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# On the Properties and Applications of Topp-Leone Kumaraswamy Inverse Exponential Distribution

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**Abstract:** *The focus of many researchers in the field of distribution theory has been on the expansion of the existing probability distributions to improve their modeling flexibility. In this paper, we introduced a new continuous probability distribution called the Topp-Leone Kumaraswamy inverse exponential distribution with four parameters. We studied the nature of the proposed distribution with the help of its mathematical and statistical properties such as quantile function, ordinary and incomplete moments, generating function and reliability. The probability density function of order statistics for this distribution was also obtained. Monte Carlo simulation was carried out to see the performance of maximum likelihood estimation of Topp-Leone Kumaraswamy Inverse Exponential distribution. In this study, we performed a classical estimation of parameters by using the technique of maximum likelihood estimate. The proposed model was applied to two real datasets and shows that it provides a better fit than other well-known distributions presented.*

***Keywords:*** *Biases, Incomplete moment, Inverse exponential, Mean square error, Quantile function.*

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## 1.0 Introduction

An inverse distribution is the distribution of the reciprocal of a random variable in probability theory and statistics. In the Bayesian sense of prior and posterior distributions for scale parameters, inverse distributions are particularly common. Inverse distributions are special cases of the class of ratio distributions in which the numerator random variable has a degenerate distribution in random variable algebra.

The inverse exponential distribution is a subclass of the inverse Weibull distribution. The inverse exponential distribution was first proposed by Keller and Kamath (1982) and it can model datasets with inverted bathtub failure rates. It's a variant of the exponential distribution with the benefit of not having a constant failure rate. The IEx distribution in terms of various system failure causes was addressed by Lin *et al.*, (1989). Using complete samples, they calculated the maximum likelihood estimator and confidence limits for the parameter and the reliability function. They also used a maintenance data set to equate this model to the inverted Gaussian and log-normal distributions. The inclusion of an extra shape parameter to obtain

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the generalized IEx distribution was discussed by Abouammoh and Alshingiti (2009).

Recent research in this area has focused on expanding existing probability distributions to improve their modeling flexibility. In line with that, several works by Oguntunde *et al.*, (2017a), Oguntunde *et al.*, (2017b), Oguntunde *et al.*, (2017c) and Oguntunde *et al.*, (2017d) used different families of distributions to extend the inverse exponential distribution. In expanding the classical distributions to improve their modeling flexibility, many authors have proposed families of distributions in literature. Some of the families of the proposed distributions include:- the Topp Leone exponentiated-G by Ibrahim *et al.*, (2020a), Topp Leone Kumaraswamy-G by Ibrahim *et al.*, (2020b), Type I Half-Logistic Exponentiated-G Family of Distributions by Bello *et al.*, (2020), Type II Half-Logistic Exponentiated-G Family of Distributions by Bello *et al.*, (2021), The Kumaraswamy-G by Cordeiro and de Castro (2011), Topp Leone-G by Al-Shomrani *et al*., (2016), Odd Chen-G family by Anzagra (2020), Power Lindley-G Family of distributions by Hassan and Nassr (2019), Modi family of distributions by Modi *et al*., (2020), A new generalized-G class by Rasheed (2020).

A new generalized family of distributions by Sule *et al.*, (2022), etc. In this context, we proposed a generalization of the inverse exponential distribution based on Ibrahim *et al.*, (2020b), which stems from the following general construction: if $H$ denotes a random variable's baseline cumulative function, then a generalized class of distributions can be defined by

$$F(x;\alpha,\lambda,\theta,\xi)=\left\{1-\left[1-H(x;\xi)^{\alpha}\right]^{2\lambda}\right\}^{\theta} \quad (1)$$

The pdf corresponding to (1) is

$$f(x;\alpha,\lambda,\theta,\xi)=2\alpha\lambda\theta h(x;\xi)H(x;\xi)^{\alpha-1}\left[1-H(x;\xi)^{\alpha}\right]^{2\lambda-1}\left\{1-\left[1-H(x;\xi)^{\alpha}\right]^{2\lambda}\right\}^{\theta-1} \quad (2)$$

where $H(x;\xi)$ is the cdf of the baseline distribution with parameter vector $\xi$.

for $x\geq 0,\alpha,\lambda,\theta,\xi\geq 0$, where equations (1) and (2) are the cdf and pdf of the TLK-G family of distributions.

The cdf and pdf of the IEx distribution are given by

$$H(x;\beta)=e^{-\left(\frac{\beta}{x}\right)} \quad (3)$$

$$h(x;\beta)=\left(\frac{\beta}{x^2}\right)e^{-\left(\frac{\beta}{x}\right)} \quad (4)$$

$x>0,\beta>0$.

This paper proposes a new continuous distribution that generalizes the inverse exponential distribution using the family of distributions derived by Ibrahim *et al.*, (2020b). This is to improve the flexibility of the baseline distribution to fit a variety of data arising from different disciplines with different shapes.

## 2.0 The Topp-Leone Kumaraswamy Inverse Exponential (TLKIEx) Distribution

This section defines a new continuous distribution called TLKIEx distribution and provide some plots of its pdf, cdf and hazard rate function (hrf). The cdf of the TLKIEx distribution is obtained by inserting equation (3) into equation (1) and it is given as:

$$F(x;\alpha,\beta,\lambda,\theta)=\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta} \quad (5)$$

$$f(x;\alpha,\beta,\lambda,\theta)=2\alpha\lambda\theta\left(\frac{\beta}{x^2}\right)\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{2\lambda-1}\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta-1} \quad (6)$$

For $x\geq 0,\alpha,\beta,\theta,\lambda>0$.

Where $\beta$ is the scale parameter and $\alpha,\theta,\lambda$ are the shape parameters respectively.

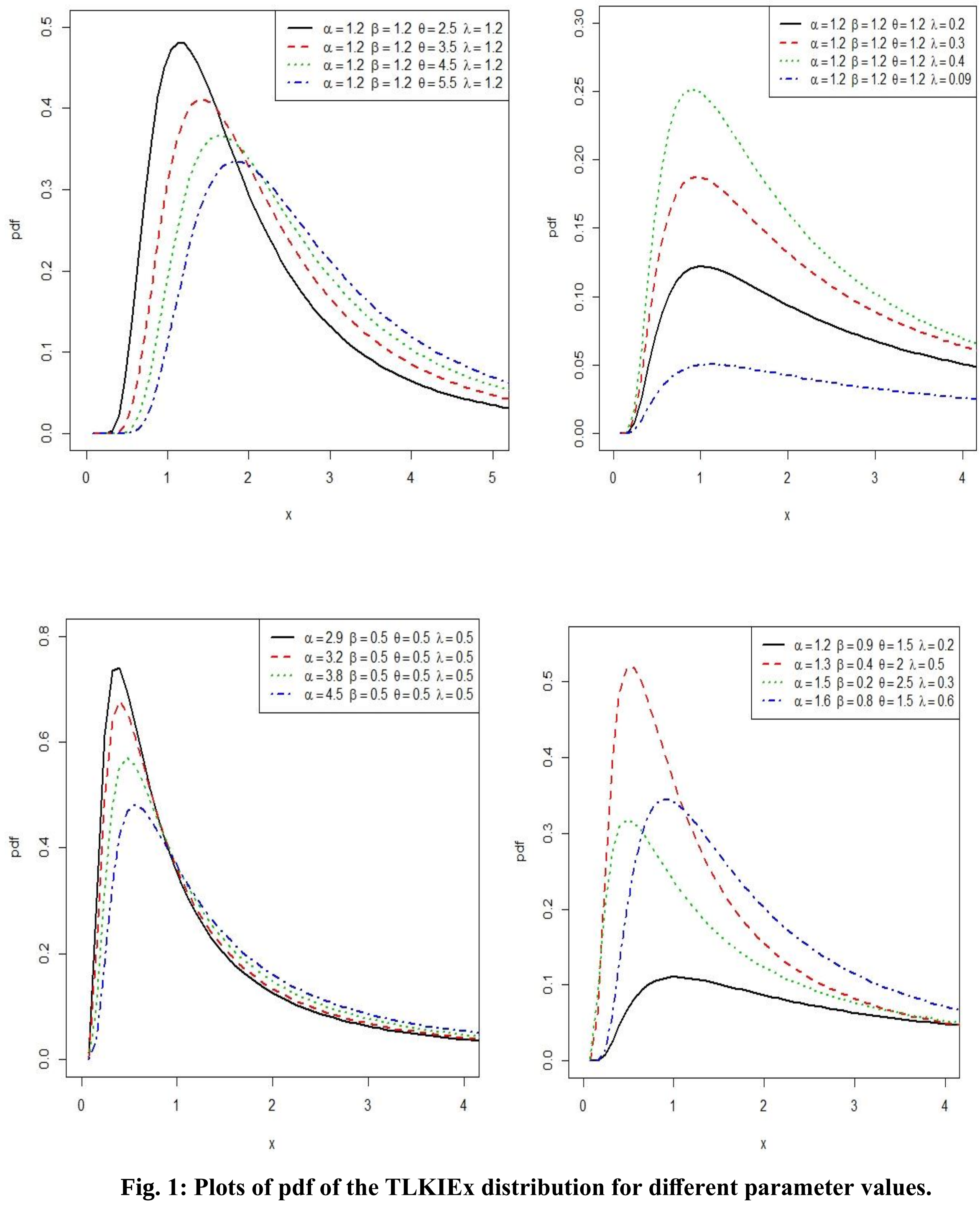


**Fig. 1: Plots of pdf of the TLKIEx distribution for different parameter values.**

## 3.0 Mathematical Properties.

This section derives some of the mathematical and statistical properties of the TLKIEx distribution such as the quantile function, moments, moment generating function, reliability measure and order statistics.

### *3.1 Moments*

The $r^{th}$ of $x$ is obtained as

$$E(X^r)=\int_0^\infty x^r f(x)dx \tag{7}$$

The $r^{th}$ moments of the TLKIEx distribution are obtained as

$$E(X^r)=\int_0^\infty x^r 2\alpha\lambda\theta\left(\frac{\beta}{x^2}\right)\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{2\lambda-1}\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta-1}dx$$

$$E(X^r)=2\alpha\lambda\theta\int_0^\infty x^r\left(\frac{\beta}{x^2}\right)\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{2\lambda-1}\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta-1}dx \tag{8}$$

Using binomial expansion on the last term in (8) with the relation

$$(1-x)^{\theta-1}=\sum_{i=1}^{\infty}\frac{(-1)^i\Gamma(\theta)}{i!\Gamma(\theta-1)}x^i \tag{9}$$

and

$$(1-x)^{\theta}=\sum_{i=1}^{\infty}\frac{(-1)^i\Gamma(\theta+1)}{i!\Gamma(\theta+1-i)}x^i \tag{10}$$

respectively.

$$\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta-1}=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\theta)}{i!\Gamma(\theta-i)}\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda i} \tag{11}$$

Substituting (11) into (8), we have

$$E(X^r)=2\alpha\lambda\theta\beta\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\theta)}{i!\Gamma(\theta-i)}\int_0^\infty x^{r-2}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{2\lambda(i+1)-1}dx \; . \tag{12}$$

Again, using the binomial expansion on the last term in (10), we have

$$\left[1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{2\lambda(i+1)-1}=\sum_{j=1}^{\infty}\frac{(-1)^j\Gamma(2\lambda(i+1))}{j!\Gamma(2\lambda(i+1)-j)}\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha j} . \tag{13}$$

Substituting (13) into (12), we have

$$E(X^r)=2\alpha\lambda\theta\beta\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\theta)}{i!\Gamma(\theta-i)}\sum_{j=1}^{\infty}\frac{(-1)^j\Gamma(2\lambda(i+1))}{j!\Gamma(2\lambda(i+1)-j)}\int_0^\infty x^{r-2}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{\alpha(j+1)}dx \; . \tag{14}$$

From (14), let $y=\alpha(j+1)\left(\frac{\beta}{x}\right)\Rightarrow x=\alpha(j+1)\left(\frac{\beta}{y}\right), dx=\frac{-dy}{\alpha(j+1)\beta x^2}$

Then,

$$E(X^r) = 2\alpha\lambda\theta\beta \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\theta)}{i!\Gamma(\theta - i)} \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(2\lambda(i+1))}{j!\Gamma(2\lambda(i+1)-j)} \int_0^{\infty} \left( \alpha(j+1)\left(\frac{\beta}{y}\right)\right)^{r-2} e^{-y} \frac{dy}{\alpha(j+1)\beta x^2} \quad (15)$$

$$E(X^r) = 2\alpha\lambda\theta\beta \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{i+j}\Gamma(\theta)\Gamma(2\lambda(i+1))}{i!\,j!\Gamma(\theta-i)\Gamma(2\lambda(i+1)-j)} \left(\alpha\beta(j+1)\right)^{r-1} \int_0^{\infty} y^{-r} e^{-y} dy \quad (16)$$

Where $\int_0^{\infty} y^{-r} e^{-y} dy = \Gamma(1-r)$

$$E(X^r) = 2\alpha\lambda\theta\beta \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{i+j}\Gamma(\theta)\Gamma(2\lambda(i+1))}{i!\,j!\Gamma(\theta-i)\Gamma(2\lambda(i+1)-j)} \left(\alpha\beta(j+1)\right)^{r-1} \Gamma(1-r) \quad (17)$$

$$E(X^r) = 2\alpha^r\lambda\theta\beta^r \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{i+j}\Gamma(\theta)\Gamma(2\lambda(i+1))}{i!\,j!\Gamma(\theta-i)\Gamma(2\lambda(i+1)-j)} \left(j+1\right)^{r-1} \Gamma(1-r) \quad (18)$$

The mean of the TLKIEx distribution is obtained by setting $r = 1$ in equation (18).

### *3.2 Moment Generating Function (MGF)*

The MGF $X$ can be obtained using the equation

$$E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx \quad (19)$$

$$E(e^{tx}) = \int_0^{\infty} e^{tx} 2\alpha\lambda\theta\left(\frac{\beta}{x^2}\right)\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{2\lambda-1}\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta-1} dx \quad (20)$$

$$e^{tx} = \sum_{m=0}^{\infty} \frac{t^m x^m}{m!} \quad (21)$$

Following the process of moments above, we have the MGF given as

$$E(e^{tx}) = 2\alpha^m\lambda\theta\beta^m \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j}\Gamma(\theta)\Gamma(2\lambda(i+1))}{i!\,j!\,m!\Gamma(\theta-i)\Gamma(2\lambda(i+1)-j)} t^m \left(j+1\right)^{m-1} \Gamma(1-m) \quad (22)$$

### *3.3 Reliability function*

The reliability function is also known as the survival function, which is the probability of an item not failing before some time. It can be defined as

$$R(x;\alpha,\beta,\lambda,\theta) = 1 - F(x;\alpha,\beta,\lambda,\theta) \quad (23)$$

$$R(x;\alpha,\beta,\lambda,\theta) = 1 - \left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta} \quad (24)$$

### *3.4 Hazard rate function*

The hazard function is given as

$$\tau(x;\alpha,\beta,\lambda,\theta)=\frac{f(x;\alpha,\beta,\lambda,\theta)}{R(x;\alpha,\beta,\lambda,\theta)} \tag{25}$$

$$\tau(x;\alpha,\beta,\lambda,\theta)=\frac{2\alpha\lambda\theta\left(\frac{\beta}{x^2}\right)\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{2\lambda-1}\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta-1}}{1-\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta}} \tag{26}$$

### *3.5 Quantile function*

The quantile function is defined as the inverse of the cdf and it is given as: $Q(u)=F^{-1}(u)$. Using the cdf of TLKIEx distribution in equation (5), we have

$$\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta}=u$$

$$u^{\frac{1}{\theta}}=1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}$$

$$1-u^{\frac{1}{\theta}}=\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}$$

$$\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2\lambda}}=1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}$$

$$1-\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2\lambda}}=\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}$$

$$\left(1-\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2\lambda}}\right)^{\frac{1}{\alpha}}=e^{-\left(\frac{\beta}{x}\right)}$$

$$-\left(\frac{\beta}{x}\right)=\log\left[\left(1-\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2\lambda}}\right)^{\frac{1}{\alpha}}\right]$$

$$x = \beta\left\{-\log\left[\left(1-\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2\lambda}}\right)^{\frac{1}{\alpha}}\right]\right\}^{-1} \tag{27}$$

The median of the TLKIEx distribution can be derived by substituting $u = 0.5$ in equation (27)

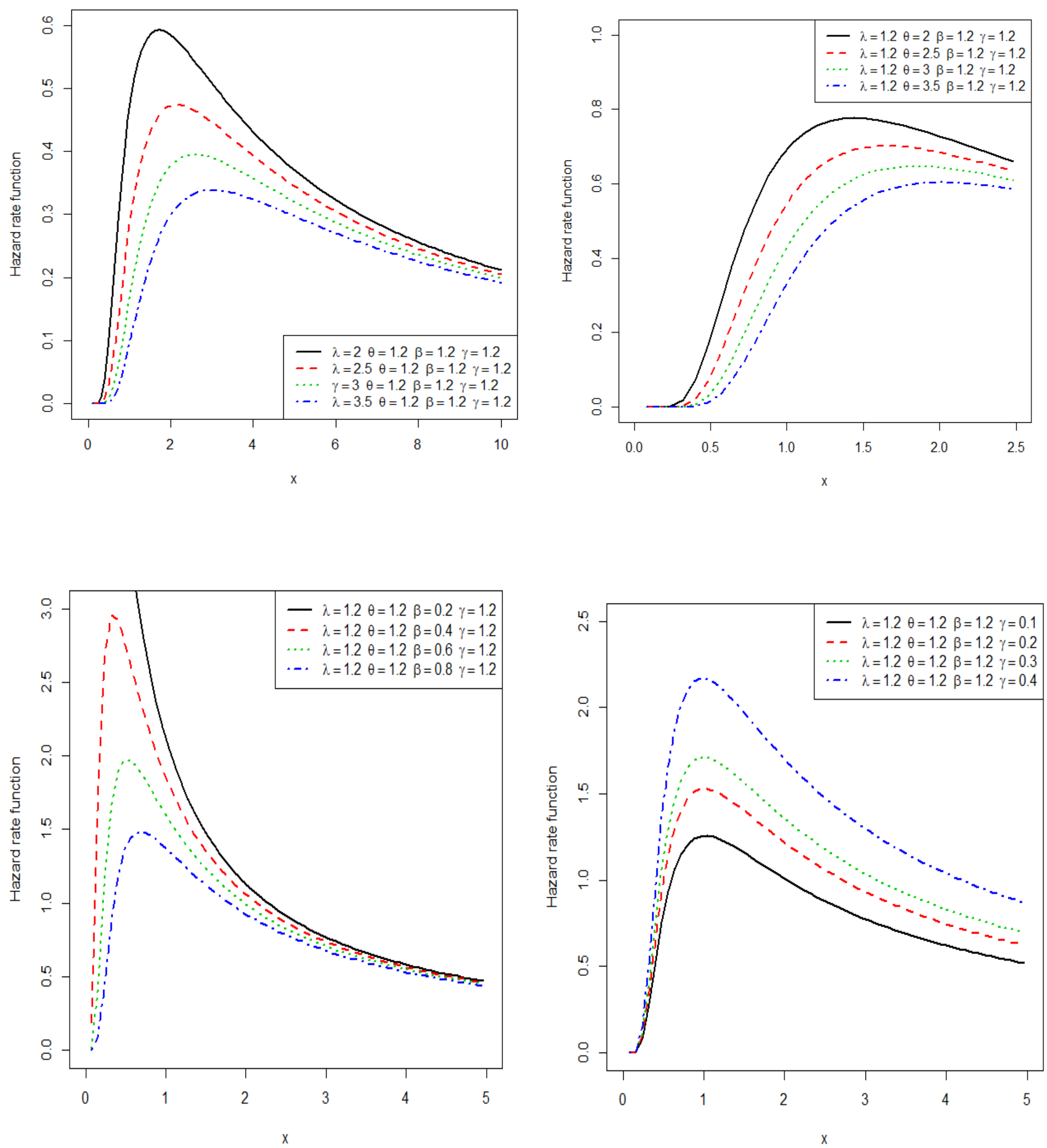


**Fig. 2: Plots of hazard rate function of the TLKIEx distribution for different parameter values.**

## 4.0 Order statistics

Let $X_1, X_2, ..., X_n$ be $n$ independent random variable from the TLKIEx distributions and let $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ be their corresponding order statistic. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1, 2, 3, ...n$ denote the cdf and pdf of the $r^{th}$ order statistics $X_{r:n}$ respectively. The pdf of the $r^{th}$ order statistics $X_{r:n}$ is given as

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(n-r+1)}{i!\,\Gamma(n-r+1-i)} [F(x)]^{r+i-1} f(x) \tag{28}$$

Using the cdf and pdf of TLKIEx distribution in equation (5) and equation (6), we have

$$f_{r:n}(x) = \frac{2\alpha\theta\lambda}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^i \left[1 - \left(1 - \left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta(r+i-1)} \frac{\beta}{x^2} \left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha} \left(1 - \left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda-1} \left[1 - \left(1 - \left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right]^{\theta-1}$$

$$f_{r:n}(x) = \frac{2\alpha\theta\lambda}{B(r, n-r+1)} \frac{\beta}{x^2} \sum_{i=0}^{n-r} \sum_{j,k=0}^{\infty} (-1)^{i+j+k} \binom{\theta(r+i)-1}{j} \binom{2\lambda(j+1)}{k} \left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha(k+1)} \tag{29}$$

Equation (29) is the $r^{th}$ order statistics of the TLKIEx distribution.
Therefore, the pdf of the minimum and maximum order statistics of the TLKIEx distribution are obtained by setting $r = 1$ and $r = n$ respectively in equation (29).

## 5.0 Estimation

In this section, we estimate the parameters of the TLKIEx distribution using maximum likelihood estimation (MLE). For a random sample, $X_1, X_2, ..., X_n$ of size $n$ from the TLKIEx $(\alpha, \beta, \theta, \lambda)$, the log-likelihood function L $(\alpha, \beta, \theta, \lambda)$ of equation (6) is given as

$$\ell(\phi) = n\log 2 + n\log\alpha + n\log\beta + n\log\lambda + \sum_{i=1}^{n} \log\left(\frac{1}{x_i^2}\right) - \sum_{i=1}^{n} \left(\frac{\beta}{x_i}\right)^{\alpha} + (2\lambda-1)\sum_{i=1}^{n} \log\left(1 - \left(e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\alpha}\right) + (\theta-1)\sum_{i=1}^{n} \log\left[1 - \left(1 - \left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{2\lambda}\right] \tag{30}$$

The components of the score vector, say $\Delta\ell(\phi) = \left(\frac{\partial \log \ell(\phi)}{\partial\alpha}, \frac{\partial \log \ell(\phi)}{\partial\beta}, \frac{\partial \log \ell(\phi)}{\partial\lambda}, \frac{\partial \log \ell(\phi)}{\partial\theta}\right)$.

Differentiating equation (30) concerning each parameter and setting the equation to zero, we have

$$\frac{\partial \log \ell(\phi)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \left(\frac{\beta}{x_i}\right)^{\alpha} \log\left(\frac{\beta}{x_i}\right) - (2\lambda-1)\sum_{i=1}^{n} \left[\frac{\beta\left(e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\alpha} \log\left(e^{-\left(\frac{\beta}{x_i}\right)}\right)}{1 - \left(e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\alpha}}\right] - (\theta-1)\sum_{i=1}^{n} \left[\frac{2\lambda\beta\left(e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\alpha} \log\left(e^{-\left(\frac{\beta}{x_i}\right)}\right)\left[1 - \left(e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\alpha}\right]^{2\lambda-1}}{1 - \left[1 - \left(e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\alpha}\right]^{2\lambda}}\right] = 0 \tag{31}$$

$$\frac{\partial \log \ell(\phi)}{\partial \beta} = \frac{n}{\beta} - \alpha \sum_{i=1}^{n} \left( \frac{\beta}{x_i} \right)^{\alpha} - \alpha(2\lambda - 1) \sum_{i=1}^{n} \left[ \frac{\left( \frac{\beta}{x_i} \right)^{\alpha} e^{-\left( \frac{\beta}{x_i} \right)}}{1 - \left( e^{-\left( \frac{\beta}{x_i} \right)} \right)^{\alpha}} \right] - (\theta - 1) \sum_{i=1}^{n} \left[ \frac{2\lambda\alpha \left( \frac{\beta}{x_i} \right)^{\alpha} e^{-\left( \frac{\beta}{x_i} \right)} \left[ 1 - \left( e^{-\left( \frac{\beta}{x_i} \right)} \right)^{\alpha} \right]^{2\lambda - 1} \left[ 1 - \left( e^{-\left( \frac{\beta}{x_i} \right)} \right)^{\alpha} \right]}{1 - \left[ 1 - \left( e^{-\left( \frac{\beta}{x_i} \right)} \right)^{\alpha} \right]^{2\lambda}} \right] = 0 \quad (32)$$

$$\frac{\partial \log \ell(\phi)}{\partial \lambda} = \frac{n}{\lambda} + 2 \sum_{i=1}^{n} \log \left[ 1 - \left( e^{-\left( \frac{\beta}{x_i} \right)} \right)^{\alpha} \right] + (\theta - 1) \sum_{i=1}^{n} \left[ \frac{\left[ 1 - \left( e^{-\left( \frac{\beta}{x_i} \right)} \right)^{\alpha} \right]^{2\lambda} \log \left[ 1 - \left( e^{-\left( \frac{\beta}{x_i} \right)} \right)^{\alpha} \right]}{1 - \left[ 1 - \left( e^{-\left( \frac{\beta}{x_i} \right)} \right)^{\alpha} \right]^{2\lambda}} \right] = 0 \quad (33)$$

$$\frac{\partial \log \ell(\phi)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log \left[ 1 - \left( 1 - \left( e^{-\left( \frac{\beta}{x} \right)} \right)^{\alpha} \right)^{2\lambda} \right] = 0 \quad (34)$$

Now, equation (31), equation (32), equation (33) and equation (34) do not have a simple form and are therefore intractable. As a result, we have to resort to the non-linear estimation of the parameters using iterative procedures.

## 5.0 Simulation study

In this Section, we perform the simulation study to see the performance of MLEs of TLKIEx distribution. The random number generation is obtained with its quantile function (qf). We note that the $u^{th}$ qf of the TLKIEx distribution is given in equation (27). Hence, if $U$ has a uniform random variable on (0, 1), then $x$ has the TLKIEx random variable.

We generated $N=10000$ samples of sizes $n=50,\ 100,\ 200$ *and* $500$ from TLKIEx distribution with its qf. Then we computed the empirical means, biases and mean squared errors (MSE) of the MLEs with

$$Bias_{\hat{\psi}} = \frac{1}{N} \sum_{i=1}^{N} \left( \hat{\psi}_i - \psi_i \right) \quad (35)$$

and

$$MSE_{\hat{\psi}} = \frac{1}{N} \sum_{i=1}^{N} \left( \hat{\psi}_i - \psi_i \right)^2, \quad (36)$$

for $\psi = (\beta, \theta, \alpha, \lambda)$

To examine the performance of the MLEs for the TLKIEx distribution, we perform a simulation study as follows:

1. Generate $N$ samples of size $n$ from the TLKIEx distribution with its qf.
2. Compute the MLEs for the $N$ samples, say $(\hat{\beta}, \hat{\theta}, \hat{\alpha}, \hat{\lambda})$, for $i = 1, 2, ..., N$
3. Compute the MLEs for $N$ samples
4. Compute the biases and mean squared errors (MSE) given in equation (35) and equation (36).

We repeat these steps for $N= 10000$ and $n = 50, 100, 200$ *and* $500$ with different values of $\psi = (\beta, \theta, \alpha, \lambda)$. Table 1 shows how the biases and MSE vary concerning $n$. As expected, the Biases and

MSEs of the estimated parameters converge to zero as *n* increases proving the consistency of the estimators.

**Table 1: Biases and MSE of the TLKIEx distribution for selected parameter values.**

| Groups | Initial values | Bias and MSE | Sample sizes n=50 | n=100 | n=200 | n=500 |
|---|---|---|---|---|---|---|
| | $\beta$ =0.5 | Bias | 0.4757282 | 0.380458 | 0.2415676 | 0.08192196 |
| | | MSE | 1.128401 | 0.8846674 | 0.5046992 | 0.3118731 |
| | $\theta$ =1 | Bias | -0.5242718 | -0.619542 | -0.7584324 | -0.9180780 |
| | | MSE | 1.149710 | 1.0108019 | 0.8783997 | 0.6661371 |
| | $\alpha$ =1 | Bias | 2.0270014 | 1.069987 | 0.7901550 | 0.29064191 |
| I | | MSE | 4.433887 | 3.5073255 | 2.2855787 | 0.8993250 |
| | $\lambda$ =1 | Bias | 1.0270014 | 0.069987 | -0.209845 | -0.7093581 |
| | | MSE | 4.074966 | 3.3408618 | 2.1548922 | 1.1079268 |
| | $\beta$ =0.5 | Bias | 0.6607848 | 0.4058381 | 0.2491716 | 0.2579595 |
| | | MSE | 2.366874 | 0.9139879 | 0.772265 | 0.5087453 |
| | $\theta$ =1.5 | Bias | -0.8392152 | -1.0941619 | -1.2508284 | -1.2820405 |
| | | MSE | 2.422754 | 1.3666966 | 1.313590 | 1.3071725 |
| | $\alpha$ =1 | Bias | 3.2580953 | 1.7659127 | 1.4749357 | -0.5941619 |
| II | | MSE | 8.116632 | 3.8690291 | 3.153124 | 1.6343640 |
| | $\lambda$ =1 | Bias | 1.7580953 | 0.2659127 | -0.0250643 | -0.5567376 |
| | | MSE | 7.639073 | 3.4527740 | 2.787003 | 1.4461530 |
| | $\beta$ =0.5 | Bias | 0.4564184 | 0.4112619 | 0.3480379 | 0.2634555 |
| | | MSE | 0.9916529 | 0.7500362 | 0.635278 | 0.4417987 |
| | $\theta$ =1.5 | Bias | -1.0435816 | -1.0887381 | -1.1509621 | -1.2365445 |
| | | MSE | 1.3653279 | 1.2564906 | 1.228647 | 1.1263980 |
| | $\alpha$ =1.1 | Bias | 3.6466509 | 1.7616832 | 1.4467925 | 1.1727391 |
| III | | MSE | 8.0577188 | 3.9690887 | 3.507378 | 2.4293541 |
| | $\lambda$ =1 | Bias | 2.2466509 | 0.3616832 | 0.0467925 | -0.2272609 |
| | | MSE | 7.4991253 | 3.5663167 | 3.195516 | 2.1525669 |
| | $\beta$ =1.5 | Bias | 0.454713 | 0.3373149 | 0.1573868 | -0.3707917 |
| | | MSE | 3.112982 | 1.772277 | 1.511235 | 0.8094963 |
| | $\theta$ =1.5 | Bias | -1.045287 | -1.1626851 | -1.3426132 | 1.7292083 |
| | | MSE | 3.252156 | 2.092611 | 2.015358 | 1.8954971 |
| | $\alpha$ =1.1 | Bias | 3.875349 | 1.8563823 | 1.3376786 | 1.0302311 |
| IV | | MSE | 7.233175 | 4.223159 | 2.807066 | 1.9164191 |
| | $\lambda$ =1 | Bias | 2.475349 | 0.4563823 | -0.0623214 | -0.3697689 |
| | | MSE | 6.553074 | 3.809977 | 2.473173 | 1.6828454 |

## 6.0 Application to real-life data set

The first data set represents the breaking strength of 100 Yarn as reported by Gomes-Silva *et al.*, (2017). The data-set consists of 63 measurements of the strengths of 1.5 cm glass fibres, which were initially collected by United Kingdom National Physical Laboratory staff. The data is presented below:
0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63,1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.
The second data set represents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports as has been used by Hosseini *et al.*, (2018). The data set is:

28.0, 98, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2,101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6,52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48, 61.2, 171.1, 113.5, 148.9, 49.9,59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2,70.0, 97.2, 123.6, 181.7, 136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9.

The pdf of the competing distributions considered are:

- Exponentiated Kumarawamy Inverse Exponential (ExKIEx)distribution by Umar *et al.,* (2017)

$$f(x;\alpha,\beta,\theta,\lambda)=\alpha\theta\lambda\beta x^{-2}\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{\lambda-1}\left[1-\left(1-\left(e^{-\left(\frac{\beta}{x}\right)}\right)^{\alpha}\right)^{\lambda}\right]^{\theta-1} \quad (37)$$

- Kumaraswamy Inverse Exponential (KIEx) distribution by Oguntunde *et al.*, (2014).

$$f(x;\alpha,\beta,\lambda)=\alpha\lambda\beta x^{-2}\left(e^{-\left(\frac{\lambda}{x}\right)}\right)^{\alpha}\left[1-\left(e^{-\left(\frac{\lambda}{x}\right)}\right)^{\alpha}\right]^{\beta-1} \quad (38)$$

- Inverse Exponential (IEx) distribution by Keller and Kamath (1982)

$$f(x;\beta)=\beta x^{-2}e^{-\left(\frac{\beta}{x}\right)}$$

Tables 2 and 3 present the estimate of each parameter and goodness of fit for the models considered. The goodness of fits considered is the Akaike Information Criteria (AIC). The smaller the AIC value the better the model. Figs. 3 and 4 present the shapes, fit and flexibility of the new model to the data sets considered. The black line represents the new model, the red line represents the ExKIEx, the green line represents the KIEx and the blue line represents the IEx distributions. It can be seen from the histogram and fitted plots that the black line which represents the TLKIEx distribution fits better in the two data sets considered.

**Table 2: The MLEs and Information Criteria of the models based on data set 1'**

| **Models** | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $-l$ | **AIC** |
|---|---|---|---|---|---|---|
| **TLKIEx** | 0.3038 | 37.6361 | 0.5069 | 337.1824 | 18.6440 | 45.2879 |
| **ExKIEx** | 0.0359 | 223.6706 | 0.8048 | 135.2661 | 21.5617 | 51.1234 |
| **KIEx** | 44.6609 | 163.2478 | - | 0.1825 | 22.0606 | 50.1211 |
| **IEx** | - | 1.4084 | - | - | 89.4392 | 180.8784 |

**Table 3: The MLEs and Information Criteria of the models based on data set 2**

| **Models** | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $-l$ | **AIC** |
|---|---|---|---|---|---|---|
| **TLKIEx** | 9.1822 | 6.4616 | 4.5909 | 2.1196 | 954.6239 | 1917.2480 |
| **ExKIEx** | 4.0728 | 2.9706 | 92.8097 | 2.9237 | 955.2849 | 1918.5700 |
| **KIEx** | 10.6439 | 8.0556 | - | 14.5206 | 955.7860 | 1917.5720 |
| **IEx** | - | 57.1300 | - | - | 1055.7730 | 2119.5450 |

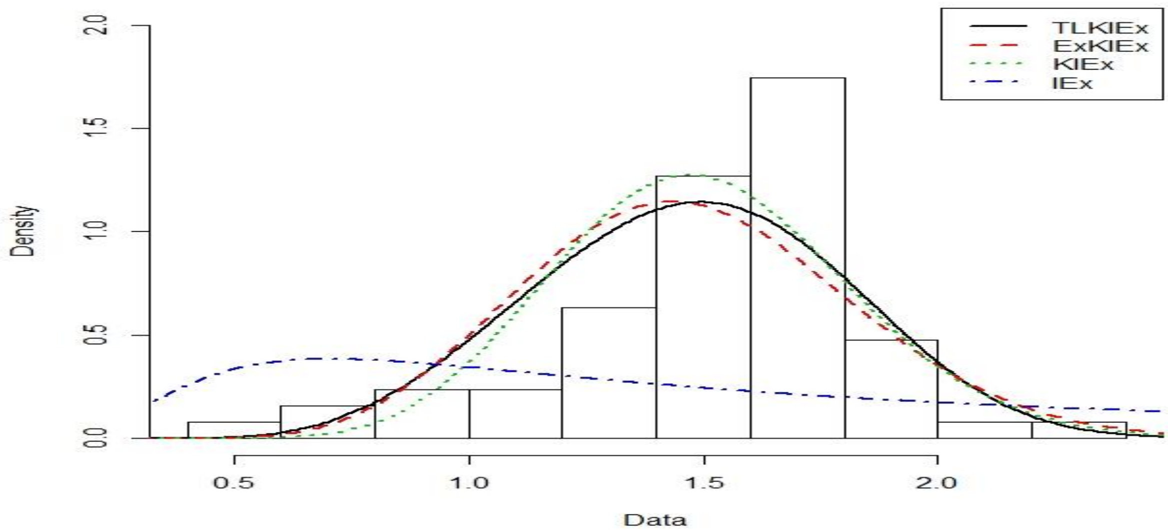


**Fig. 3: Histogram and fitted pdfs for the TLKEx, KEx, TLEx, ExEx and Ex models to the data set 1.**

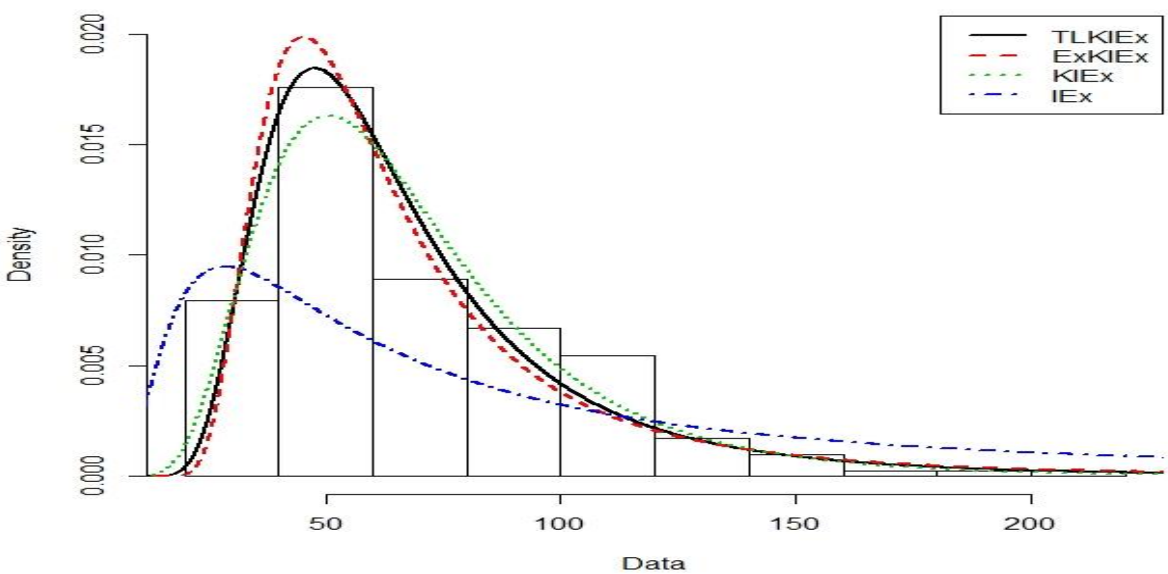


**Fig. 4: Histogram and fitted pdfs for the TLKEx, KEx, TLEx, ExEx and Ex models to the data set 2**

## 7.0 Conclusion

This paper has derived a new distribution called the Topp-Leone Kumaraswamy inverse exponential distribution that extends the inverse exponential distribution by adding extra shape parameters. Some properties of the new distribution were derived such as the survival function, hazard rate function, quantile function, median and order statistics. The shapes of the proposed distribution were shown by plotting the graphs of the pdf and hazard rate function. The estimation of the model parameters by the method of the maximum likelihood was done using a package in *R* known as *AdequacyModel*. Monte Carlo simulation was carried out to see the performance of MLEs of the TLKIEx distribution and as expected, the Biases and MSEs of the estimated parameters converge to zero as *n* increases proving the consistency of the estimator. Application of the new distribution to two real data sets was carried out to see the performance and flexibility of the new model. The results of the analysis presented in Tables 1 and 2 showed that the Topp-Leone Kumaraswamy inverse exponential distribution is quite effective and superior in fitting the two data sets considered. Also, the fit and flexibility of the new model can be seen from the histogram and fitted pdf plots for the two data sets, it can be deduced that the new model fits the two data sets better than the competing models considered.

As also seen from the plots of the pdf and hazard rate function, the new model can be applied in different areas due to the different shapes exhibited by the new model.


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## 9.0 References


Abouammoh, A. M. & Alshingiti, A. M. (2009). Reliability of generalised inverted exponential distribution, Journal *of Statistical Computation and Simulation*, 79, pp. 1301-1315.

Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. & Shahbaz, M. Q., (2016). Topp-Leone family of distributions: Some properties and application, *Pakistan Journal of Statistics and Operation Research*, XII, 3, pp. 443-451.

Anzagra, L., Sarpong, S. & Nasiru, S., (2020). Odd Chen-G family of distributions, *Annals of Data Science*, doi.org/10.1007/s40745-020-00248-2.

Bello, O. A., Doguwa, S. I., Yahaya, A. & Jibril, H. M. (2020). A type I Half-Logistic Exponentiated-G family of distributions: properties and applications, *Communication in Physical Sciences*, 7, 3, pp. 147-163.

Bello, O. A., Doguwa, S. I., Yahaya, A. & Jibril, H. M. (2021). A type II Half-Logistic exponentiated-G family of distributions with applications to survival analysis, *FUDMA Journal of Sciences*, 5, 3, pp. 177-190.

Cordeiro, G. M., & de Castro, M., (2011). A new family of generalized distributions, *Journal of Statistical Computation and Simulation*, 81, 7, pp. 883–898.

Gomes-Silva, F., Percontini, A., Brito, E., Ramos, M. W., Silva, R. V. & Cordeiro, G. M. (2017), The odd Lindley-G family of distributions, *Austrian Journal of Statistics*, 46, 65-87.

Hassan, S. & Nassr, S. G., (2019). Power Lindley-G Family of Distributions, *Annals of Data Science*, 6, pp. 189-210.

Hosseini, B., Afshari, M. & Alizadeh M. (2018). The generalized odd gamma-G family of distributions: properties and applications. *Austrian Journal of Statistics*, 47, 2, pp. 69-89.

Ibrahim, S., Doguwa, S. I., Audu, I. & Jibril, H. M. (2020a). On the Topp Leone exponentiated-G family of distributions: properties and applications, *Asian Journal of Probability and Statistics*; 7, 1, pp. 1-15.

Ibrahim, S., Doguwa S. I., Audu, I. & Jibril, H. M., (2020b). The Topp Leone Kumaraswamy-G family of distributions with applications to cancer disease data, *Journal of Biostatistics and Epidemiology*, 6, 1, pp. 37-48.

Keller, A. Z. & Kamath, A. R. (1982). Reliability analysis of CNC Machine Tools. *Reliability Engineering*, 3, pp. 449-473.

Lin, C. T., Duran, B. S. & Lewis, T. O. (1989). Inverted gamma as life distribution, *Microelectron Reliability*, 29, 4, pp. 619-626.

Modi, K., Kumar, D. & Singh, Y., (2020). A new family of distribution with application on two real data sets on survival problem. *Science and Technology Asia*, 25, 1, pp, 1-10.

Oguntunde, P. E., Babatunde, O. S. & Ogunmola, A. O. (2014). Theoretical analysis of the Kumaraswamy inverse exponential distribution, *International Journal of Statistics and Applications*, 4, 2, pp. 113-116.

Oguntunde, P. E., Adejumo, A. O. & Owoloko, E. A. (2017a). Application of Kumaraswamy inverse exponential distribution to real lifetime data. *International Journal of*

*Applied Mathematics and Statistics*, 56, 5, pp. 34–47.

Oguntunde, P. E., Adejumo, A. O. & Owoloko, E. A. (2017b). *On the flexibility of the transmuted inverse exponential distribution*. Lecture Notes on engineering and computer science: Proceeding of the World Congress on Engineering. Held at London, UK. On the 5th – 7th of July 2017, pp. 123–126.

Oguntunde, P. E., Adejumo, A. O. & Owoloko, E. A. (2017c). *On the exponentiated generalized inverse exponential distribution*. Lecture Notes on engineering and computer science: Proceeding of the World Congress on Engineering. Held at London, UK. On the 5th – 7th of July 2017, pp. 80–83

Oguntunde, P. E., Adejumo, A. O. & Owoloko, E. A. (2017d). *The Weibull-inverted exponential distribution: A generalization of the inverse exponential distribution*. Proceeding of the World Congress on Engineering. Held at London, UK. On the 5th – 7th of July 2017, pp. 16–19

Rasheed, N. (2020). A new generalized-G class of distributions and its applications with Dagun distribution. *Research Journal of Mathematical and Statistical Science*, 8(3): 1-13.

Sule, I., Lawal, H. O. & Bello, O. A. (2022). Properties of a new generalized family of distributions with applications to relief times of patient's data, *Journal of Statistical Modeling and Analytics*, 4, 1, pp. 39 - 55.

Umar, N., Abba, B. & Mohammad, A. S. (2017). *A note on the exponentiated kumaraswamy Inverse Exponential Distribution*. Proceedings of the 3rd YUMSCIC, pp. 314-318.

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