Solutions of Second Order-Linear Ordinary Differential Equation with Variable Coefficients by Iterative Method

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Abstract: *The purpose of this study was to introduce an iterative method to solve second order linear ordinary differential equations with the variable coefficient for ordinary and singular points. This method was used to solve some examples of the equations. The solutions obtained proved that the method is effective, accurate and also reduced the large volume of the computational work that is generally associated with popular power series methods.*

Keywords: *Differential equations, ordinary point, singular point, iterative method*

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1.0 Introduction

Linear differential equations are used to model many physical phenomena in science and engineering, such as modeling of transmission of infectious diseases, population growth rate in an ecosystem, rate of cooling of a system, fluid dynamics, aerodynamics, etc.

Second-order linear differential equations with variable coefficients play important roles in the modeling of the aforementioned physical phenomena. However, the second-order linear differential equations of the form given below (equation 1) are difficult to solve:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0. \qquad (1)$$

where $P(x), Q(x)$ and $R(x)$ are functions of the independent variable x.

Therefore, the solution for equation 1 becomes the main subject of consideration for most authors (Dass, 2008; Joseph and Mihir, 2015; Stroud and Dexter, 2011, 2017; William and Richard, 2005)). The authors presented power series method of solution for the equation 1 which required large amount of tedious work and calculations. In spite amount of computations Nuran and Mustafa (2005) and Disu, Ishola and Olorunnishola (2013) further extended the power series solution method to Riccati and non-linear first order differential equations respectively.

More so, investigated perturbation parameter to obtain an asymptotes expansion solution of linear differential equations with constant coefficients. However, the method also required large volume of work and computations.

In an effort to overcome this problem, we propose an iterative method to solve second order linear ordinary differential equations with variable coefficient for both ordinary and singular points.

2.0 Second-Order Differential Equations with Variable Coefficients

A second order differential equation with variable coefficients is a differential equation where the coefficients are functions of the independent variable as shown in equation 1.

Suppose that we wish to solve equation 1 in a neighborhood of a point x_0. The solution of equation 1in an interval containing point x_0 is closely related to the behavior of $P(x)$ in the interval and the equivalent standard form of equation 1 can be written as equation 2,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \quad (2)$$

where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$.

If the functions $p(x)$ and $q(x)$ are analytic at $x = 0$, x is called an ordinary point for the Eq. (2). Otherwise, it is a singular point or singularity.

2.1 The basic concept of iterative method

A liner differential equation can be written as equation 3

$$Ly + Ry = f \ , \quad (3)$$

where L is the highest order-derivative which is assumed to be invertible, R linear differential operator and f is a source term.

The application of the inverse operator L^{-1} to both side of equation 3 yields equation 4

$$y = g - L^{-1}(Ry), \quad (4)$$

where g is the term obtained from the integration of the source term (f).

Under the assumption expressed by equation 5, the solution to equation 4 can be obtained by the substitution of equation 4 in equation 3 to obtain equation 6

$$y = \sum_{i=0}^{\infty} y_i \quad (5)$$

$$\sum_{i=0}^{\infty} y_i = g - \sum_{j=0}^{i} L^{-1}(Ry) \quad (6)$$

Therefore the recursive terms generated from equation 6 are given as

$$y_0 = g, y_{m+1} = L(y_m), \quad m = 0,\ 1,\ 2, \quad (7)$$

3.0 Solutions of the Second-Order Differential Equations with Variable Coefficients for Ordinary Point

Case 1: Consider $y'' - xy = 0$

$\Rightarrow y'' = xy$

$y = \iint \{xy\}dxdx + Bx + A$

The zeroth component and recurrence relation are

$y_0 = A + Bx$

$y_{n+1} = \iint \{xy_n\}dxdx$

The successive approximations are

$$y_1 = \iint \{x(A + Bx)\}dxdx = \frac{Ax^3}{2\times3} + \frac{Bx^4}{3\times4}$$

$$y_2 = \iint \{xy_1)\}dxdx = \iint \left\{\frac{Ax^4}{2\times3} + \frac{Bx^5}{3\times4})\right\}dxdx = \frac{Ax^6}{(2\times3)(5\times6)} + \frac{Bx^7}{(3\times4)(6\times7)}$$

$$y_3 = \iint \{xy_2)\}dxdx = \iint \left\{\frac{Ax^7}{(2\times3)(5\times6)} + \frac{Bx^8}{(3\times4)(6\times7)}\right\}dxdx = \frac{Ax^9}{(2\times3)(5\times6)(8\times9)} + \frac{Bx^{10}}{(3\times4)(6\times7)(9\times10)}$$

$$y_4 = \iint \{xy_3)\}dxdx = \iint \left\{\frac{Ax^{10}}{(2\times3)(5\times6)(8\times9)} + \frac{Bx^{11}}{(3\times4)(6\times7)(9\times10)})\right\}dxdx = \frac{Ax^{12}}{(2\times3)(5\times6)(8\times9)(11\times12)} + \frac{Bx^{13}}{(3\times4)(6\times7)(9\times10)(12\times13)}$$

Then, the solution is

$$y = y_0 + y_1 + y_2 + y_3 + y_4 + \cdots = A + Bx + \frac{Ax^3}{2\times3} + \frac{Bx^4}{3\times4} + \frac{Ax^6}{(2\times3)(5\times6)} + \frac{Bx^7}{(3\times4)(6\times7)} + \frac{Ax^9}{(2\times3)(5\times6)(8\times9)} + \frac{Bx^{10}}{(3\times4)(6\times7)(9\times10)} + \frac{Ax^{12}}{(2\times3)(5\times6)(8\times9)(11\times12)} + \frac{Bx^{13}}{(3\times4)(6\times7)(9\times10)(12\times13)} + \cdots$$

$$= A\left\{1 + \frac{x^3}{2\times3} + \frac{x^6}{(2\times3)(5\times6)} + \frac{x^9}{(2\times3)(5\times6)(8\times9)} + \frac{x^{12}}{(2\times3)(5\times6)(8\times9)(11\times12)} + \cdots\right\} + B\left\{x + \frac{x^4}{3\times4} + \frac{x^7}{(3\times4)(6\times7)} + \frac{x^{10}}{(3\times4)(6\times7)(9\times10)} + \frac{x^{13}}{(3\times4)(6\times7)(9\times10)(12\times13)} \ldots\right\}$$

Case 2: Consider $y'' + x^2y = 0$

$$\Rightarrow y'' = -x^2y$$

$$y = -\iint \{x^2y\}dxdx + Bx + A$$

$$\Longrightarrow y = A + Bx - \iint \{x^2y\}dxdx$$

$$y_0 = A + Bx$$

$$y_{n+1} = -\iint \{x^2 y_n\}dxdx$$

The successive approximations are:

$$y_1 = \iint \{-x^2 y_0\}dxdx = \iint \{-Ax^2 - Bx^3)\}dxdx = -\frac{Ax^4}{3\times4} - \frac{Bx^5}{4\times5}$$

$$y_2 = \iint \{-x^2 y_1\}dxdx = \iint \left\{\frac{Ax^6}{3\times4} + \frac{Bx^7}{4\times5}\right\} dxdx = \frac{Ax^8}{(3\times4)(7\times8)} + \frac{Bx^9}{(4\times5)(8\times9)}$$

$$y_3 = \iint \{-x^2 y_2\}dxdx = \iint \left\{-\frac{Ax^{10}}{(3\times4)(7\times8)} - \frac{Bx^{11}}{(4\times5)(8\times9)}\right\} dxdx = -\frac{Ax^{12}}{(3\times4)(7\times8)(11\times12)} - \frac{Bx^{13}}{(4\times5)(8\times9)(12\times13)}$$

$$y_4 = \iint \{-x^2 y_3\}dxdx = \iint \left\{\frac{Ax^{14}}{(3\times4)(7\times8)(11\times12)} + \frac{Bx^{15}}{(4\times5)(8\times9)(12\times13)}\right\} dxdx = \frac{Ax^{16}}{(3\times4)(7\times8)(11\times12)(15\times16)} + \frac{Bx^{17}}{(4\times5)(8\times9)(12\times13)(16\times17)}$$

The solution is

$$y = y_0 + y_1 + y_2 + y_4 + \cdots = A + Bx - \frac{Ax^4}{3\times4} - \frac{Bx^5}{4\times5} + \frac{Ax^8}{(3\times4)(7\times8)} + \frac{Bx^9}{(4\times5)(8\times9)} - \frac{Ax^{12}}{(3\times4)(7\times8)(11\times12)} - \frac{Bx^{13}}{(4\times5)(8\times9)(12\times13)} + \frac{Ax^{16}}{(3\times4)(7\times8)(11\times12)(15\times16)} + \frac{Bx^{17}}{(4\times5)(8\times9)(12\times13)(16\times17)}$$

$$= A\left\{1 - \frac{x^4}{3\times4} + \frac{x^8}{(3\times4)(7\times8)} - \frac{x^{12}}{(3\times4)(7\times8)(11\times12)} + \frac{x^{16}}{(3\times4)(7\times8)(11\times12)(15\times16)} + \cdots\right\} + B\left\{x - \frac{x^5}{4\times5} + \frac{x^9}{(4\times5)(8\times9)} - \frac{x^{13}}{(4\times5)(8\times9)(12\times13)} + \frac{x^{17}}{(4\times5)(8\times9)(12\times13)(16\times17)} + \cdots\right\}$$

Case 3: Consider $y'' - xy' + y = 0$

$$\Rightarrow y'' = x\frac{dy}{dx} - y$$

$$y = \iint \left\{x\frac{dy}{dx} - y\right\} dxdx + Bx + A$$

$$\Longrightarrow y_0 = A + Bx$$

$$y_{n+1} = \iint \left\{x\frac{dy_n}{dx} - y_n\right\} dxdx$$

The successive approximations are:

$$y_1 = \iint \left\{x\frac{dy_0}{dx} - y_0\right\} dxdx$$

$$= \iint \left\{x\frac{d}{dx}(A + Bx) - (A + Bx)\right\} dxdx = \iint \{Bx - A - Bx)\}dxdx = \iint \{-A)\}dxdx = -\frac{Ax^2}{2!}$$

$$y_2 = \iint \left\{x\frac{dy_1}{dx} - y_1\right\} dxdx = \iint \left\{x\frac{d}{dx}\left(-\frac{Ax^2}{2!}\right) - (-\frac{Ax^2}{2!})\right\} dxdx = \iint \left\{x\left(-\frac{2Ax}{2!}\right) -\right\} dxdx$$

$$= \iint \left\{-\frac{2Ax^2}{2!} + \frac{Ax^2}{2!}\right\} dxdx = \iint \left\{-\frac{Ax^2}{2!}\right\} dxdx = -\frac{Ax^4}{4!}$$

$$y_3 = \iint \left\{x\frac{dy_2}{dx} - y_2\right\} dxdx = \iint \left\{x\frac{d}{dx}\left(-\frac{Ax^4}{4!}\right) - (-\frac{Ax^4}{4!})\right\} dxdx = \iint \left\{x\left(-\frac{4Ax^3}{4!}\right) + \frac{Ax^4}{4!}\right\} dxdx$$

$$= \iint \left\{-\frac{4Ax^4}{4!} + \frac{Ax^4}{4!}\right\} dxdx = \iint \left\{-\frac{3Ax^4}{4!}\right\} dxdx = -\frac{3Ax^6}{6!}$$

$$y_4 = \iint \left\{x\frac{dy_3}{dx} - y_3\right\} dxdx = \iint \left\{x\frac{d}{dx}\left(-\frac{3Ax^6}{6!}\right) - (-\frac{3Ax^6}{6!})\right\} dxdx = \iint \left\{x\left(-\frac{18Ax^5}{6!}\right) + \frac{3Ax^6}{6!}\right\} dxdx = \iint \left\{-\frac{18Ax^6}{6!} + \frac{3Ax^6}{6!}\right\} dxdx = \iint \left\{-\frac{15Ax^6}{6!}\right\} dxdx = -\frac{15Ax^8}{8!}$$

$$y_5 = \iint \left\{x\frac{dy_4}{dx} - y_4\right\} dxdx = \iint \left\{x\frac{d}{dx}\left(-\frac{15Ax^8}{8!}\right) - (-\frac{15Ax^8}{8!})\right\} dxdx = \iint \left\{x\left(-\frac{120Ax^7}{8!}\right) + \frac{15Ax^8}{8!}\right\} dxdx = \iint \left\{-\frac{120Ax^8}{8!} + \frac{15Ax^8}{8!}\right\} dxdx = \iint \left\{-\frac{105Ax^8}{8!}\right\} dxdx = -\frac{105Ax^{10}}{10!}$$

The solution is:

$y = y_0 + y_1 + y_2 + y_3 + \cdots = A + Bx - \frac{Ax^2}{2!} - \frac{Ax^4}{4!} - \frac{3Ax^6}{6!} - \frac{15Ax^8}{8!} - \frac{105Ax^{10}}{10!} + \cdots = Bx + A\left\{1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{3x^6}{6!} - \frac{15x^8}{8!} - \frac{105x^{10}}{10!} - \cdots\right\}$

Case 4: $y'' + 2xy' + 2y = 0$

$\Rightarrow y'' = -2x\frac{dy}{dx} - 2y$

$y = -\iint \left\{2x\frac{dy}{dx} + 2y\right\} dxdx + a_1x + a_0$

$\Longrightarrow y = a_0 + a_1x - \iint \left\{2x\frac{dy}{dx} + 2y\right\} dxdx$

$y_0 = a_0 + a_1x$

$y_{n+1} = -\iint \left\{2x\frac{dy_n}{dx} + 2y_n\right\} dxdx$

The successive approximations are:

$y_1 = -\iint \left\{2x\frac{dy_0}{dx} + 2y_0\right\} dxdx = -\iint \left\{2x\frac{d}{dx}(a_0 + a_1x) + 2(a_0 + a_1x)\right\} dxdx$
$= -\iint \{2a_1x + 2a_0 + 2a_1x\}dxdx = -\iint \{2a_0 + 4a_1x\}dxdx = -\frac{2a_0x^2}{2} - \frac{4a_1x^3}{6} = -a_0x^2 - \frac{2a_1x^3}{3}$

$y_2 = -\iint \left\{2x\frac{dy_1}{dx} + 2y_1\right\} dxdx = -\iint \left\{2x\frac{d}{dx}\left(-a_0x^2 - \frac{2a_1x^3}{3}\right) + 2(-a_0x^2 - \frac{2a_1x^3}{3})\right\} dxdx$
$= -\iint \left\{2x\left(-2a_0x - \frac{6a_1x^2}{3}\right) + 2(-a_0x^2 - \frac{2a_1x^3}{3})\right\} dxdx = -\iint \left\{-4a_0x^2 - \frac{12a_1x^3}{3} - 2a_0x^2 - \frac{4a_1x^3}{3})\right\} dxdx = \iint \left\{6a_0x^2 + \frac{16a_1x^3}{3}\right\} dxdx = \frac{6a_0x^4}{12} + \frac{16a_1x^5}{3\times20} = \frac{a_0x^4}{2} + \frac{4a_1x^5}{15}$

$y_3 = -\iint \left\{2x\frac{dy_2}{dx} + 2y_2\right\} dxdx = -\iint \left\{2x\frac{d}{dx}\left(\frac{a_0x^4}{2} + \frac{4a_1x^5}{15}\right) + 2(\frac{a_0x^4}{2} + \frac{4a_1x^5}{15})\right\} dxdx$
$= -\iint \left\{2x\left(\frac{4a_0x^3}{2} + \frac{20a_1x^4}{15}\right) + 2(\frac{a_0x^4}{2} + \frac{4a_1x^5}{15})\right\} dxdx = -\iint \left\{\frac{8a_0x^4}{2} + \frac{40a_1x^5}{15} + \frac{2a_0x^4}{2} + \frac{8a_1x^5}{15}\right\} dxdx = -\iint \left\{\frac{10a_0x^4}{2} + \frac{48a_1x^5}{15}\right\} dxdx = -\frac{10a_0x^6}{2\times5\times6} - \frac{48a_1x^7}{15\times6\times7} = -\frac{a_0x^6}{6} - \frac{2a_1x^7}{35}$

The solution is

$y = y_0 + y_1 + y_2 + y_3 + \cdots = a_0 + a_1x - a_0x^2 - \frac{2a_1x^3}{3} + \frac{a_0x^4}{2} + \frac{4a_1x^5}{15} - \frac{a_0x^6}{6} - \frac{2a_1x^7}{35}$
$= a_0\left\{1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots\right\} + a_1\left\{x - \frac{2x^3}{3} + \frac{4x^5}{15} - \frac{2x^7}{35} + \cdots\right\}$

4.0 Solutions of Second Order Differential Equations with Variable Coefficients for Singular Point

Case 1: Consider $(x-1)y'' + 2y' = 0$

The equation is a singular point at point $x = 1$.

$xy'' - y'' + 2y' = 0$

$-y'' = -xy'' - 2y'$

$y'' = xy'' + 2y'$

This is equivalent to

$y = a_0 + xa_1 + \iint \{xy'' + 2y'\}dxdx$

The zeroth component y_0 and recurrence relation are:

$y_0 = a_0 + xa_1$

$y_{n+1} = \iint \left\{x\frac{d^2y_n}{dx^2} + 2\frac{dy_n}{dx}\right\} dxdx$

The successive approximations are

$y_1 = \iint \left\{x\frac{d^2y_0}{dx^2} + 2\frac{dy_0}{dx}\right\} dxdx$

$y_1 = \iint \left\{x\frac{d^2}{dx^2}(a_0 + xa_1) + 2\frac{d}{dx}(a_0 + xa_1)\right\} dxdx = \iint (2a_1)dxdx = a_1x^2$

$y_2 = \iint \left\{x\frac{d^2y_1}{dx^2} + 2\frac{dy_1}{dx}\right\} dxdx = \iint \left\{x\frac{d^2}{dx^2}(a_1x^2) + 2\frac{d}{dx}(a_1x^2)\right\} dxdx = \iint \{(2a_1x + 2a_1x)\}dxdx$
$= \iint (6a_1x)dxdx = a_1x^3$

$y_3 = \iint \left\{x\frac{d^2y_2}{dx^2} + 2\frac{dy_2}{dx}\right\} dxdx = \iint \left\{x\frac{d^2}{dx^2}(a_1x^3) + 2\frac{d}{dx}(a_1x^3)\right\} dxdx = \iint \{(6a_1x^2 + 6a_1x^2)\}dxdx = \iint (12a_1x^2)dxdx = a_1x^4$

The solution is:

$y = y_0 + y_1 + y_2 + y_3 + \cdots = a_0 + xa_1 + a_1x^2 + a_1x^3 + a_1x^4 + \cdots = a_0 + a_1\{x + x^2 + x^3 + x^4 + \cdots\}$

Case 2: Consider $(1 - x^2)y'' - 2xy' = 0$

The equation has two singular points $x = -1, 1$.

$y'' - x^2y'' - 2xy' = 0$

$y'' = x^2y'' + 2xy'$

This is equivalent to

$y = a_0 + xa_1 + \iint \{x^2y'' + 2xy'\}dxdx$

The zeroth component y_0 and recurrence relation are:

$y_0 = a_0 + xa_1$

$y_{n+1} = \iint \left\{x^2\frac{d^2y_n}{dx^2} + 2x\frac{dy_n}{dx}\right\} dxdx$

The successive approximations are

$y_1 = \iint \left\{x^2\frac{d^2y_0}{dx^2} + 2x\frac{dy_0}{dx}\right\} dxdx = \iint \left\{x^2\frac{d^2}{dx^2}(a_0 + xa_1) + 2x\frac{d}{dx}(a_0 + xa_1)\right\} dxdx = \iint (2xa_1)dxdx = \frac{a_1x^3}{3}$

$y_2 = \iint \left\{x^2\frac{d^2y_1}{dx^2} + 2x\frac{dy_1}{dx}\right\} dxdx = \iint \left\{x^2\frac{d^2}{dx^2}(\frac{a_1x^3}{3}) + 2x\frac{d}{dx}(\frac{a_1x^3}{3})\right\} dxdx$

$= \iint \left\{\frac{6a_1x^3}{3} + \frac{6a_1x^3}{3})\right\} dxdx = \iint \left\{\frac{12a_1x^3}{3}\right\} dxdx = \iint \{4a_1x^3\}dxdx = \frac{4a_1x^5}{20} = \frac{a_1x^5}{5}$

$y_3 = \iint \left\{x^2\frac{d^2y_2}{dx^2} + 2x\frac{dy_2}{dx}\right\} dxdx = \iint \left\{x^2\frac{d^2}{dx^2}(\frac{a_1x^5}{5}) + 2x\frac{d}{dx}(\frac{a_1x^5}{5})\right\} dxdx$

$= \iint \left\{\frac{20a_1x^5}{5} + \frac{10a_1x^5}{5})\right\} dxdx = \iint \left\{\frac{30a_1x^5}{5}\right\} dxdx = \iint \{6a_1x^5\}dxdx = \frac{a_1x^7}{7}$

The solution is

$y = y_0 + y_1 + y_2 + y_3 + \cdots = a_0 + xa_1 + \frac{a_1x^3}{3} + \frac{a_1x^5}{5} + \frac{a_1x^7}{7} + \cdots = a_0 + a_1\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$

Case 3: Consider $(x^2 - 4)y'' + 3xy' + y = 0$

The equation has two singular points at points $x = -2,\ 2$

$(x^2 - 4)y'' + 3xy' + y = 0$

$x^2y'' - 4y'' + 3xy' + y = 0$

$-4y'' = -x^2y'' - 3xy' - y$

$4y'' = x^2y'' + 3xy' + y$

$y'' = \frac{1}{4}(x^2y'' + 3xy' + y)$

This is equivalent to the integral equation

$y = \frac{1}{4}\iint \{x^2y'' + 3xy' + y\}dxdx + Bx + A$

$\Rightarrow y = A + Bx + \frac{1}{4}\iint \left\{x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y\right\} dxdx$

The zeroth component y_0 and recurrence relation are:

$y_0 = A + Bx$

$y_n = \frac{1}{4}\iint \left\{x^2\frac{d^2y_n}{dx^2} + 3x\frac{dy_n}{dx} + y_n\right\} dxdx$

The successive approximations are

$y_1 = \frac{1}{4}\iint \left\{x^2\frac{d^2y_0}{dx^2} + 3x\frac{dy_0}{dx} + y_0\right\} dxdx = \frac{1}{4}\iint \left\{x^2\frac{d^2}{dx^2}(A + Bx) + 3x\frac{d}{dx}(A + Bx) + (A + Bx)\right\} dxdx$

$= \frac{1}{4}\iint \{3Bx + A + Bx)\}dxdx = \frac{1}{4}\iint \{4Bx + A)\}dxdx = \frac{1}{4}\left\{\frac{4Bx^3}{2\times 3} + \frac{Ax^2}{2}\right\} = \frac{Bx^3}{6} + \frac{Ax^2}{8}$

$$y_2 = \frac{1}{4}\iint \left\{x^2\frac{d^2y_1}{dx^2} + 3x\frac{dy_1}{dx} + y_1\right\}dxdx \quad = \frac{1}{4}\iint \left\{x^2\frac{d^2}{dx^2}\left(\frac{Bx^3}{6} + \frac{Ax^2}{8}\right) + 3x\frac{d}{dx}\left(\frac{Bx^3}{6} + \frac{Ax^2}{8}\right) + \left(\frac{Bx^3}{6} + \frac{Ax^2}{8}\right)\right\}dxdx = \frac{1}{4}\iint \left\{x^2\frac{d}{dx}\left(\frac{3Bx^2}{6} + \frac{2Ax}{8}\right) + 3x\left(\frac{3Bx^2}{6} + \frac{2Ax}{8}\right) + \left(\frac{Bx^3}{6} + \frac{Ax^2}{8}\right))\right\}dxdx$$

$$= \frac{1}{4}\iint \left\{\frac{6Bx^3}{6} + \frac{2Ax^2}{8} + \frac{9Bx^3}{6} + \frac{6Ax^2}{8} + \frac{Bx^3}{6} + \frac{Ax^2}{8}\right\}dxdx \; = \iint \left\{\frac{6Bx^3}{4\times6} + \frac{2Ax^2}{4\times8} + \frac{9Bx^3}{4\times6} + \frac{6Ax^2}{4\times8} + \frac{Bx^3}{4\times6} + \frac{Ax^2}{4\times8}\right\}dxdx = \iint \left\{\frac{9Ax^2}{4\times8} + \frac{16Bx^3}{4\times6}\right\}dxdx = \frac{9Ax^4}{(4\times8)(3\times4)} + \frac{16Bx^5}{(4\times6)(4\times5)} = \frac{3Ax^4}{128} + \frac{Bx^5}{30}$$

$$y_3 = \frac{1}{4}\iint \left\{x^2\frac{d^2y_2}{dx^2} + 3x\frac{dy_2}{dx} + y_2\right\}dxdx \quad = \frac{1}{4}\iint \left\{x^2\frac{d^2}{dx^2}\left(\frac{3Ax^4}{128} + \frac{Bx^5}{30}\right) + 3x\frac{d}{dx}\left(\frac{3Ax^4}{128} + \frac{Bx^5}{30}\right) + \left(\frac{3Ax^4}{128} + \frac{Bx^5}{30}\right)\right\}dxdx \; = \iint \left\{\frac{36Ax^4}{4\times128} + \frac{20Bx^5}{4\times30} + \frac{36Ax^4}{4\times128} + \frac{15Bx^5}{4\times30} + \frac{3Ax^4}{4\times128} + \frac{Bx^5}{4\times30}\right\}dxdx \; = \iint \left\{\frac{75Ax^4}{4\times128} + \frac{36Bx^5}{4\times30}\right\}dxdx = \frac{75Ax^6}{(4\times128)(5\times6)} + \frac{36Bx^7}{(4\times30)(6\times7)} = \frac{5Ax^6}{1024} + \frac{Bx^7}{140}$$

The solution is:

$$y = y_0 + y_1 + y_2 + y_3 + \cdots = A + Bx + \frac{Ax^2}{8} + \frac{Bx^3}{6} + \frac{3Ax^4}{128} + \frac{Bx^5}{30} + \frac{5Ax^6}{1024} + \frac{Bx^7}{140} + \cdots$$

$$= A\left\{1 + \frac{x^2}{8} + \frac{3x^4}{128} + \frac{5x^6}{1024} + \cdots\right\} + B\left\{x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{x^7}{140} + \cdots\right\}$$

Case 4: Consider $(1 - x^2)y'' - 2xy' + 2y = 0$

The equation has two singular points at points $x = -1, 1$

$y'' - x^2y'' - 2xy' + 2y = 0$

$y'' = x^2y'' + 2xy' - 2y$

This is equivalent to

$y = a_0 + xa_1 + \iint \{x^2y'' + 2xy' - 2y\}dxdx$

The zeroth component y_0 and recurrence relation are:

$y_0 = a_0 + xa_1$

$$y_{n+1} = \iint \left\{x^2\frac{d^2y_n}{dx^2} + 2x\frac{dy_n}{dx} - 2y_n\right\}dxdx$$

The successive approximations are

$$y_1 = \iint \left\{x^2\frac{d^2y_0}{dx^2} + 2x\frac{dy_0}{dx} - 2y_0\right\}dxdx = \iint \left\{x^2\frac{d^2}{dx^2}(a_0 + xa_1) + 2x\frac{d}{dx}(a_0 + xa_1) - 2(a_0 + xa_1)\right\}dxdx = \iint (2xa_1 - 2a_0 - 2xa_1)dxdx = \iint -2a_0dxdx = -\frac{2a_0x^2}{2} = -a_0x^2$$

$$y_2 = \iint \left\{x^2\frac{d^2y_1}{dx^2} + 2x\frac{dy_1}{dx} - 2y_1\right\}dxdx = \iint \left\{x^2\frac{d^2}{dx^2}(-a_0x^2) + 2x\frac{d}{dx}(-a_0x^2) - 2(-a_0x^2)\right\}dxdx = \iint \{-2a_0x^2 - 4a_0x^2 + 2a_0x^2\}dxdx = \iint \{-4a_0x^2\}dxdx = -\frac{4a_0x^4}{3\times4} = -\frac{a_0x^4}{3}$$

$$y_3 = \iint \left\{x^2\frac{d^2y_2}{dx^2} + 2x\frac{dy_2}{dx} - 2y_2\right\}dxdx = \left\{x^2\frac{d^2}{dx^2}\left(-\frac{a_0x^4}{3}\right) + 2x\frac{d}{dx}\left(-\frac{a_0x^4}{3}\right) - 2\left(-\frac{a_0x^4}{3}\right)\right\}dxdx = \iint \left\{-\frac{12a_0x^4}{3} - \frac{8a_0x^4}{3} + \frac{2a_0x^4}{3}\right\}dxdx = \iint \left\{-\frac{18a_0x^4}{3}\right\}dxdx = -\frac{18a_0x^6}{3\times5\times6} = -\frac{a_0x^6}{5}$$

The solution is:

$$y = y_0 + y_1 + y_2 + y_3 + \cdots = a_0 + xa_1 - a_0x^2 - \frac{a_0x^4}{3} - \frac{a_0x^6}{5} - \cdots = a_1x + a_0\left\{1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \cdots\right\}$$

5.0 Conclusion

In this study, the iterative method was applied for solving second order linear ordinary differential equation with variable coefficients. The method was applied only with the use of integration and differentiation. Results obtained affirmed that the iterative method is very powerful and efficient in finding the series solutions for the differential equations. It is worth mentioning that the method reduced the large volume of the computational work that is generally associated with popular power series methods while still maintaining high accuracy.

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Conflict of Interest

The authors declared no conflict of interest