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Topp Leone Exponential – Generalized Inverted Exponential Distribution Properties and Application

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Abstract: *In this paper, a new distribution called Topp Leone Exponential – Generalized Inverted Exponential distribution (TLE-GIE) is developed to improve on the flexibility of Topp Leone generalized inverted exponential (TL-GIE) distribution. The TLE-GIE distribution was developed by extending the generalized inverted exponential distribution with Topp Leone Exponential G family of distribution. The respective density and distribution functions of this new distribution (TLE-GIE) were derived including some mathematical properties such as moments, quantile function, renyi entropy and order statistics. A simulation study conducted, by the consideration of the Maximum Likelihood Estimate (MLE) method shows that the estimated parameters of TLE-GIE are consistent as the BIAS and RMSE approach zero. Finally, three real data sets were used to validate the results obtained from the MLE method. The results obtained indicated that the TLE-GIE distribution provided a better fitness of the data sets than the TL-GIE and other competitive distributions. Perhaps, this new distribution may be useful for the modelling real life data sets that may behave exponentially.*

Keywords: *Topp Leone Exponential – Generalized Inverted Exponential distribution; Mathematical properties; Simulation study; Application to real life data sets.*

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1.0 Introduction

The field of statistical inference and modeling real-life data sets with probability distributions; have received great attention from researchers. These models (probability distributions) are of great importance in applications in many fields such as medicine, engineering, biological science, management, and public health among others.

Although, the exponential distribution is used to model a lifetime data, it is generally known as a constant failure rate, memoryless property

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and limited to modelling some unimodal data. However, exponential distribution has been extended by various families of distributions as aimed to improve its flexibility. These distribution functions are more flexible to model real data sets, for example; Generalized Exponential – Exponential (GEE) distribution by Gupta and Gundu (1999). Generalization of the Exponential (GE) distribution was introduced by Eugene *et al*. (2002), Beta – Exponential (BE) distribution was also developed by Kotz (2006). Other distributions are the Exponentiated Exponential (ETE) distribution has also been reported by Gupta and Gundu (2009), Transmuted Exponentiated Exponential (TEE) distribution proposed by Merovci (2013), Gamma Exponentiated Exponential (GEE) distribution by Ristic and Balakrishnan (2012), Exponentiated Exponential distribution by Nadarajah (2011), Beta Generalized Exponential (BGE) distribution by Souza *et al.*, (2010), Exponentiated Kumaraswamy Exponential (EKE) distribution by Jailson and Ana (2015), Top leone Exponential Exponential (TLEE) distribution by Sanusi, *et al.*, (2020).

Thus, an exponential distribution has been transformed by researchers using different transformation techniques. The common technique is to modify or introduce additional tuning parameters to an exponential distribution, to improve its theoretical and practical sense. This has led to the development of inverse or inverted exponential distribution developed by Keller *et al.* (1982) and it has been studied and discussed as a life model.

However, researchers have extended the transformed exponential distribution with a series of families of distributions such as; the Exponential Inverse Exponential (EIE) distribution by Oguntunde *et al.* (2017), the Topp Leone Exponentiated Inverse Exponential (TLET-IE) distribution developed by Ibrahim *et al*. (2020). However, the Inverted Exponential (IE) distribution has been generalized to yield the Generalized Inverted Exponential (GIE) distribution developed by Abouammoh and Alshingiti (2009). Though, this Generalized Inverted Exponential (GIE) distribution is one of the generalization models of exponential as it is flexible to contain different forms of the hazard function. The Generalized Inverse Exponential distribution has been extended with Topp Leone G family of distributions; that is, Topp Leone Generalized Inverted Exponential (TL-GIE) distribution by Zakeia *et al.* (2020). Although, the advantages of the TL-G family include the simplicity and pliancy of its functions, the following notable first-order stochastic dominance property involving the CDF of the exponentiated-G family: $H(x) \geq G(x)^{\lambda}; x \in \Re$ and can generate flexible models thanks to the additional parameter λ. Hence, it is motivated by offering a suitable alternative to the exponentiated – G family, while keeping a similar degree of simplicity. Abdulhakim et al. (2020).

Also, Generalized inverted exponential has been extended with some other families of distributions such as the Beta Generalized Inverted Exponential (B-GIE) and its application was reported by Hanna *et al.* (2017), Exponentiated Generalized Inverted Exponential (E-GIE) distribution by Oguntunde *et al.* (2014), Modified Generalized Inverted Exponential (M-GIE) distribution by Abouammoh and Alshangiti (2016), Poison Generalized Inverted Exponential (P-GIE) distribution by Ramesh *et al.* (2021).

Therefore, in this paper; a new distribution called Topp Leone Exponential - Generalized Inverted – Exponential (TLE-GIE) is developed to improve on the flexibility of Topp Leone Generalized Inverted Exponential (TL-GIE) distribution developed by Zakeia *et al.* (2020). The densities of this new TLE-GIE distribution are defined. Also, some mathematical properties are derived. Simulation study to confirm the parameter consistency of this new distribution base on the method of maximum likelihood estimate (MLE) is discussed. The flexibility of this distribution is illustrated in an application to three real life data sets previously used to test the flexibility of TL-GIE distribution. The remaining sections of the paper; are organized as follows. The cdf, pdf and densities graphs of the new distribution are defined in section 2. The useful expansion of TLE-GIE is discussed in section 3. Some mathematical properties of the TLE-GIE are discussed in section 4. The maximum likelihood estimate obtained for the parameters of TLE-GIE is presented in section 5. A simulation study with the MLE method on the efficiency of TLE-GIE is presented in section 6. Applications to three real data sets for the model are shown in Section 7 and Section 8 concludes the paper.

2. 0 Topp Leone Exponential – Generalized Inverted Exponential

In this section, the cumulative distribution function (cdf) and probability density function (pdf) of the new distribution called Topp Leone Exponential – Generalized Inverted Exponential (TLE-GIE) are defined.

Thus, the cdf and pdf of Topp Leone Exponential G family of distributions developed by Sanusi *et al.* (2020) are defined as follows:

$$F_{TLE-G}\left(x;\sigma,\lambda,\beta\right)=\left[1-\exp\left\{-2\lambda\left(\frac{G\left(x;\beta\right)}{\bar{G}\left(x;\beta\right)}\right)\right\}\right]^{\sigma} \tag{1}$$

and

$$f_{TLE-G}\left(x;\sigma,\lambda,\beta\right)=\frac{2\sigma\lambda g\left(x,\beta\right)}{\left(\bar{G}\left(x;\beta\right)\right)^{2}}\exp\left\{-2\lambda\left(\frac{G\left(x;\beta\right)}{\bar{G}\left(x;\beta\right)}\right)\right\}\left[1-\exp\left\{-2\lambda\left(\frac{G\left(x;\beta\right)}{\bar{G}\left(x;\beta\right)}\right)\right\}\right]^{\sigma-1} \tag{2}$$

respectively.

Also, the cdf and pdf of the Generalized inverted exponential distribution developed by Abouammoh and Alshingiti (2009). are given as follows:

$$G\left(x\right)=1-\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta} \tag{3}$$

and

$$g\left(x\right)=\left(\frac{\theta\gamma}{x^{2}}\right)\exp\left\{-\frac{\gamma}{x}\right\}\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta-1} \tag{4}$$

$\vartheta, x \succ 0$, respectively.

Note: equation (5) is considered, after the substitution of both equations (3) and (4) into (1) and (2).

$$\frac{1-\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta}}{\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta}}=\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1 \tag{5}$$

Therefore, the cdf and pdf of the new distribution called TLE-GIE are respectively defined as

$$F_{TLEGIE}\left(x;\sigma,\lambda,\theta,\gamma\right)=\left[1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{\sigma} \tag{6}$$

$$f_{TLE-GIE}\left(x;\sigma,\lambda,\theta,\gamma\right)=\frac{2\sigma\lambda\gamma\theta\exp\left\{-\frac{\gamma}{x}\right\}\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta-1}}{x^{2}\left[\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta}\right]^{2}}\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}$$
$$\times\left[1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{\sigma-1} \tag{7}$$

Henceforth, a random variable X with density function given in equation (7) follows $TLE-GIE\left(x,\phi\right)$ where $\phi=\left(\sigma,\lambda,\theta,\gamma\right)$ is a vector with the following parameters, the survival function $S\left(x,\phi\right)$, hazard function $h\left(x,\phi\right)$, inverse hazard function $\tau\left(x,\phi\right)$ and the cumulative hazard function $H\left(x,\phi\right)$ for TLE-GIE distribution are given by

$$S\left(x;\phi\right)=1-\left[1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{\sigma} \tag{8}$$

$$h\left(x;\phi\right)=\frac{2\sigma\lambda\gamma\theta\exp\left\{-\frac{\gamma}{x}\right\}\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\left[1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{\sigma-1}}{x^{2}\left[\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{2}\right]^{-(\theta+1)}1-\left[1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{\sigma}} \tag{9}$$

$$\tau\left(x;\phi\right)=\frac{2\sigma\lambda\gamma\theta\exp\left\{-\frac{\gamma}{x}\right\}\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta-1}\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}}{x^{2}\left[\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta}\right]^{2}\left[1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{-\sigma}} \tag{10}$$

$$H\left(x;\phi\right)=-\ln\left[1-\left[1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{\sigma}\right] \tag{11}$$

The two figures above are the respective probability density and Hazard functions plots showing the behavioral shape of the new distribution (TLE-GIE). Figure 1 shows the flexibility of the new distribution as it can capture any data distribution that may behave asymmetrically with a long tail towards the right-hand side. While figure 2 explains the hazard function which is a downward bathtub shape

3. 0 Important Expansion

This section introduced a useful expansion for the pdf of TLE-GIE.

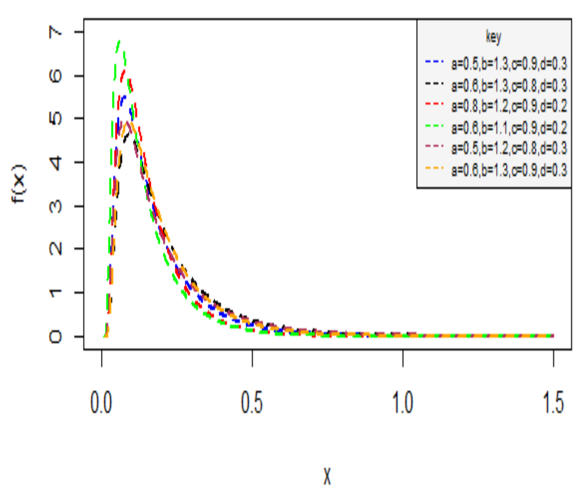

Fig. 1: pd function of TLE-GIE

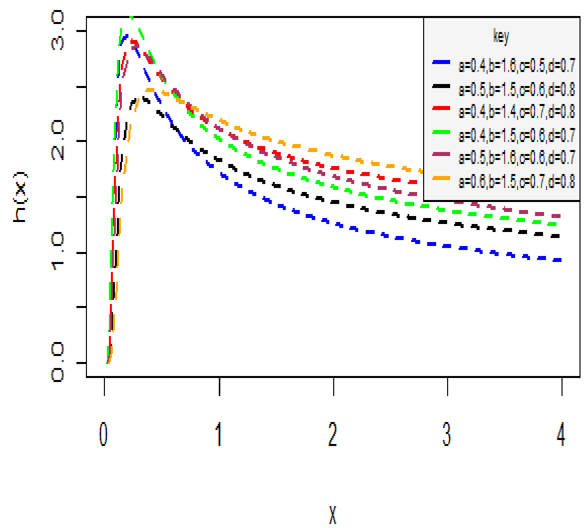

Fig. 2: Hzd function of TLE-GIE

Using generalized binomial and Taylor series expansions in equation (12), thus if $|x| \prec 1$ and $k \succ 0$ is a real noninteger, the power series holds:

$$(1-x)^{k-1} = \sum_{a=0}^{\infty} \frac{(-1)^a \Gamma(k)}{a!\Gamma(k-a)} x^a \qquad (12)$$

thus, applying the idea of equation (12) to the last term in (7), this becomes;

$$f_{TLE-GIE}(x;\sigma,\lambda,\theta,\gamma) = \frac{2\sigma\lambda\gamma\theta \exp\left\{-\frac{\gamma}{x}\right\}\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta-1}}{x^2\left[\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta}\right]^2} \sum_{a=1}^{n} (-1)^a \frac{\Gamma(\sigma)}{a!\Gamma(\sigma-a)} \qquad (13)$$

$$\times\left[\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{a+1}$$

In equation (13) above, apply power series to the term:
$\left[\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]^{a+1} = \left[\exp\left\{-2\lambda(a+1)\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right]$ so equation (13) becomes:

$$= \frac{2\sigma\lambda\gamma\theta \exp\left\{-\frac{\gamma}{x}\right\}\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta-1}}{x^2\left[\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta}\right]^2} \sum_{a,b=1}^{n} (-1)^{a+b} \frac{\Gamma(\sigma)(a+1)^b(2\lambda)^b}{a!b!\Gamma(\sigma-a)} \left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)^b \tag{14}$$

$$= \frac{2\sigma\lambda\gamma\varphi \exp\left\{-\frac{\gamma}{x}\right\}\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta-1}}{x^2\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{2\theta}} \sum_{a,b,c=1}^{n} (-1)^{a+b+c} \frac{\Gamma(\sigma)(a+1)^b(2\lambda)^b b!}{a!b!c!\Gamma(\sigma-a)(b-c)!} \left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta(b-c)}$$

$$= \frac{2\sigma\lambda\gamma\theta \exp\left\{-\frac{\gamma}{x}\right\}}{x^2\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{2\theta}} \sum_{a,b,c=1}^{n} (-1)^{a+b+c} \frac{\Gamma(\sigma)(a+1)^b(2\lambda)^b b!}{a!b!c!\Gamma(\sigma-a)(b-c)!} \left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{\theta[1-(b-c)]-1} \tag{15}$$

$$= \frac{2\sigma\lambda\gamma\theta \left[\exp\left\{-\frac{\gamma}{x}\right\}\right]^{d+1}}{x^2\Gamma\left[\theta[1-(b-c)]-d\right]} \sum_{a,b,c,d=1}^{n} (-1)^{a+b+c+d} \frac{\Gamma(\sigma)(a+1)^b(2\lambda)^b b!\Gamma\left[\theta[1-(b-c)]\right]}{a!b!c!d!(b-c)!\Gamma(\sigma-a)} \left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-2\theta} \tag{16}$$

Consider equation (17) below and apply it to the last term of equation (16) above, therefore equation (16) becomes: $(1-x)^{-k} = \sum_{b=0}^{\infty} \frac{\Gamma(k+b)}{b!\Gamma(k)} x^b \qquad |x| \prec 1, b \succ 0$ (17)

$$= \frac{2\sigma\lambda\gamma\theta \left[\exp\left\{-\frac{\gamma}{x}\right\}\right]^{e+d+1}}{x^2\Gamma\left[\theta[1-(b-c)]-d\right]} \sum_{a,b,c,d,e=1}^{n} (-1)^{a+b+c+d} \frac{\Gamma(\sigma)(a+1)^b(2\lambda)^b b!\Gamma\left[\theta[1-(b-c)]\right]\Gamma\left[(2\theta+e)\right]}{a!b!c!d!e!\Gamma(\sigma-a)(b-c)!\Gamma[2\theta]}$$

$$= \frac{2\sigma\lambda\gamma\theta\Gamma\left[\theta[1-(b-c)]\right]\Gamma\left[(2\theta+e)\right]}{x^2\Gamma\left[\theta[1-(b-c)]-d\right]} \sum_{a,b,c,d,e=1}^{n} (-1)^{a+b+c+d} \frac{\Gamma(\sigma)(a+1)^b(2\lambda)^b b!}{a!b!c!d!e!\Gamma(\sigma-a)(b-c)!\Gamma[2\theta]}$$

$$\times \exp\left\{-(e+d+1)\frac{\gamma}{x}\right\}$$

$$= \sum_{a,b,c,d,e=1}^{\infty} \psi_{a,b,c,d,e}\, \gamma \exp\left\{-(e+d+1)\frac{\gamma}{x}\right\} \tag{18}$$

$$where:\ \psi_{a,b,c,d,f,g} = (-1)^{a+b+c+d} \frac{2\sigma\lambda\gamma\theta\Gamma\left[\theta[1-(b-c)]\right]\Gamma\left[(2\theta+e)\right]\Gamma(\sigma)(a+1)^b(2\lambda)^b b!}{x^2 a!b!c!d!e!\Gamma(\sigma-a)(b-c)!\Gamma[2\theta]\Gamma\left[\theta[1-(b-c)]-d\right]}$$

4.0 Mathematical Properties

This section provides some mathematical properties of the TLE-GIE distribution such as moments, quantile function, rényi entropy, order statistics and the probability weighted moment.

4.1 Moments

Suppose X is a random variable with TLE-GIE distribution, then the raw moment, say $\mu_n^{'}$, is given by

$$\mu_n^{'} = E\left(x^n\right) = \int_{-\infty}^{\infty} x^n f_{TLEGIE}\left(x;\sigma,\lambda,\theta,\gamma\right)dx \tag{19}$$

$$= \sum_{a,b,c,d,e=1}^{\infty} \psi_{a,b,c,d,e} \int_{-\infty}^{\infty} x^{n-2}\gamma \exp\left\{-\left(e+d+1\right)\frac{\gamma}{x}\right\}\left(x\right)dx \tag{20}$$

$$\psi_{a,b,c,d,e} = \sum_{a,b,c,d,e=1}^{n} \left(-1\right)^{a+b+c+d} \frac{2\sigma\lambda\gamma\theta\Gamma\left[\theta\left[1-\left(b-c\right)\right]\right]\Gamma\left[\left(2\theta+e\right)\right]\Gamma\left(\sigma\right)\left(a+1\right)^b}{a!b!c!d!e!\Gamma\left[\theta\left[1-\left(b-c\right)\right]-d\right]}$$

$$\times\frac{\left(2\lambda\right)^b b!}{\Gamma\left(\sigma-a\right)\left(b-c\right)!\Gamma\left[2\theta\right]}$$

4.2 Quantile function

The quantile function of TLE-GIE is derived as:

$$x = \frac{-\gamma}{\log\left[1-\left(1-\left(\frac{k}{1+k}\right)\right)^{\frac{1}{\theta}}\right]} \tag{21}$$

$$\text{where: } k = \left(-\frac{1}{2\lambda}\right)\log\left(1-u^{\frac{1}{\sigma}}\right)$$

4.3 Renyi entropy

It is used as indices of diversity and quantifies the uncertainty or randomness of a system. The Renyi Entropy for TLE-GIE distribution is defined by

$$I_R\left(v\right) = \left(1-v\right)^{-1}\log\int_{-\infty}^{\infty} f^v\left(x\right)dx \qquad \text{for } v \succ 0 \text{ and } v \neq 1$$

$$I_R\left(v\right) = \left(1-v\right)^{-1}\log\sum_{a,b,c,d,e}^{\infty} \zeta_{a,b,c,d,e}\int_{-\infty}^{\infty} \exp\left\{-\left(d+v+e\right)\frac{\gamma}{x}\right\}dx \tag{22}$$

where:

$$\zeta_{a,b,c,d,e} = \left(-1\right)^{a+b+c+d} \frac{\left(2\sigma\lambda\gamma\theta\right)^v\left[v\left(\sigma-1\right)\right]!\left[2\lambda\left(a+v\right)\right]^b b!\left[\theta\left(v-b+c\right)-v\right]!\Gamma\left(2v\theta+e\right)}{a!b!c!d!e!x^{2v}\left(b-c\right)!\left[v\left(\sigma-1\right)-a\right]!\left[\left(\theta\left(v-b+c\right)-v\right)-d\right]!\Gamma\left(2v\theta+e\right)}$$

4.4 Order statistics

Let $X_{1;n} \le X_{2;n} \le \ldots \le X_{n;n}$ be the ordered sample from a continuous population with pdf f(x) and cdf F(x). The pdf of $X_{k;n}$, the kth order statistics of TLE-GIE distribution is given by

$$f_{i;n} = \sum_{j=0}^{n-i} \sum_{a,b,c,d,e=0}^{\infty} \Phi_{a,b,c,d,e} \Omega_{i,j} \exp\left\{ -\frac{(1+d+e)\gamma}{x} \right\} \tag{23}$$

where:

$$\Phi_{a,b,c,d,e} = (-1)^{a+b+c+d} \frac{2\sigma\lambda\gamma\theta b!\Gamma\left[\sigma\left[1+\left(j+i-1\right)\right]\right]\left[2\lambda\left(a+1\right)\right]^{b}\Gamma\left[\theta\left(1+c-b\right)\right]}{x^{2}a!b!c!d!e!\Gamma\left[\sigma\left[1+\left(j+i-1\right)\right]-a\right]}$$

$$\times \frac{\Gamma\left(2\theta+e\right)}{\Gamma\left[\theta\left(1+c-b\right)-d\right]\Gamma\left(2\theta\right)\left(b-c\right)!}$$

$$\Omega_{i,j} = (-1)^{j} \frac{\binom{n-1}{j}}{\beta\left(i,n-i+1\right)}$$

5.0 Parameter Estimation

Several approaches are used to estimate a parameter, but the maximum likelihood method is the most commonly used among others. Therefore, the maximum likelihood estimators of the unknown parameters of the TLE-GIE distribution from complete samples are determined. Let $X_1, \ldots, X_n$ be observed values from the TLE-GIE distribution with a vector of parameters ϕ. The Log-likelihood function can be expressed as

$$\begin{aligned} l(\phi) &= n\log(2) + n\log(\sigma) + n\log(\lambda) + n\log(\theta) + n\log(\gamma) - 2\sum_{i=1}^{n}\log(x) - \sum_{i=1}^{n}\left(\frac{\gamma}{x}\right) \\ &+ (\theta-1)\sum_{i=1}^{n}\log\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right] - 2\theta\sum_{i=1}^{n}\log\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right] - 2\lambda\sum_{i=1}^{n}\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right) \\ &+ (\sigma-1)\sum_{i=1}^{n} log\left(1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right) \end{aligned} \tag{24}$$

Differentiating equation (24) with respect to $\sigma, \lambda, \varphi, \gamma$ respectively and equating them to 0, we have:

$$\frac{n}{\sigma} + \sum_{i=1}^{n} log\left(1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right) = 0$$

$$\begin{aligned}&\frac{n}{\lambda}-2\sum_{i=1}^{n}\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)-2(\sigma-1)\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\\&\times\sum_{i=1}^{n}\frac{1}{\left(1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right)}=0\end{aligned}$$

$$\begin{aligned}&\frac{n}{\theta}+\sum_{i=1}^{n}\log\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]-2\sum_{i=1}^{n}\log\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]-2\lambda\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-(1+\theta)}\sum_{i=1}^{n}\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\\&+(\sigma-1)\sum_{i=1}^{n}\frac{\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-(1+\theta)}\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}}{\left(1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right)}=0\end{aligned}$$

$$\begin{aligned}&\frac{n}{\gamma}-\sum_{i=1}^{n}\left(\frac{1}{x}\right)-(\theta-1)\sum_{i=1}^{n}\frac{x^{-1}\exp\left\{-\frac{\gamma}{x}\right\}}{\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]}+2\theta x^{-1}\exp\left\{-\frac{\gamma}{x}\right\}\sum_{i=1}^{n}\frac{x^{-1}\exp\left\{-\frac{\gamma}{x}\right\}}{\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]}\\&-2\lambda x^{-1}\exp\left\{-\frac{\gamma}{x}\right\}\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}\sum_{i=1}^{n}\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\\&-(\sigma-1)\sum_{i=1}^{n}\frac{x^{-1}\exp\left\{-\frac{\gamma}{x}\right\}\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta-1}\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}}{\left(1-\exp\left\{-2\lambda\left(\left[1-\exp\left\{-\frac{\gamma}{x}\right\}\right]^{-\theta}-1\right)\right\}\right)}=0\end{aligned}$$

6.0 Simulation Study

In this section, a simulation study for the different sample size to on the efficiency of MLE method to determine the parameters' consistency of TLE-GIE is carried out. Thus, a random variable X from TLE-GIE is generated using R Software. Samples of size n = 10; 20; 40; 60; 80 and 100 from TLE-GIE distribution for some selected combination of parameters are used. This process is repeated N = 1000 time to calculate mean estimate, means squared error and bias. The results obtained are given in Table 1. From the Table, it is observed that when sample size increases the mean of the MLE approaches the initial values. Bias and Root Mean Squared Error (RMSE) decrease towards zero that is the parameters of TLE-GIE distribution are consistent. Therefore, the maximum likelihood method works very well to estimate the parameters of TLE-GIE distribution.

Table 1: Means, Bias and RMSEs for the TLE-GIE parameter estimates when $\sigma = 1.7, \lambda = 1.4, \phi = 0.6, \gamma = 1.3$

n	MLE				
		σ	λ	θ	γ
10	**Mean**	1.7531	1.2833	1.6560	1.5705
10	**Bias**	0.2531	-0.1167	0.0560	0.2705
10	**RMSE**	0.8062	0.6392	0.7113	0.4209
20	**Mean**	1.6248	1.2892	1.4845	1.5263
20	**Bias**	0.1248	-0.1108	-0.1155	0.2263
20	**RMSE**	0.5264	0.5448	0.5477	0.3440
40	**Mean**	1.5371	1.2860	1.3806	1.5037
40	**Bias**	0.0371	-0.1140	-0.2194	0.2037
40	**RMSE**	0.3450	0.4640	0.4523	0.2904
60	**Mean**	1.5074	1.2729	1.3536	1.5022
60	**Bias**	0.0074	-0.1271	-0.2464	0.2022
60	**RMSE**	0.2787	0.4265	0.4167	0.2625
80	**Mean**	1.4893	1.2593	1.3406	1.4982
80	**Bias**	-0.0107	-0.1407	-0.2594	0.1982
80	**RMSE**	0.2436	0.4030	0.3913	0.2519
100	**Mean**	1.4959	1.2532	1.3295	1.4887
100	**Bias**	-0.0041	-0.1468	-0.2705	0.1887
100	**RMSE**	0.2156	0.3981	0.3822	0.2334

7.0 Application to Data Sets

The flexibility of TLE-GIE in application to real life data sets is illustrated by comparing its performance with some other existing distributions. The parameters are estimated using the maximum likelihood method. The goodness-of-fit statistic and the MLE's for the models' parameters are presented in Tables 2, 3 and 4. To compare the fitted models, this paper used some goodness-of-fit measures which include the Akaike information criterion (*AIC*) and consistent Akaike information criterion (*CAIC*).

The fits of this new TLE-GIE distribution with other competitive distributions such as Topp Leone generalized inverted exponential (TLGIE) distribution, Topp-Leone Inverted Exponential (TLIE) distribution, Topp-Leone Standard Inverted Exponential (TLSIE) distribution, Inverted Exponential (IE) distribution and Topp-Leone Generalized Standard Inverted Exponential (TLGSIE) distribution are compared. Their PDFs are available in the literature. Throughout the results presented in the three tables, the distribution with the lowest AIC and CAIC values is TLE-GIE distribution compares to other competitive distributions; that is, TLE-GIE best fits the three data sets.

The three data sets considered are:

Data Set 1

The first data set that we considered has been used by Atallah *et al.* (2014). They represent 40 patients suffering from blood cancer (Leukemia) from one ministry of health hospital in Saudi Arabia. The ordered life time (in years) are given as follows: 0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.370, 2.532, 2.693, 2.805, 2.910, 2.912, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 4.381.

Data Set 2

The second data set consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes used by Proschan (1963). The actual data are: 194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 7.

Data Set 3

This data set consists of the waiting times (in seconds), between 65 successive eruptions of the Kiama Blowhole. These values were recorded, with the aid of digital watch on 12 July 1998 by Jim Irish and have been referenced by Pinho *et al.* (2015). The actual data are: 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77,36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29,8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

In Tables 2 to 4, the values of log-likelihood (LL), AIC and CAIC, those that are minimum; are more favorable to TLE-GIE distribution than other existing distribution, this indicates that the new model (TLE-GIE) is the best. It is depicted from the results that our proposed model provides best fit compare to other competitive sub-models. Thus, TLE-GIE is more reliable with these types of data sets.

Table 2. Parameters Estimation for various distributions depending on data set 1.

Models	Parameters' estimate						
	σ	λ	θ	γ	LL	AIC	CAIC
TLE-GIE	0.4752	0.0001	4.8260	0.4969	-65.7926	139.585	140.728
TL-GIE	0.418685	2.19025	7.26267		-82.2875	170.575	171.242
TL-IE	0.589171	4.55247			-85.5231	175.046	175.37
TL-SIE	4.55482				-90.3942	182.788	182.894
IE	2.00825				-91.1589	184.318	184.423
TL-GSIE	3.2155	0.755551			-88.1251	180.25	180.575

Table 3. Parameters Estimation for various distributions depending on data set 2.

Models	Parameters' estimate						
	σ	λ	θ	γ	LL	AIC	CAIC
TLE-GIE	9.5184	0.0578	0.3392	0.0047	-1033.621	2075.242	2075.72
TLGIE	8.84653	0.361313	1.11353		-1065.13	2136.25	2136.38
TL-IE	1.20401	22.9514			-1164.41	2332.83	2332.89
TL-SIE	106.161				-1379.43	2762.86	2762.92
IE	19.9992				-1082.51	2167.01	2167.03

Table 4. Parameters Estimation for various distributions depending on data set 3

Models	Parameters' estimate						
	σ	λ	θ	γ	LL	AIC	CAIC
TLE-GIE	29.4949	0.1523	0.3596	0.0254	-293.958	595.917	596.584
TL-GIE	2.06861	0.77448	14.7643		-295.07	596.14	596.54
TLSIE	283.888				-304.914	611.828	611.893
IE	20.4134				-299.175	600.351	600.415

8. 0 Conclusion

The generalized inverted exponential distribution is extended with Topp-Leone Exponential G family of distributions and resulted in Topp Leone Exponential Generalized Inverted Exponential (TLE-GIE) distribution with an additional four parameters. The mathematical properties of this compound distribution are derived based on the Binomial theorem expansion. The efficiency performance of the four parameters is consistent, however, tested through a simulation study with the method of maximum likelihood Estimate. Finally, this new TLE-GIE distribution is applied to three real data sets to assess its flexibility over some existing distributions. It is significantly observed that this new distribution best fit the data sets compared to the other distributions used in this study. Topp Leone Exponential Generalized Inverted Exponential (TLE-GIE) distribution improved its flexibility over Topp Leone Generalized Inverted Exponential (TL-GIE) distribution which only possesses the characteristics feature of top leone and generalized inverted exponential. Thus, the additional characteristics feature of exponential distribution in this new distribution (TLE-GIE) helps it to capture the exponential trend in the three data sets which TL-GIE cannot capture; with this additional characteristics feature, this has increase and boosts the flexibility of this new distribution called Topp Leone Exponential – Generalized inverted Exponential (TLE-GIE) distribution.

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