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The Inverse Lomax Chen Distribution: Properties and Applications

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Abstract: *Many researchers in the field of distribution theory have been expanding or generalizing existing probability distributions to improve their modeling flexibility. In this paper, we introduced a new continuous probability distribution called the inverse Lomax Chen distribution with four parameters. We studied the nature of the proposed distribution with the help of its mathematical and statistical properties such as quantile function, ordinary moments, generating function and reliability. The distribution of order statistics for this distribution was also obtained. Monte Carlo simulation was carried out to see the performance of MLEs of the inverse Lomax Chen distribution. We performed a classical estimation of parameters by using the technique of maximum likelihood estimate. The proposed model was applied to three real datasets and the results show that the proposed distribution provides a better fit than its comparators*

Keywords: *Biases, Glass Fibres, Inverse Lomax Chen, Maximum Likelihood Estimate, Mean Square Error, Quantile Function*

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1.0 Introduction

The real-life world phenomena largely describe the application of statistical distributions in modeling lifetime data. These distributions are very helpful and their theory is explored widely and new distributions are being produced. The goal of statistical parametric modeling is to find the best model for a set of data gathered from experiments, observational studies, surveys, and other sources.

Most modeling strategies are centered on determining the most appropriate probability distribution that explains the data set's underlying structure. There is, however, no one probability distribution that corresponds to all data sets. As a result, there has been a need to expand or construct new classical distributions (Nasiru, 2018).

The exponential, gamma and lognormal distributions may be used to simulate monotonic hazard rates. These distributions, however, have several flaws. For starters, none of their hazard rate functions have bathtub forms. Only monotonically increasing, decreasing, or constant hazard rates are seen in these distributions. The bathtub-shaped hazard rate is the most realistic.

This happens in almost all real-world systems. For example, when a population is separated into subpopulations with early failures, wear-out failures, and more or less continual failures, such forms emerge. As a result, a perfect bathtub is made up of two change points and a constant component that is contained within the change points. Bathtub shape's utility is well known in a variety of fields. To examine real datasets with bathtub failure rates, many parametric probability distributions have been devised.

Chen (2000) presented a bathtub-shaped or increasing failure rate (IFR) function for a novel two-parameter lifetime distribution. Chaubey and Zhang (2015) proposed an extension of Chen's (2000) family of distributions based on Lehman alternatives, Gupta *et al*., (1998), which was shown to be a viable alternative to the generalized and exponentiated Weibull families for modeling survival data. Khan *et al*. (2015) introduced a new distribution called the transmuted exponentiated Chen (TEC) and looked at some of its statistical features using survival data. The density and hazard functions' analytical shapes were determined by the authors. For lifetime data, the TEC distribution shows an increasing and declining hazard function. Khan *et al*. (2018) examined various structural aspects of the Kumaraswamy exponentiated Chen (KE-CHEN) distribution for modeling a bathtub-shaped hazard rate function. Tarvirdizade and Ahmadpour (2019) created the Weibull–Chen (W–C) distribution, which is constructed by compounding the Weibull and Chen distributions and has growing, decreasing, and bathtub-shaped hazard rate functions. The new distribution is more versatile in terms of modeling bathtub-shaped hazard rate data, and its hazard rate function is straightforward. Quantiles, moments, order statistics, and Renyi entropy were among the statistical properties explored by the authors.

Recent research in this area has focused on expanding existing probability distributions to improve their modeling flexibility. Some families of distributions proposed in the literature include Inverse Lomax-G by Falgore and Doguwa (2020), Topp Leone exponentiated-G by Ibrahim et al. (2020a), Topp Leone Kumaraswamy-G by Ibrahim et al. (2020b), The Kumaraswamy-G by Cordeiro and DeCastro (2011), Modi family of continuous probability distributions by Modi et al., (2020), Odd Chen-G by El-Morshedy et al., (2020).

In this context, we proposed a generalization of the Chen distribution based on the inverse Lomax-G family of distributions proposed by Falgore and Doguwa (2020), which stems from the following general construction: if *G* denotes a random variable's baseline cumulative function, then a generalized class of distributions can be defined by

$$F(x;\alpha,\beta,\mu)=\left[1+\frac{\beta(1-G(x;\mu))}{G(x;\mu)}\right]^{-\alpha} \qquad (1)$$

The pdf corresponding to (1) is

$$f(x;\alpha,\beta,\mu)=\frac{\alpha\beta g(x;\mu)}{G(x;\mu)^2}\left[1+\frac{\beta(1-G(x;\mu))}{G(x;\mu)}\right]^{-\alpha-1} \qquad (2)$$

where $G(x;\mu)$ is the cdf of the baseline distribution with parameter vector μ.

for $x\geq 0,\alpha,\beta,\mu\geq 0$, where equations (1) and (2) are the cdf and pdf of the IL-G family of distributions.

The cdf and pdf of the Chen distribution are given by

$$G(x;\lambda,b)=1-e^{\lambda(1-e^{x^b})} \qquad (3)$$

$$g(x;\lambda,b)=\lambda bx^{b-1}e^{x^b}e^{\lambda(1-e^{x^b})} \qquad (4)$$

$x>0,\lambda,b>0$.

2.0 The Inverse Lomax Chen (ILC) Distribution

This section defines a new continuous distribution called ILC distribution and provide some plots of its pdf, cdf and hazard rate function (hrf). The cdf of the ILC distribution is obtained by inserting (3) into (1) given as:

$$F(x;\alpha,\beta,\lambda,b)=\left[1+\frac{\beta(e^{\lambda(1-e^{x^b})})}{1-e^{\lambda(1-e^{x^b})}}\right]^{-\alpha}$$

(5)

$$f(x;\alpha,\beta,\lambda,b)=\frac{\alpha\beta\lambda b x^{b-1}e^{x^b}e^{\lambda(1-e^{x^b})}}{\left(1-e^{\lambda(1-e^{x^b})}\right)^2}\left[1+\frac{\beta(e^{\lambda(1-e^{x^b})})}{1-e^{\lambda(1-e^{x^b})}}\right]^{-\alpha-1}$$

(6)

For $x\geq 0,\alpha,b,\beta,\lambda>0$.

where b is the scale parameter and α,β,λ are the shape parameters respectively.

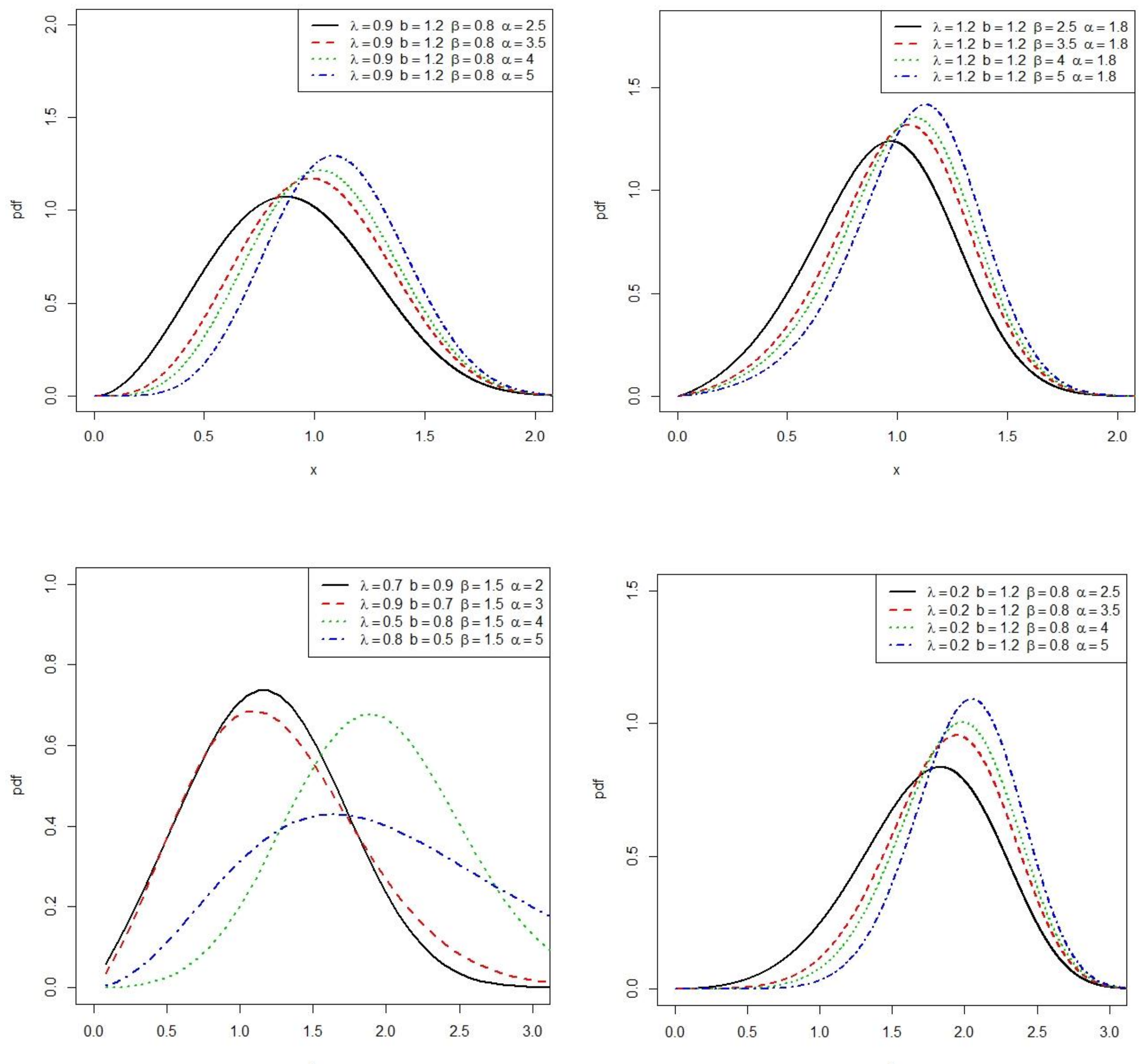
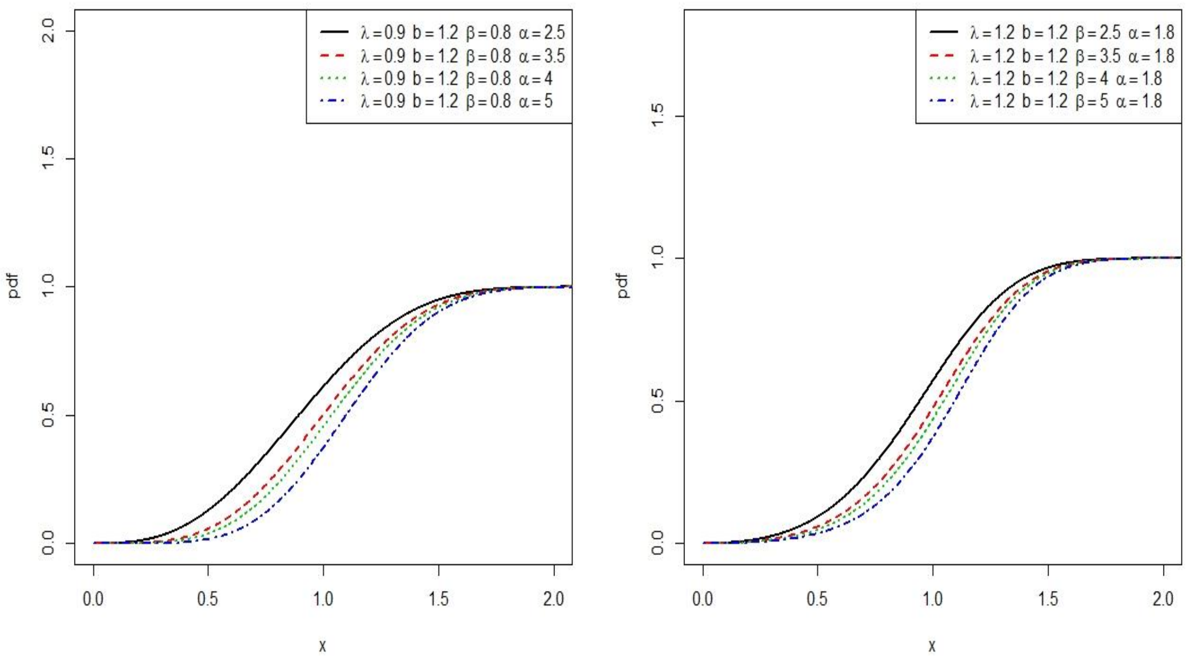


Fig. 1: Plots of pdf of the ILC distribution for different parameter values.



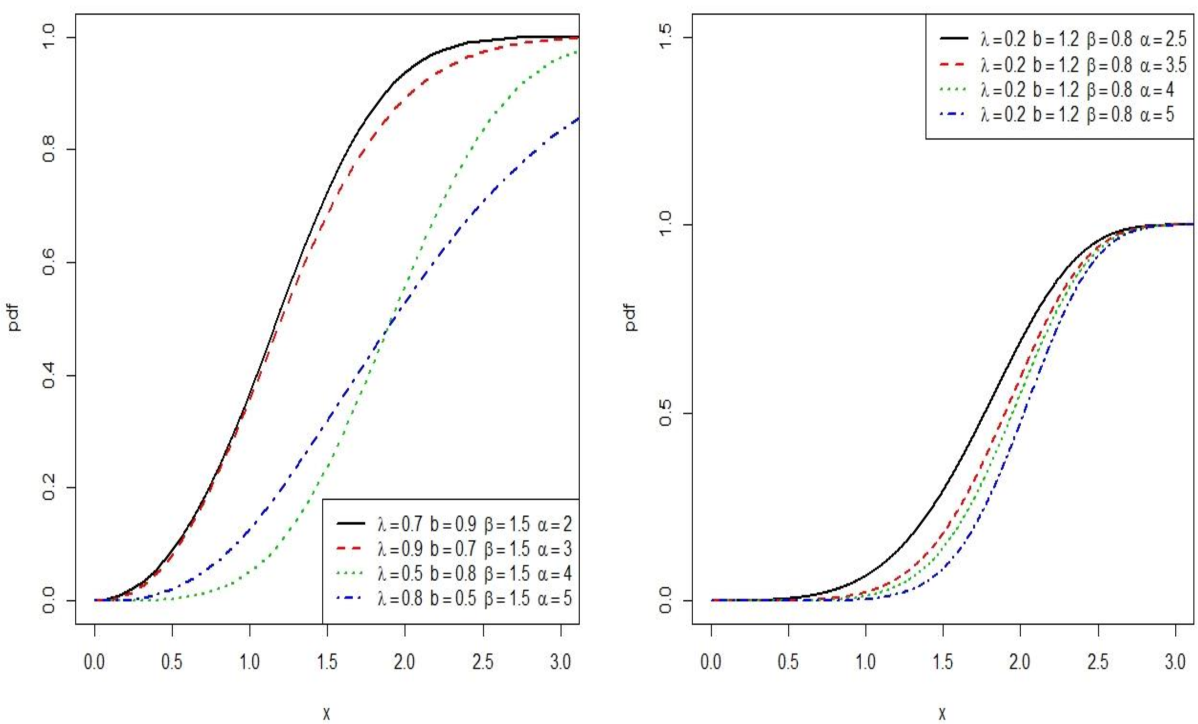


Fig. 2: Plots of cdf of the ILC distribution for different parameter values.

3. 0 Important Representation.

This section provides an expansion for (6) using the generalized binomial expansion given as

$$\left[1+z\right]^{-b}=\sum_{i=0}^{\infty}\binom{-b}{i}z^{i} \qquad (7)$$

Using the last term in the equation (6) in relation to equation (7), we have

$$\left[1+\frac{\beta(e^{\lambda(1-e^{x^{b}})})}{1-e^{\lambda(1-e^{x^{b}})}}\right]^{-\alpha-1}=\sum_{i=0}^{\infty}\binom{-\alpha-1}{-1}\left[\frac{\beta(e^{\lambda(1-e^{x^{b}})})}{1-e^{\lambda(1-e^{x^{b}})}}\right]^{i} \qquad (8)$$

$$\left[1+\frac{\beta(e^{\lambda(1-e^{x^{b}})})}{1-e^{\lambda(1-e^{x^{b}})}}\right]^{-\alpha-1}=\sum_{i=0}^{\infty}\binom{-\alpha-1}{-1}\beta^{i}\left[(e^{\lambda(1-e^{x^{b}})})\right]^{i}\left[1-e^{\lambda(1-e^{x^{b}})}\right]^{-i} \qquad (9)$$

The substitution of equation (9) into equation (6) yields equation 10

$$f(x;\alpha,\beta,\lambda,b)=\alpha\beta\lambda b x^{b-1}e^{x^{b}}e^{\lambda(1-e^{x^{b}})}\sum_{i=0}^{\infty}\binom{-\alpha-1}{-i}\beta^{i}\left[(e^{\lambda(1-e^{x^{b}})})\right]^{i}\left[1-e^{\lambda(1-e^{x^{b}})}\right]^{-i-2} \qquad (10)$$

Also, the expansion of the last term in equation (10 leads to equation 11 as follows

$$\left[1-e^{\lambda(1-e^{x^{b}})}\right]^{-i-2}=\sum_{j=0}^{\infty}(-1)^{j}\binom{-i-2}{-j}\left[(e^{\lambda(1-e^{x^{b}})})\right]^{j} \qquad (11)$$

Equation 11 is also substituted into equation 10 to obtained equation 12 and upon expansion, equation 13 was obtained ,

$$f(x;\alpha,\beta,\lambda,b)=\alpha\beta\lambda b x^{b-1}e^{x^{b}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\binom{-i-2}{-j}\binom{-\alpha-1}{-i}\beta^{i}\left[(e^{\lambda(1-e^{x^{b}})})\right]^{i+j+1} \qquad (12)$$

$$\left[(e^{\lambda(1-e^{x^{b}})})\right]^{i+j+1}=\sum_{k=0}^{\infty}(-1)^{k}\lambda^{k}\binom{i+j+1}{k}(1-e^{x^{b}})^{k} \qquad (13)$$

Equation 14 was obtained from the expansion of the last term in equation (13), while equation 15 was obtained by the substitution of equations 13 and 14 into equation 12

$$(1-e^{x^{b}})^{k}=\sum_{l=0}^{\infty}(-1)^{l}\binom{k}{l}\left[e^{x^{b}}\right]^{l} \qquad (14)$$

$$f(x;\alpha,\beta,\lambda,b)=\alpha\beta^{i+1}\lambda^{k+1}b\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+k+l}\binom{-i-2}{-j}\binom{-\alpha-1}{-i}\binom{i+j+1}{k}\binom{k}{l}x^{b-1}\left[e^{x^{b}}\right]^{l+1} \qquad (15)$$

Equation (15) is the important representation of the pdf of Inverse Lomax Chen distribution from which we can obtain some of the properties of the distribution.

4.0 Properties of ILC Distribution

Some of the mathematical and statistical properties of the ILC distribution such as the quantile function, moments, moment generating function, reliability measure and order statistics are presented in this section as follows

4.1 Moments

The r^{th} moment of x is obtained as

$$E(X^r) = \int_0^\infty x^r f(x) dx \tag{16}$$

The r^{th} moments of the ILC distribution are obtained as

$$E(X^r) = \int_0^\infty x^r \alpha \beta^{i+1} \lambda^{k+1} b \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty (-1)^{j+k+l} \binom{-i-2}{-j} \binom{-\alpha-1}{-i} \binom{i+j+1}{k} \binom{k}{l} x^{b-1} \left[e^{x^b} \right]^{l+1} dx$$

$$E(X^r) = \alpha \beta^{i+1} \lambda^{k+1} b \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty (-1)^{j+k+l} \binom{-i-2}{-j} \binom{-\alpha-1}{-i} \binom{i+j+1}{k} \binom{k}{l} \int_0^\infty x^{r+b-1} \left[e^{x^b} \right]^{l+1} dx \tag{17}$$

The solution to equation 17 are as follow

Let $y = x^{b(l+1)}$

$$\frac{dy}{dx} = b(l+1) x^{b(l+1)-1}$$

$$\int_0^\infty x^{r+b-1} e^y \frac{dy}{b(l+1) x^{b(l+1)-1}}$$

$$\frac{\left(b(l+1)\right)^{r+b-1}}{\left(b(l+1)\right)^{b(l+1)}} \int_0^\infty y^{\frac{r-bl-2}{2}} e^y dy$$

$$\int_0^\infty y^{\frac{r-bl-2}{2}} e^y dy = \Gamma\left(\frac{r-bl-2}{2} + 1 \right)$$

Therefore, the moment of inverse Lomax Chen distribution is given by equation 18

$$E(X^r) = \alpha \beta^{i+1} \lambda^{k+1} b \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty (-1)^{j+k+l} \binom{-i-2}{-j} \binom{-\alpha-1}{-i} \binom{i+j+1}{k} \binom{k}{l} \frac{\left(b(l+1)\right)^{r+b-1}}{\left(b(l+1)\right)^{b(l+1)}} \Gamma\left(\frac{r-bl-2}{2} + 1 \right) \tag{18}$$

Equation (18) is the r^{th} moment of the ILC distribution. The mean of the distribution will be obtained by setting r=1 in (18).

4.2 Moment generating function(MGF)

The mgf of X can be obtained using the equation

$$E(e^{tx}) = \int_0^\infty e^{tx} f(x) dx \tag{19}$$

$$E(e^{tx}) = \int_0^\infty e^{tx} \alpha \beta^{i+1} \lambda^{k+1} b \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty (-1)^{j+k+l} \binom{-i-2}{-j} \binom{-\alpha-1}{-i} \binom{i+j+1}{k} \binom{k}{l} x^{b-1} \left[e^{x^b} \right]^{l+1} dx \tag{20}$$

$$e^{tx} = \sum_{m=0}^\infty \frac{t^m x^m}{m!} \tag{21}$$

Following the process of moments above, we have the MGF given as

$$E(e^{tx}) = \alpha \beta^{i+1} \lambda^{k+1} b \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty (-1)^{j+k+l} \binom{-i-2}{-j} \binom{-\alpha-1}{-i} \binom{i+j+1}{k} \binom{k}{l} \frac{t^m}{m!} \frac{\left(b(l+1)\right)^{m+b-1}}{\left(b(l+1)\right)^{b(l+1)}} \Gamma\left(\frac{m-bl-2}{2} + 1 \right) \tag{22}$$

4.3 Reliability function

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$R(x;\alpha,\beta,\lambda,b)=P(X>x)=1-F(x;\alpha,\beta,\lambda,b) \tag{23}$$

$$R(x;\alpha,\beta,\lambda,b)=1-\left[1+\frac{\beta(e^{\lambda(1-e^{x^b})})}{1-e^{\lambda(1-e^{x^b})}}\right]^{-\alpha} \tag{24}$$

4.4 Hazard rate function

$$\tau(x;\alpha,\beta,\lambda,b)=\frac{f(x;\alpha,\beta,\lambda,b)}{R(x;\alpha,\beta,\lambda,b)} \tag{25}$$

$$\tau(x;\alpha,\beta,\lambda,b)=\frac{\dfrac{\alpha\beta\lambda b x^{b-1}e^{x^b}e^{\lambda(1-e^{x^b})}}{\left(1-e^{\lambda(1-e^{x^b})}\right)^2}\left[1+\dfrac{\beta(e^{\lambda(1-e^{x^b})})}{1-e^{\lambda(1-e^{x^b})}}\right]^{-\alpha-1}}{1-\left[1+\dfrac{\beta(e^{\lambda(1-e^{x^b})})}{1-e^{\lambda(1-e^{x^b})}}\right]^{-\alpha}} \tag{26}$$

4.5 Quantile function

The quantile function is defined as the inverse of the cdf and it is given as: $Q(u)=F^{-1}(u)$. Using the cdf of ILC distribution in (3.65), we have

$$F(x;\alpha,\beta,\lambda,b)=\left[1+\frac{\beta(e^{\lambda(1-e^{x^b})})}{1-e^{\lambda(1-e^{x^b})}}\right]^{-\alpha}=u$$

$$u^{\frac{1}{-\alpha}}=1+\frac{\beta(e^{\lambda(1-e^{x^b})})}{1-e^{\lambda(1-e^{x^b})}}$$

$$u^{\frac{1}{-\alpha}}-1=\frac{\beta(e^{\lambda(1-e^{x^b})})}{1-e^{\lambda(1-e^{x^b})}}$$

$$\beta(e^{\lambda(1-e^{x^b})})=\left(u^{\frac{1}{-\alpha}}-1\right)\left(1-e^{\lambda(1-e^{x^b})}\right)$$

$$\beta(e^{\lambda(1-e^{x^b})})+u^{\frac{1}{-\alpha}}e^{\lambda(1-e^{x^b})}=u^{\frac{1}{-\alpha}}-1+e^{\lambda(1-e^{x^b})}$$

$$\beta(e^{\lambda(1-e^{x^b})})+u^{\frac{1}{-\alpha}}e^{\lambda(1-e^{x^b})}-e^{\lambda(1-e^{x^b})}=u^{\frac{1}{-\alpha}}-1$$

$$e^{\lambda(1-e^{x^b})}\left(\beta+u^{\frac{1}{-\alpha}}-1\right)=u^{\frac{1}{-\alpha}}-1$$

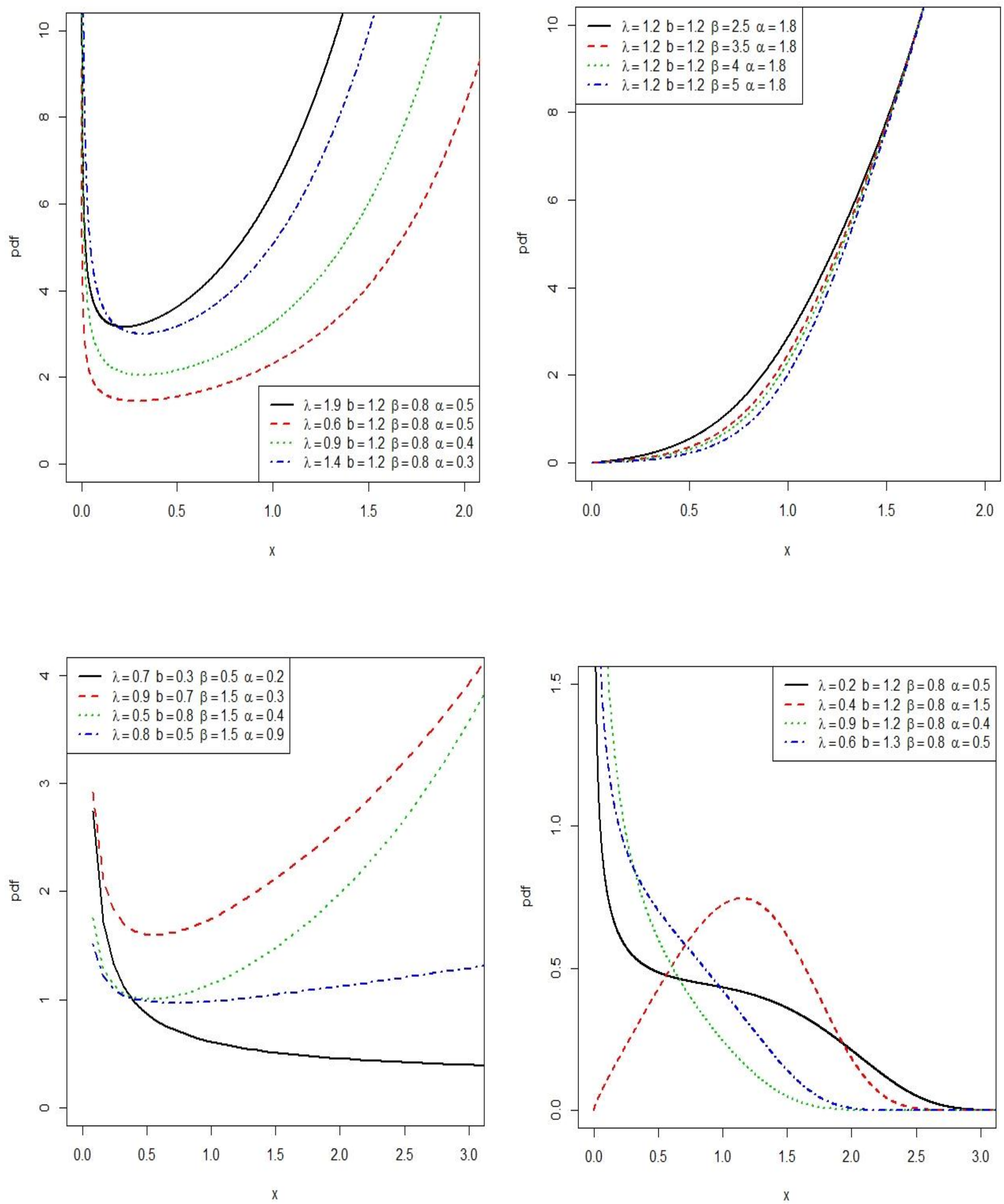


Fig. 2: Plots of hazard rate function of the ILC distribution for different parameter values.

$$e^{\lambda(1-e^{x^b})} = \frac{u^{\frac{1}{-\alpha}} - 1}{\left(\beta + u^{\frac{1}{-\alpha}} - 1\right)}$$

$$\lambda(1-e^{x^b}) = log\left[\frac{u^{\frac{1}{-\alpha}} - 1}{\left(\beta + u^{\frac{1}{-\alpha}} - 1\right)}\right]$$

$$1-e^{x^b} = \frac{log\left[\frac{u^{\frac{1}{-\alpha}} - 1}{\left(\beta + u^{\frac{1}{-\alpha}} - 1\right)}\right]}{\lambda}$$

$$e^{x^b} = 1 - \frac{log\left[\frac{u^{\frac{1}{-\alpha}} - 1}{\left(\beta + u^{\frac{1}{-\alpha}} - 1\right)}\right]}{\lambda}$$

$$x^b = log\left[1 - \frac{log\left[\frac{u^{\frac{1}{-\alpha}} - 1}{\left(\beta + u^{\frac{1}{-\alpha}} - 1\right)}\right]}{\lambda}\right]$$

$$x = \left\{log\left[1 - \frac{log\left[\frac{u^{\frac{1}{-\alpha}} - 1}{\left(\beta + u^{\frac{1}{-\alpha}} - 1\right)}\right]}{\lambda}\right]\right\}^{\frac{1}{b}} \quad (27)$$

The median of the ILC distribution can be derived by substituting $u = 0.5$ in (27) as follows:

$$x = \left\{log\left[1 - \frac{log\left[\frac{0.5^{\frac{1}{-\alpha}} - 1}{\left(\beta + 0.5^{\frac{1}{-\alpha}} - 1\right)}\right]}{\lambda}\right]\right\}^{\frac{1}{b}} \quad (28)$$

5.0 Order Statistics

Let $X_1, X_2, ..., X_n$ be n independent random variable from the ILC distributions and let $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ be their corresponding order statistic. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1, 2, 3, ...n$ denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. The pdf of the r^{th} order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x;\alpha,\beta,\lambda,b) = \frac{1}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i [F(x;\alpha,\beta,\lambda,b)]^{r+i-1} f(x;\alpha,\beta,\lambda,b) \quad (29)$$

Using the cdf and pdf of ILC distribution, we have

$$f_{r:n}(x;\alpha,\beta,\lambda,b) = \frac{\alpha\beta^{j+1}\lambda^{l+1}b}{B(r,n-r+1)} \sum_{i=0}^{n-r}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{i+k+l+m} \binom{-\alpha(r+i+1)}{j}\binom{-j-2}{k}\binom{j+k+1}{l}\binom{l}{m} x^{b-1}\left[e^{x^b}\right]^{m+1}$$

(30)

Equation (30) is the r^{th} order statistics of the ILC distribution.

Therefore, the pdf of the minimum and maximum order statistics of the ILC distribution are obtained by setting $r=1$ and $r=n$ respectively in (30).

6.0 Parameter Estimation

In this section, we estimate the parameters of the ILC distribution using maximum likelihood estimation (MLE). For a random sample, $X_1, X_2, ..., X_n$ of size n from the ILC $(\alpha, \beta, \lambda, b)$, the log-likelihood function L $(\alpha, \beta, \lambda, b)$ of (6) is given as

$$\mathrm{l}(\phi) = nlog(\alpha) - nlog(\beta) + nlog(\lambda) + nlog(b) + (b-1)\sum_{i=1}^{n} log(x_i) + \sum_{i=1}^{n}(x_i^b) + \lambda\sum_{i=1}^{n}(1-e^{x_i^b}) - 2\sum_{i=1}^{n}\left[1-e^{\lambda(1-e^{x_i^b})}\right] - (\alpha-1)\sum_{i=1}^{n} log\left[1+\frac{\beta(e^{\lambda(1-e^{x_i^b})})}{1-e^{\lambda(1-e^{x_i^b})}}\right] \tag{31}$$

The components of the score vector, say $\nabla \mathrm{l}(\phi) = \left(\frac{\partial log \mathrm{l}(\phi)}{\partial \lambda}, \frac{\partial log \mathrm{l}(\phi)}{\partial \beta}, \frac{\partial log \mathrm{l}(\phi)}{\partial \alpha}, \frac{\partial log \mathrm{l}(\phi)}{\partial b}\right)$.

Differentiating (31) with respect to each parameter, we have

$$\frac{\partial log \mathrm{l}(\phi)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} log\left[1+\frac{\beta(e^{\lambda(1-e^{x_i^b})})}{1-e^{\lambda(1-e^{x_i^b})}}\right] = 0 \tag{32}$$

$$\frac{\partial log \mathrm{l}(\phi)}{\partial \beta} = \frac{n}{\beta} - (\alpha-1)\sum_{i=1}^{n}\left\{\frac{\left[\frac{(e^{\lambda(1-e^{x_i^b})})}{1-e^{\lambda(1-e^{x_i^b})}}\right]}{\left[1+\frac{\beta(e^{\lambda(1-e^{x_i^b})})}{1-e^{\lambda(1-e^{x_i^b})}}\right]}\right\} = 0 \tag{33}$$

$$\frac{\partial log \mathrm{l}(\phi)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}(1-e^{x_i^b}) - 2\sum_{i=1}^{n}\left[\frac{(e^{\lambda(1-e^{x_i^b})})(1-e^{x_i^b})}{1-(e^{\lambda(1-e^{x_i^b})})}\right] - (\alpha-1)\sum_{i=1}^{n}\left\{\frac{\left[\frac{\beta(e^{\lambda(1-e^{x_i^b})})(1-e^{x_i^b})(1-e^{\lambda(1-e^{x_i^b})}) - (e^{\lambda(1-e^{x_i^b})})}{\left(1-e^{\lambda(1-e^{x_i^b})}\right)^2}\right]}{\left[1+\frac{\beta(e^{\lambda(1-e^{x_i^b})})}{1-e^{\lambda(1-e^{x_i^b})}}\right]}\right\} = 0 \tag{34}$$

$$\frac{\partial log \mathrm{l}(\phi)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} log(x_i) + \sum_{i=1}^{n} X_i^b log(x_i) - 2\lambda\sum_{i=1}^{n}\left[\frac{e^{x_i^b} x_i^b log(x_i)(e^{\lambda(1-e^{x_i^b})})}{1-e^{\lambda(1-e^{x_i^b})}}\right] - (\alpha-1)2\lambda\beta\sum_{i=1}^{n}\left\{\frac{\left[\frac{e^{x_i^b} x_i^b log(x_i)(e^{\lambda(1-e^{x_i^b})})\left(1-e^{x_i^b}\right) - e^{\lambda(1-e^{x_i^b})}}{\left(1-e^{\lambda(1-e^{x_i^b})}\right)^2}\right]}{\left[1+\frac{\beta(e^{\lambda(1-e^{x_i^b})})}{1-e^{\lambda(1-e^{x_i^b})}}\right]}\right\} = 0 \tag{35}$$

Now, equations (32), (33), (34) and (35) do not have a simple form and are therefore intractable. As a result, we have to resort to non-linear estimation of the parameters using iterative procedures.

7. 0 Simulation Study

In this section, we perform the simulation study to see the performance of MLEs of ILC distribution. The random number generation is obtained with its quantile function. We note that the u^{th} quantile function of the ILC distribution is given in (27). Hence, if U has a uniform random variable on (0, 1), then x has the ILC random variable.

We generated *N=10000* samples of sizes *n=20, 50, 100, 250 and 500* from ILC distribution with its quantile function. Then we computed the empirical means, biases and mean squared errors (MSE) of the MLEs with

$$Bias_{\hat{\psi}} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\psi}_i - \psi_i\right) \qquad (36)$$

and

$$MSE_{\hat{\psi}} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\psi}_i - \psi_i\right)^2, \qquad (37)$$

for $\psi = (\beta, \alpha, \lambda, b)$

To examine the performance of the MLEs for the ILC distribution, we perform a simulation study as follows:

1. Generate N samples of size n from the ILC distribution with its quantile function.
2. Compute the MLEs for the N samples, say $(\hat{\beta}, \hat{\alpha}, \hat{\lambda}, \hat{b})$, for $i = 1, 2, ..., N$
3. Compute the MLEs for N samples
4. Compute the biases and mean squared errors MSE given in (35) and (36).

We repeat these steps for *N= 10000* and *n = 20, 50, 100, 250 and 500* with different values of $\psi = (\beta, \alpha, \lambda, b)$. Table 1 shows how the biases and MSE vary with *n*. As expected, the Biases and MSEs of the estimated parameters converge to zero as *n* increases which proves the consistency of the estimators.

Table 1: Biases and MSE of the ILC distribution for selected parameter values.

Initial values	**Bias and MSE**	**Sample sizes**				
		n=20	**n=50**	**n=100**	**n=250**	**n=500**
α **=0.5**	Bias	4.8589	-0.0119	-0.0271	-0.0220	-0.0151
	MSE	987.3814	0.2446	0.0360	0.0201	0.0121
β **=0.7**	Bias	0.4689	0.2312	0.1300	0.0834	0.0513
	MSE	0.6061	0.1563	0.0809	0.0502	0.0282
λ **=0.5**	Bias	1.9754	-0.0424	-0.0060	-0.0067	-0.0071
	MSE	1.9754	0.3178	0.2594	0.1550	0.0924
b **=0.6**	Bias	2.2144	0.5163	0.2968	0.1297	0.0552
	MSE	405.2246	11.1458	1.9379	0.7012	0.2424

8. 0 Real-life Application

In this section, we fit the ILC distribution to data set 1, data set 2 and data set 3 and for illustrative purposes also present a comparative study with the fits of TEC, EC and C models. These applications prove empirically the flexibility of the proposed distributions in modeling real life data sets. All the computations are performed using the *R* software.

The first data set represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. The data set was given by Gross and Clark (1975). The data set consists of twenty (20) observations and it is as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The second data set was given by Lee (1992) and it represents the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital a period from 1929 to 1938. It has also been applied by Ramos et al., (2013). The data set is as follows:
0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 1.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.
The third data set represents the breaking strength of 100 Yarn as reported by Gomes-Silva et al., (2017). The data set consists of 63 measurements of the strengths of 1.5 cm glass fibres, which were initially collected by United Kingdom National Physical Laboratory staff. The data is presented below:
0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.
The pdf of the comparators considered are:

- Transmuted Exponentiated Chen (TEC) Distribution (Khan et al. (2016)).

$$f(x)=\alpha\beta\theta x^{\beta-1}e^{\left(x^{\beta}+\alpha\left(1-e^{x^{\beta}}\right)\right)}\left(1-e^{\alpha\left(1-e^{x^{\beta}}\right)}\right)^{\theta-1}\left[1+\lambda-2\lambda\left(1-e^{\alpha\left(1-e^{x^{\beta}}\right)}\right)\right]^{\theta} \quad (38)$$

- Extented Chen (EC) distribution (Chaubey and Zang (2015)).

$$f(x)=\beta\alpha\theta x^{\beta-1}e^{\left(x^{\beta}+\alpha\left(1-e^{x^{\beta}}\right)\right)}\left[1-e^{\alpha\left(1-e^{x^{\beta}}\right)}\right]^{\theta-1} \quad (39)$$

The model selection is carried out using the AIC (Akaike information criterion) and the CAIC (consistent Akaike information criteria).

$$AIC=-2l+2k \quad (40)$$

$$CAIC=AIC+\frac{2k(k+1)}{(n-k-1)} \quad (41)$$

where l denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size.
The model with a minimum value of AIC or CAIC is chosen as the best model to fit the data sets considered.

Table 2: The MLEs and Information Criteria of the models based on data set 1

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{b}$	l	**AIC**	**CAIC**
ILC	311.3293	138.8220	5.2802	-	0.2187	-15.5453	39.0906	41.7573
TEC	1.1603	0.5039	0.5869	25.8577	-	-16.6857	41.3714	44.0381
EC	2.9249	0.3325	-	671.5116	-	-16.6857	39.7436	42.4104
C	-	-	0.9523	-	0.1369	-24.5700	53.1401	53.8460

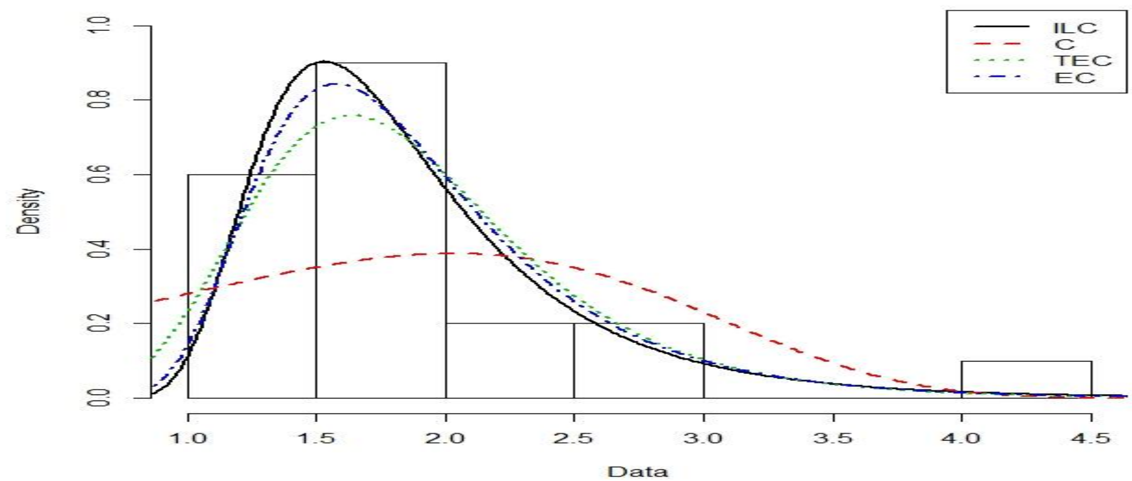


Fig. 3: Histogram and fitted pdfs for the ILC, TEC, EC and C models to the data set 1

Table 3: The MLEs and Information Criteria of the models based on data set 2

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{b}$	l	**AIC**	**CAIC**
ILC	1.8449	3.0331	0.1825	-	0.2506	-579.3084	1166.6170	1166.9620
TEC	0.0142	0.3512	0.0146	0.9951	-	-581.7857	1167.5710	1167.6930
EC	0.0859	0.2817	-	0.9951	-	-580.8960	1167.792	1167.9041
C	-	-	0.3389	-	0.0214	-581.7857	1167.571	1167.6730

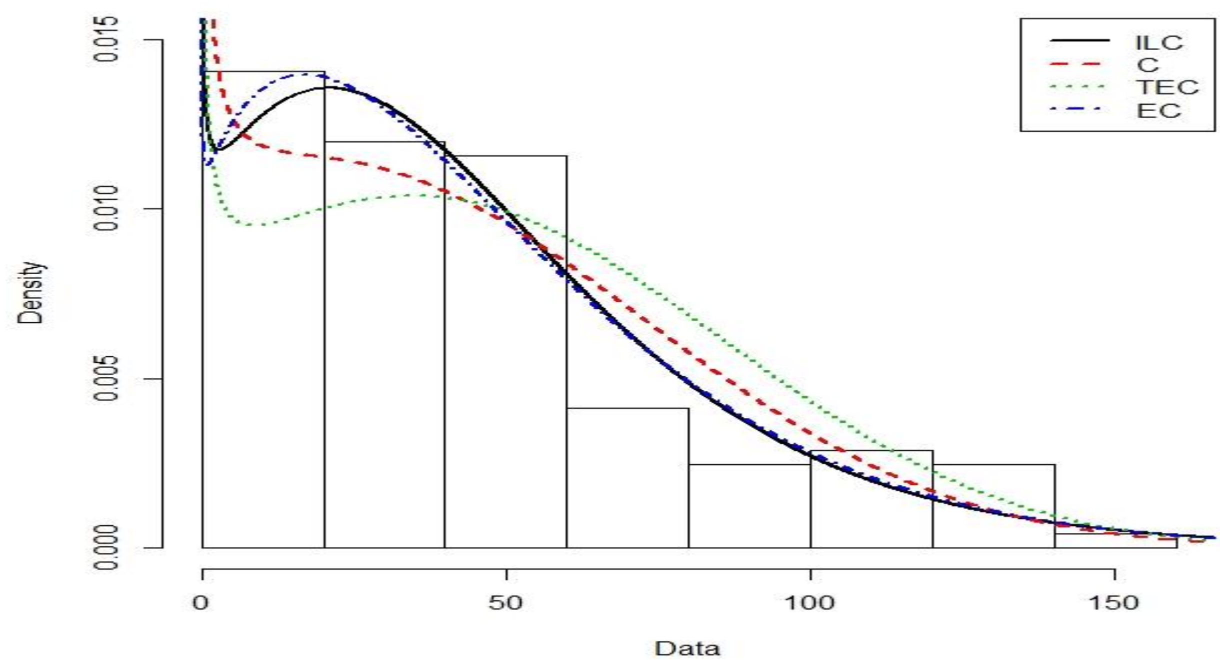


Fig. 4: Histogram and fitted pdfs for the ILC, TEC, EC and C models to the data set 2.

Table 4: The MLEs and Information Criteria of the models based on data set 3

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{b}$	l	**AIC**	**CAIC**
ILC	1.5719	3.6272	0.3462	-	1.4863	-13.0803	34.1605	34.8502
TEC	0.2372	1.5861	0.6292	1.7796	-	-13.7696	35.4191	36.1088
EC	0.1726	1.6831	-	1.9494	-	-14.2733	34.5465	34.9533
C	-	-	1.9603	-	0.0721	-16.4613	36.9447	37.1227

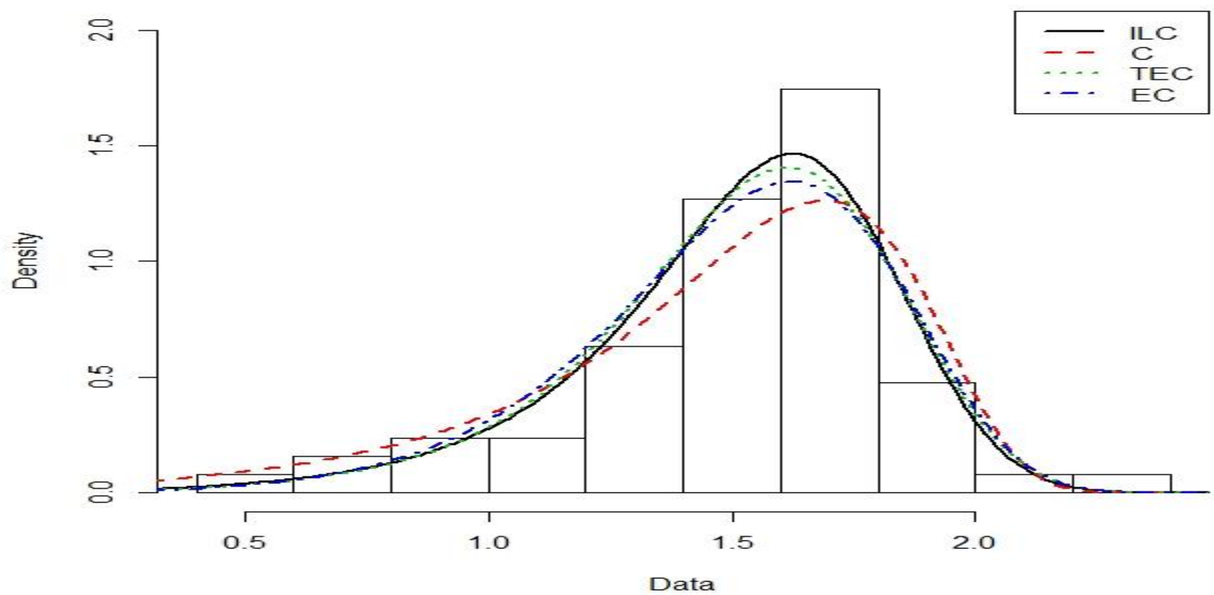


Fig. 4: Histogram and fitted pdfs for the ILC, TEC, EC and C models to the data set 3

Figs. 3 and 4 present the shapes, fit and flexibility of the new model about the data sets considered. The black line represents the new model, the red line represents the baseline distribution, the green line represents the TEC and the blue line represents the EC distributions. It can be seen from the histogram and fitted plots that the black line which represents the proposed distribution fits better in the three data sets considered.

9 Conclusion

This paper has derived a new distribution called the inverse Lomax Chen distribution with four parameters that extends the Chen distribution. Some properties of the new distribution were derived such as the survival function, hazard rate function, quantile function, median and order statistics. The shapes of the proposed distribution were shown by plotting the graphs of the pdf and hazard rate function. It can be seen from the hazard rate plots that the shape of the new distribution has increased, decreasing, constant and bathtub shapes. The estimation of the model parameters by the method of the maximum likelihood was carried out using a package in *R* known as *AdequacyModel*. Monte Carlo simulation was carried out to see the performance of MLEs of the inverse Lomax Chen distribution and as expected, the Biases and MSEs of the estimated parameters converge to zero as *n* increases which proves the consistency of the estimators. Application of the new distribution to three real data sets was carried out and the results are presented in Table 1, Table 2 and Table 3. The results indicate that the inverse Lomax Chen distribution is quite effective and superior in fitting the three data sets considered. Also, the flexibility of the proposed distribution can be seen from the histogram and fitted pdf plots for the three data sets and it is evident that the new model fits the three data sets better than the competing distributions considered.

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