l+1\right)^{2}}\right)$$

$$-\frac{2\mu}{\hbar^{2}}\left(-A_{2}+\frac{A_{0}}{\phi}\right)^{2}\left(4n+1+\sqrt{\left(2l+1\right)^{2}}\right)^{-2}-A_{2}\phi \qquad (30)$$

3.0 Results and Discussion

The mass spectra of the heavy mesons such as charmonium and bottomonium is predicted by applying the following relation (Inyang *et al.,* 2021).

$$M=2m+E_{nl}, \qquad (31)$$

where m is heavy quark mass, and E_{nl} is energy eigen values.

Substituting Eq. (26) into Eq. (31) we obtain the mass spectra for HHP as:

$$M=2m+\sqrt{\frac{-\hbar^{2}A_{2}\phi^{3}}{12\mu}}\left(4n+2+\sqrt{\left(2l+1\right)^{2}}\right)$$

$$-\frac{2\mu}{\hbar^{2}}\left(A_{1}-A_{2}+\frac{A_{0}}{\phi}\right)^{2}\left(4n+1+\sqrt{\left(2l+1\right)^{2}}\right)^{-2}+\frac{A_{0}}{2}-A_{2}\phi \qquad (32)$$

We predict the mass spectra of heavy mesons such as charmonium and bottomonium for different quantum states using Eq. (32). The free parameters of Eq. (32) were then obtained by solving two algebraic equations in the case of charmonium and bottomonium, respectively.

For bottomonium $b\bar{b}$ and charmonium $c\bar{c}$, we adopt the numerical values of these masses as m_b = 4.823 GeV and m_c = 1.209 GeV (Olive *et al.,* 2014). Then, the corresponding reduced mass is μ_b = 2.4115 GeV and μ_c = 0.6045 GeV, respectively. The experimental data were taken from (Tanabashi *et al.,* 2018). We note that the prediction of mass spectra of charmonium and bottomonium are in good agreement with experimental data and the work of other researchers, as presented in Tables 1 and 2. Our predictions are improved compared to other theoretical predictions.

Table1. Mass spectra of charmonium in (GeV) (m_c =1.209 GeV, μ = 0.6045 GeV, A_0 = 1.422 GeV, A_1 = 2.949 GeV, A_2 = - 0.009 GeV, ϕ = 1.52 GeV, $\hbar$ = 1)

State	Present work	Abu-Shady, 2016	Ciftci,and Kisoglu,2018	Tanabashi *et al.,* 2018
1S	3.096	3.096	3.096	3.096
2S	3.686	3.686	3.672	3.686
1P	3.525	3.255	3.521	3.525
2P	3.772	3.779	3.951	3.773
3S	4.040	4.040	4.085	4.040
4S	4.263	4.269	4.433	4.263
1D	3.770	3.504	3.800	3.770
2D	4.159	-	-	4.159
1F	3.874	-	-	-

Table 2. Mass spectra of bottomonium in (GeV) (m_b =4.823 GeV, μ = 2.4115 GeV, A_0 = - 0.323 GeV, A_1 = 2.110 GeV, A_2 = - 0.031 GeV, ϕ = 1.52 GeV, $\hbar$ = 1)

State	Present work	Abu-Shady, 2016	Ciftci,and Kisoglu,2018	Tanabashi *et al.,* 2018
1S	9.460	9.460	9.462	9.460
2S	10.023	10.023	10.027	10.023
1P	9.898	9.619	9.963	9.899
2P	10.256	10.114	10.299	10.260
3S	10.355	10.355	10.361	10.355
4S	10.580	10.567	10.624	10.580
1D	10.164	9.864	10.209	10.164
2D	10.306	-	-	-
1F	10.209	-	-	-

4. 0 Conclusion

In this study, we modeled the adopted HHP to interact in quark-antiquark system. The solutions of the Schrödinger equation for energy eigenvalues using the series expansion method were obtained. The present result was applied to predict heavy-meson masses of charmonium and bottomonium for different quantum states. The result agreed with experimental data and are improved compared to other theoretical predictions.

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Conflict of Interest

The authors declares no conflict of interest,