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Derivation of a New Odd Exponential-Weibull Distribution

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Abstract: *The study of statistical distributions has led to the development of numerous extensions of well-known continuous distributions to enhance their flexibility and applicability across various fields. In this paper, we introduce a new three-parameter distribution known as the Odd Exponential-Weibull (OE-W) distribution, which extends the traditional Weibull distribution by incorporating additional parameter. We thoroughly investigate the mathematical properties of the OE-W distribution, deriving explicit formulas for the quantile function, moments, moment-generating function, survival function, hazard function, entropy, and order statistics. Parameter estimation is conducted using the maximum likelihood estimation (MLE) method, which is known for its robustness. To assess the reliability and accuracy of these parameter estimates, a Monte Carlo simulation study is performed. The simulation results indicate that the MLE method consistently yields reliable and accurate estimates for the parameters of the OE-W distribution. The introduction of this new distribution provides a valuable tool for modeling and analyzing data in fields such as reliability engineering and survival analysis, where flexible and accurate probability distributions are crucial.*

Keywords: *Odd-Exponential-G, Weibull, Quantile function survival function, Maximum likelihood, Order Statistics*.

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1.0 Introduction

Statistical probability distributions are very important in explaining real-world situations. Statistical distributions are in general, used in describing real-world occurrences. In line with the usefulness of these statistical distributions, their theory is largely studied and applied to many scenarios in real life. One of the main challenges of statisticians is to find the appropriate and efficient statistical distribution in modelling natural life events in the form of known probability distributions. Probability distributions pose important usefulness in modelling natural life phenomena that are characterized by uncertainty and riskiness. It was noted that the interest in developing and modifying existing distributions to form more flexible statistical distributions remains an arguable topic in the statistics profession; numerous generalized forms and classes of distributions have emerged and applied to describe various phenomena a common feature of these generalized distributions is that they have more parameters. In several fields of study such as medicine, engineering, and finance, estimating and analyzing lifetime data is fundamental, some lifetime distributions have been employed to model such kinds of data, and the feature of the techniques used in a statistical analysis depends heavily on the assumed probability distributions given this, significant efforts have been used in the development of large classes of typical probability distributions along with pertinent statistical procedures. Researchers in many fields of study have contributed to the development and application of new probability distribution models.

Mudholkar *et al.*, (1996) introduced the Exponentiated Weibull distribution, followed by the modified Weibull extension by Xie *et al.*, (2002). Other notable developments include the flexible Weibull extension (FWEx) by Bebbington *et al.*, (2007), the beta-modified Weibull by Silva *et al.*, (2010), the Kumaraswamy Weibull by Cordeiro *et al.*, (2010), and the transmuted Weibull by Aryal and Tsokos (2011). Zhang and Xie (2011) proposed the truncated Weibull distribution, while Cordeiro *et al.*, (2013) introduced the exponentiated generalized Weibull. Merovci and Elbatal (2013) developed the McDonald modified Weibull, and Hanook *et al.*, (2013) presented the beta inverse Weibull. Elbatal and Aryal (2013) introduced the transmuted additive Weibull, and Cordeiro *et al.*, (2014) proposed both the McDonald Weibull and Kumaraswamy modified Weibull. Afify *et al.*, (2014) introduced the transmuted complementary Weibull geometric, and Nofal *et al.*, (2016) presented the Kumaraswamy transmuted exponentiated additive Weibull. Nofal *et al.*, (2017) developed the generalized transmuted Weibull, and Aryal *et al*. (2017) introduced the Topp-Leone generated Weibull. Additional contributions include the Kumaraswamy complementary Weibull geometric by Afify *et al.*, (2017), the Marshall-Olkin additive Weibull by Afify *et al.*, (2018), the Zubair–Weibull by Ahmad (2018), the alpha power transformed Weibull by Ahmad *et al.*, (2019), the Topp Leone exponentiated Weibull by Ibrahim (2021), the Type II Half-Logistic Exponentiated Weibull by Akanji *et al.*, (2023), and the Type I Half-Logistic Exponentiated Weibull by Bello *et al.*, (2023).

Based on the inference drawn by Bourguignon *et al*. (2014), the cumulative distribution function (CDF) and probability density function (PDF) for the Odd Exponential-G family of distributions are defined as follows:

$$F_{OEG}\left(x;\delta,\xi\right)=1-\exp\left\{-\frac{\delta M\left(x;\xi\right)}{1-M\left(x;\xi\right)}\right\};\ \forall x;\delta,\xi>0 \tag{1}$$

$$f_{OEG}\left(x;\delta,\xi\right)=\frac{\delta m\left(x;\xi\right)}{\left(1-M\left(x;\xi\right)\right)^{2}}\exp\left\{-\frac{\delta M\left(x;\xi\right)}{1-M\left(x;\xi\right)}\right\};\ \forall x;\delta,\xi>0 \tag{2}$$

where $\delta>0$ is the scale parameter and $x>0$, $m\left(x;\xi\right)$ and $M\left(x;\xi\right)$ are the pdf and CDF and ξ is the parameters' vector of the baseline distribution.

Based on the work published by y Sadiq *et al.*, (2023), a random variable X is considered to follow a Weibull distribution with a scale parameter a and a shape parameter b if its PDF and CDF are expressed as follows:

$$m\left(x\,;\alpha\,,\beta\right)=\beta\alpha^{-\beta}x^{\beta-1}\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\quad;\quad x,\alpha,\beta>0 \tag{3}$$

$$M\left(x;\alpha,\beta\right)=1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\};\quad x,\alpha,\beta>0 \tag{4}$$

The proposed study is expected to contribute to the field of statistical distributions by providing a new, flexible extension of the Odd Exponential-Weibull (OE-W) distribution. This new distribution can potentially offer better fit and modelling capabilities for various types of data, particularly in reliability engineering and survival analysis.

2.0 The Odd Exponential-Weibull Distribution

Let the baseline distribution M has Weibull distribution with pdf and cdf as in equations (3) and (4), then the PDF and CDF of Odd Exponential-Weibull distribution (OE-W) are defined by inserting equations (3) and (4) in equation (1) and equation (2) respectively.

$$f_{OEW}\left(x;\delta,\alpha,\beta\right)=\frac{\delta\left(\beta\alpha^{-\beta}x^{\beta-1}\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{2}}\exp\left\{-\frac{\delta\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}\right\};\ \forall x;\delta,\alpha,\beta>0 \tag{5}$$

$$F_{OEW}\left(x;\delta,\alpha,\beta\right)=1-\exp\left\{-\frac{\delta\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}\right\};\ \forall x;\delta,\alpha,\beta>0 \tag{6}$$

3.0 Suitable Expansion for the Odd Exponential-Weibull (OE-W) Distribution

Here, we consider the individual terms in the given PDF and CDF of the OE-W distribution via some standard mathematical expansion, which comprises the generalized binomial expansion for negative and positive power, the power series expansion and so on for instance,

$$e^{-ax}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}(ax)^{i} \tag{7}$$

$$(1-x)^{ab}=\sum_{j=0}^{\infty}(-1)^{j}\binom{ab}{j}(x)^{j} \tag{8}$$

However, equation (6) using the expansion series in equation (8) is as,

$$F_{OEW}\left(x;\delta,\alpha,\beta\right)=1-\exp\left\{-\frac{\delta\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}\right\}$$

$$1-\exp\left\{-\frac{\delta\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}\right\}=\sum_{i=0}^{\infty}(-1)^{i}\binom{1}{i}\exp\left\{-\frac{\delta\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}\right\}^{i} \tag{9}$$

Using equation (7)

$$\exp\left\{-\frac{\delta\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}\right\}^{i}=\sum_{j=0}^{\infty}\frac{(-1)^{j}}{j!}\left(\frac{\delta\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}\right)^{ij} \tag{10}$$

Also using equation (8)

$$\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{ij}=\sum_{k=0}^{\infty}(-1)^{k}\binom{ij}{k}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k} \tag{11}$$

Combining equations (9), (10) and (11) we have,

$$F_{OEW}(x;\delta,\alpha,\beta)=\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1}{i}\binom{ij}{k}\frac{\delta^{ij}}{j!}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{-ij}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k}$$

$$=\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1}{i}\binom{ij}{k}\frac{\delta^{ij}}{j!}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k-ij}$$

$$F_{OEW}(x;\delta,\alpha,\beta)=\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1}{i}\binom{ij}{k}\frac{\delta^{ij}}{j!}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k-ij} \tag{12}$$

Therefore, equation (13) reduces to,

$$F_{OEW}(x;\delta,\alpha,\beta)=\sum_{i,j,k=0}^{\infty}A_{i,j,k}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k-ij} \tag{13}$$

where $A_{i,j,k}=\dfrac{\binom{1}{i}\binom{ij}{k}\delta^{ij}(-1)^{i+j+k}}{j!}$

Differentiating equation (13) w.r.t. x we have the corresponding PDF as,

$$f_{OEW}(x;\delta,\alpha,\beta)=\sum_{i,j=0}^{\infty}A_{i,j,k}(k-ij)\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{(k-ij)-1} \tag{14}$$

Now $A_{i,j,k}=\dfrac{\beta\alpha^{-\beta}x^{\beta-1}\binom{1}{i}\binom{ij}{k}\delta^{ij+1}(-1)^{i+j+k}}{j!}$

Therefore, equations (13) and (14) are the reduced CDF and PDF of the OE-W Distribution.

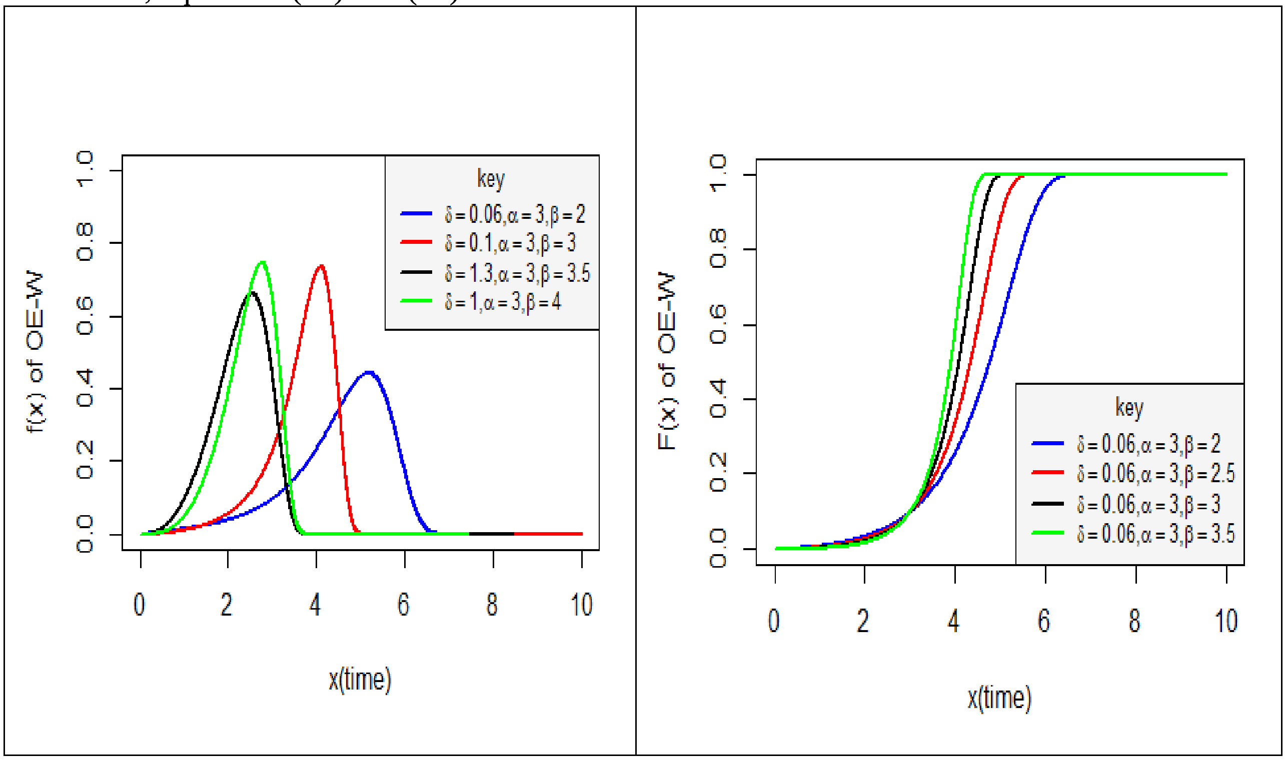

Fig 1: Plots of PDF and CDF of OE-W distribution for different values of parameters.

4.0 Mathematical Properties of OE-W Distribution

This section explores various mathematical properties of OE-W distribution.

4.1 Moments of the OE-W Distribution

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the rth moments for the OE-W distribution.

$$E(X^r) = \int_0^\infty x^r f(x;\delta,\alpha,\beta)dx \tag{15}$$

The rth moments for OE-W distribution are derived by substituting equation (14) into equation (15) we obtain

$$E(X^r) = \sum_{i,j=0}^{\infty} A_{i,j,k}\left(k-ij\right)\int_0^\infty x^r \left(\exp\left\{-\left(\frac{x}{\alpha}\right)^\beta\right\}\right)^{(k-ij)-1} dx \tag{16}$$

4.2 Moment Generating Function of OE-W Distribution

The moment-generating function of a random variable X which follows the OE-W distribution by using equation (14) we have,

$$M_X^{OE-W}(t) = E\left(e^{tx}\right) \tag{17}$$

$$= \int_0^\infty e^{tx} f_{OE-W}\left(x;\delta,\alpha,\beta\right)dx$$

$$M_X^{OE-W}(t)=\sum_{i,j=0}^{\infty}A_{i,j,k}(k-ij)\int_0^{\infty}e^{tx}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{(k-ij)-1}dx \tag{18}$$

4.3 Entropies of OE-W Distribution

The entropy of any random variable X is a measure of indecisiveness, variability, and details innate to the probable results of the variable. It is defined mathematically by Renyi (1961) using equation (14) we have,

$$I_R(\varpi)=\frac{1}{1-\varpi}\log\left(\int_0^{\infty}f_{OE-W}^{\varpi}(x;\delta,\alpha,\beta)\,dx\right)$$

$$=\frac{1}{1-\varpi}\log\left(\int_0^{\infty}\left(\sum_{i,j=0}^{\infty}A_{i,j,k}(k-ij)\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{(k-ij)-1}\right)^{\varpi}dx\right) \tag{19}$$

where $\varpi>0$ and $\varpi\neq 1$

The nth entropy is defined by

$$I_{nth}(\varpi)=\frac{1}{\varpi-1}\log\left(1-\int_0^{\infty}f_{OE-W}^{\varpi}(x;\delta,\alpha,\beta)\,dx\right)$$

$$=\frac{1}{1-\varpi}\log\left(1-\int_0^{\infty}\left(\sum_{i,j=0}^{\infty}A_{i,j,k}(k-ij)\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{(k-ij)-1}\right)^{\varpi}dx\right) \tag{20}$$

where $\varpi>0$ and $\varpi\neq 1$

4.4 Survival Function of OE-W Distribution

The survival function of a random variable X which follows the OE-W distribution is given by

$$S_{OEW}(x;\delta,\alpha,\beta)=\exp\left\{-\frac{\delta\left(1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}\right\} \tag{21}$$

4.5 Hazard Function of OE-W Distribution

The hazard function of a random variable X which follows the OE-Weibull distribution is given by

$$h_{OEW}(x;\delta,\alpha,\beta)=\frac{\delta\beta\alpha^{-\beta}x^{\beta-1}}{\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}} \tag{22}$$

4.6 Quantile Function of the OE-Weibull Distribution

The quantile function of the OE-W distribution is obtained by inverting the CDF in equation (6). Since the variable U is uniformly distributed on (0,1), then, we equate U with the CDF as;

$$x_u = (-\phi)\left(\log\left(1-\left(\frac{\log(1-u)}{(\log(1-u)-\delta)}\right)\right)\right)^{\frac{1}{\omega}} \tag{23}$$

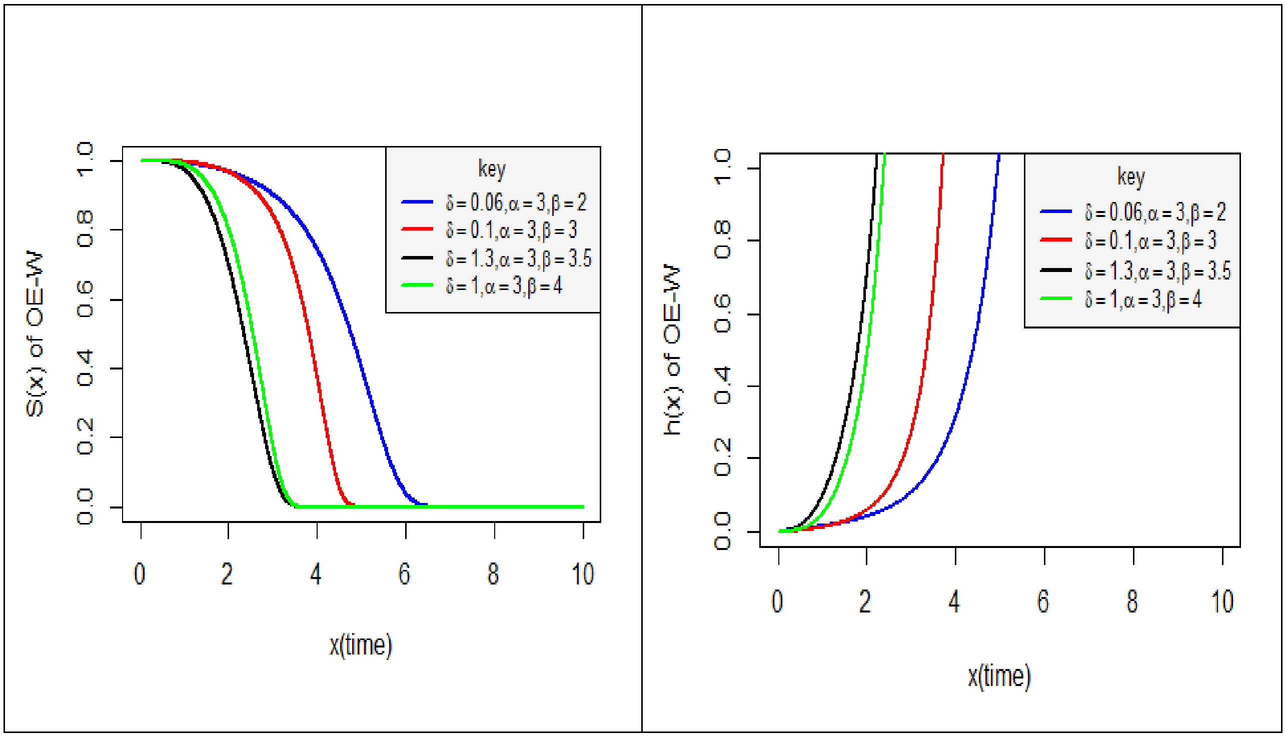

Fig 2: Plots of Survival and hazard functions of the OE-W distribution for different valves of parameter

5.0 Order Statistics of OE-W Distribution

Suppose $X_1, X_2, X_3, \ldots, X_n$ is a random sample from the OE-W distribution and $X_{r:n}$ represents the r[th] order statistic, then, using equations (13) and (14) we have

$$f_{r:n}(x;\delta,\alpha,\beta)=\frac{n!}{\left[(r-1)!(n-r)!\right]}\left[f_{OE-W}(x;\delta,\alpha,\beta)\right]\left[F_{OE-W}(x;\delta,\alpha,\beta)\right]^{r-1}\left[1-F_{OE-W}(x;\delta,\alpha,\beta)\right]^{n-r}$$

$$=\frac{n!}{\left[(r-1)!(n-r)!\right]}\left[\sum_{i,j=0}^{\infty}A_{i,j,k}\,(k-ij)\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{(k-ij)-1}\right]\left[\sum_{i,j,k=0}^{\infty}A_{i,j,k}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k-ij}\right]^{r-1}\left[1-\sum_{i,j,k=0}^{\infty}A_{i,j,k}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k-ij}\right]^{n-r} \quad (24)$$

The equation above is called the r[th] order statistics for the OE-W distribution.

Let r = n, then the probability density function of the maximum order statistics of OE-W distribution is

$$f_{n:n}\left(x;\delta,\alpha,\beta\right)=n\left[\sum_{i,j=0}^{\infty}A_{i,j,k}\left(k-ij\right)\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{(k-ij)-1}\right]\left[\sum_{i,j,k=0}^{\infty}A_{i,j,k}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k-ij}\right]^{n-1} \quad (25)$$

Also, let r = 1, then the probability density function of the minimum order statistics of OE-W distribution is

$$f_{1:n}\left(x;\delta,\alpha,\beta\right)=n\left[\sum_{i,j=0}^{\infty}A_{i,j,k}\left(k-ij\right)\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{(k-ij)-1}\right]\left[1-\sum_{i,j,k=0}^{\infty}A_{i,j,k}\left(\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)^{k-ij}\right]^{n-1} \quad (26)$$

6.0 Estimation of Parameters for OE-Weibull Distribution

Suppose that $x_1, x_2, x_3, \ldots, x_n$ are the observed values from the developed OE-W distribution with parameters δ, α and β. Suppose that $\Phi=\left[\delta,\alpha,\beta\right]^T$ is the $\left[m\times 1\right]$ vector of the parameters. The log-likelihood function of Φ using equation (5) is expressed by;

$$\ell_n\left(\Phi\right) = \sum_{i=1}^{n}\left(\log\delta+\log\beta-\beta\log\alpha+\left(\beta-1\right)\log x_i-\left(\frac{x_i}{\alpha}\right)^{\beta}-\delta\left(\frac{1-\exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}}{\exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}}\right)\right) \quad (27)$$

Taking the partial derivative of equation (24) w.r.t. the parameters $\left(\delta,\alpha \text{ and } \beta\right)$ and setting them to zero gives:

$$\frac{\partial\ell_n\left(\Phi\right)}{\partial\alpha}=\sum_{i=1}^{n}\left(-\frac{\alpha}{\beta}+\frac{\beta x_i^{\beta}}{\alpha^{\beta+1}}+\frac{\delta\beta x_i^{\beta}\left(\exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}\right)}{\alpha^{\beta+1}\exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}}\right) \quad (28)$$

$$\frac{\partial\ell_n\left(\Phi\right)}{\partial\delta}=\sum_{i=1}^{n}\left(\frac{1}{\delta}-\frac{\left(1-\exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}\right)}{\exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}}\right) \quad (29)$$

$$\frac{\partial\ell_n\left(\Phi\right)}{\partial\beta}=\sum_{i=1}^{n}\left(\frac{1}{\beta}+\log x_i-\log\alpha-\left(\frac{x_i}{\alpha}\right)^{\beta}\log\left(\frac{x_i}{\alpha}\right)+\frac{\delta\left(1-\exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}\right)\log\left(\frac{x_i}{\alpha}\right)}{\exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}}\right) \quad (30)$$

The MLEs of the parameters $\left(\delta, \alpha \text{ and } \beta\right)$, say $\left(\hat{\delta}, \hat{\alpha} \text{ and } \hat{\beta}\right)$ are the simultaneous solution of equations (28), (29), and (30) i.e. $\frac{\partial \ell_n(\Phi)}{\partial \delta} = 0, \frac{\partial \ell_n(\Phi)}{\partial \alpha} = 0$ and $\frac{\partial \ell_n(\Phi)}{\partial \beta} = 0$.

These equations are intractable and non-linear and can only be solved using the numerical iterative method.

7.0 Results and Discussions

In this section, we present the Monte Carlo Simulations for the extended models. The extensive class of computational techniques known as "Monte Carlo simulations" utilizes repeated random sampling to generate numerical solutions. The core principle underlying Monte Carlo simulations is their ability to tackle problems that may be theoretically complex or analytically intractable.

7.1 M.L.E for the OE-Weibull Distribution

The consistency of the estimated parameters within the developed distribution of the Odd Exponential-Weibull distribution was rigorously evaluated through a comprehensive simulation study. Utilizing the Monte Carlo simulation technique, we assessed the bias and root mean square error (RMSE) of the parameters derived from the maximum likelihood estimates (MLEs). Specifically, the log-likelihood function, as expressed in equation (27), and the quantile function, outlined in equation (23), served as the basis for generating simulated datasets across various sample sizes n = 20, 50, 100, 250, 500, 1000, 2000, 3000, 4000, 5000, and 10000, with each sample undergoing 1000 replications. The simulation was conducted under the assumption of predetermined parameter values for the Odd Exponential-Weibull distribution. This approach allowed for a detailed analysis of the estimation procedure's reliability and the robustness of the parameter estimates across different sample sizes. The initial assumed parameter values for the simulation are given as; $\delta = 1, \alpha = 2, \beta = 1$

Table 1: Simulation Results of the Odd Exponential-Weibull Distribution

Sample Sizes	Parameter	Estimates	Bias	RMSE
	δ	1.2301	0.2301	0.8572
20	α	2.0884	0.0884	0.8177
	β	1.1015	0.1015	0.2663
	δ	1.1956	0.1956	0.7023
50	α	2.1261	0.1261	0.6948
	β	1.0450	0.0450	0.1586
	δ	1.1142	0.1142	0.4921
100	α	2.0799	0.0799	0.5150
	β	1.0248	0.0248	0.1161
	δ	1.0827	0.0827	0.3859
250	α	2.0672	0.0672	0.4237
	β	1.0141	0.0141	0.0816

	δ	1.0591	0.0591	0.2746
500	α	2.0535	0.0535	0.3147
	β	1.0084	0.0084	0.0590
	δ	1.0388	0.0388	0.2077
1000	α	2.0349	0.0349	0.2469
	β	1.0046	0.0046	0.0457
	δ	1.0252	0.0252	0.1701
2000	α	2.0223	0.0223	0.2041
	β	1.0035	0.0035	0.0360
	δ	1.0228	0.0228	0.1426
3000	α	2.0212	0.0212	0.1703
	β	1.0035	0.0035	0.0307
	δ	1.0201	0.0201	0.1291
4000	α	2.0197	0.0197	0.1544
	β	1.0029	0.0029	0.0273
	δ	1.0166	0.0166	0.1111
5000	α	2.0168	0.0168	0.1342
	β	1.0025	0.0025	0.0242
	δ	1.0121	0.0121	0.0856
10000	α	2.0124	0.0124	0.1043
	β	1.0019	0.0019	0.0185

Table 1 presents the simulation results for the parameters of the Odd Exponential-Weibull (OE-Weibull) distribution across varying sample sizes. Parameter estimates for d, s and b are presented along with their respective biases and root mean square errors (RMSEs). As the sample size increases from 20 to 10,000, we observe consistent improvements in the precision of parameter estimation. Specifically, for d, s and b, biases decrease substantially with increasing sample size, indicating robustness and convergence of the estimation method (MLE). Similarly, RMSE values decrease, reflecting reduced variability and enhanced accuracy in parameter estimation as the sample size gets large. These simulation results highlight the reliability and efficiency of the proposed Odd Exponential-Weibull distribution in practical applications, particularly in contexts requiring robust survival analysis models. This interpretation highlights the key findings regarding parameter estimation accuracy and the impact of sample size on the performance of the Odd Exponential-Weibull distribution.

8.0 Conclusion

The Odd Exponential-Weibull (OE-W) distribution is introduced as a novel extension of the Weibull distribution, incorporating two additional parameters to create a more flexible three-parameter model. This study explores its mathematical properties, including quantile functions, moments, moment-generating functions, survival functions, hazard functions, entropy, and order statistics. Parameter estimation using maximum likelihood is rigorously tested through Monte

Carlo simulations, which demonstrate the reliability and accuracy of the estimates. The results reveal that the OE-W distribution offers a robust tool for modeling and analyzing data, particularly in fields requiring precise survival analysis and reliability assessmet.

The introduction of the OE-W distribution presents a valuable addition to the family of statistical distributions, providing enhanced flexibility for data modeling. The simulation results confirm that maximum likelihood estimation is effective in providing accurate parameter estimates, making the OE-W distribution a reliable choice for practical applications.It is recommended that researchers consider applying the OE-W distribution in various fields, such as reliability engineering and survival analysis, to leverage its enhanced modeling capabilities. Further empirical studies and applications should be conducted to explore its performance in different real-world scenarios and to assess its comparative advantages over other distributions.

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Authors' contributions

This research was a joint effort by all authors. Musa Ndamadu Farouq took charge of the study design, statistical analysis, protocol development, and the initial drafting of the manuscript. Nwaze Obini Nweze, Monday Osagie Adenomon, and Mary Unekwu Adehi supervised the data analysis and conducted the literature review. All authors reviewed and approved the final manuscript.