Communication in Physical Sciences, 2023, 9(4): 483-499

Analytical Solutions of the Schrodinger Equation with q-Deformed Modified Mobius Square Potential Using the Nikiforov-Uvarov Method

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Received: 06 May 2023/Accepted 12 August 2023/Published 28 August 2023

Abstract: *In this study, a new potential named the "q-deformed modified Mobius square potential (MMSP)" is proposed. The Schrodinger equation with this model is solved using the famous Nikiforov-Uvarov (NU) method to obtain the energy equation and wave function of this system. The effects of the potential parameters and deformation parameter* $(q>0 \text{ and } q<0)$ *is analysed numerically and graphically. Findings reveal that the energy of the system increases as the deformation parameter increases. This implies that the deformation could be used as a regulator or booster to manipulate the energy spectra of the system. The findings from this study will apply to atomic and molecular physics and chemical physics.*

Keywords: *Schrödinger Equation; Nikiforov-Uvarov method; q-deformed Modified Mobius Square Potential; Energy Spectra*

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1.0 Introduction

The Schrödinger equation (SE) is one of the most famous wave equations in Physics (Inyang et al., 2021a; Inyang et al., 2022; William et al., 2020). The reason is that the equation enables us to study the interactions of a particle in a quantum system in a non-relativistic regime (Akpan et al., 2021; Ntibi *et al.*, 2020; Inyang et al., 2021b). The equation is considered the central result in the study of quantum systems since it is endowed with the necessary information needed to describe a quantum system under consideration. The Schrödinger wave equation is one of the most powerful equations in modern Physics and Chemistry. In addition to these fields, the Schrödinger equation has applications in developing research areas such as quantum information (Ayedun et al., 2022; Inyang et al., 2023a) and Quantum finance (Baaquie, 2007). In quantum mechanics, the solutions to the SE for a given system can be obtained if we know the form of the underlying interaction potential. This goes to show in principle that potential plays a great role in physics and particularly quantum mechanics (Schiff,1955). A confining potential is, therefore, a mathematical representation of those forces that bind the particles of a system in a specific region. Confining potentials such as Coulomb, Kratzer, Cornell, and others have many forms depending upon the interaction of particles within the system and have been studied (Sameer & Majid, 2012; Thompson *et al.*, 2021; Inyang *et al.*, 2021c; Omugbe *et al.*,2023; Inyang et al., 2023b). A careful perusal of research works on the analytical solutions of the SE reveals that various forms of interaction potentials have extensively been employed to unravel the underlying dynamics of a variety of systems with varying degrees of success. Confining potentials play a vital role in studying a host of physical systems in diverse fields of physics like solid state physics, atomic and molecular physics, nuclear and particle physics and chemical physics (Okon et al., 2023; William et al., 2023; Omugbe et al., 2022; Inyang et al., 2022a; Edet et al., 2019; Edet et al., 2020). For example, the combination of harmonic and Coulomb potential has been used to bind the two electrons in a two-dimensional quantum dot (QD) region, the Kratzer and the pseudoharmonic potentials are considered as diatomic molecular potentials and the Cornell potential is extensively used as inter quark potential (Ibekwe *et al*., 2022a; Ibekwe *et al*., 2022b; Omugbe *et a*l., 2021; Ibekwe e*t al*., 2021).

Also, the analytic solutions of SE for a general potential can be proved an asset, as such general results may be utilized to obtain solutions for some variants of the general potential, which can act as suitable models for some physical systems. Therefore, the possibility of enhancing the domain of applications of these potentials is explored and some of its variants, particularly in QD systems, meson particles, and diatomic molecules produce good results that are comparable to results investigated via other alternative potentials and methods. Given the foregoing, we propose a model called q-deformed Modified Mobius Square Potential (MMSP) given as;

$$V\left(r\right)=-V_1\left(\frac{A+Be^{-2\delta r}}{1-qe^{-2\delta r}}\right)^2 \qquad (1)$$

where V_1, A, B, and δ are the depth of the potential, the range of the potential, the length of the molecular bond, and an adjustable screening parameter, respectively. q is the deformation parameter. The q-MMSP is a short-range potential it is a general case of the modified Mobius square potential. Hence, this work is an extension of the work of (Okorie et al., 2018).

Due to the need to expand our knowledge of solvable potentials and their possible utility in studying physical systems, in the present research, attention is focused on q-deformed potentials. The inclusion of "q" a term in a potential is generally desirable to improve the accuracy of theoretical predictions. q-deformed hyperbolic potentials, which were proposed by Arai more than two decades ago, destroy the symmetry of the system and consequently the symmetry of the solution. They present promising applications for modelling the atom-trapping potentials in Bose-Einstein condensates or vibrational spectra of diatomic molecules (Arai ,1991). We have seen that several authors who studied various quantum systems reported interesting results about the effect of the deformation parameter on the energy spectra, thermal properties, wave function etc. of such systems. Eğrifes *et al*.,2000 results aided in improving the accuracy of the deformed hyperbolic potentials. Peña *et al*,.2020 noted that depending on the choice of the parameters involved in the exponential type, several q-deformed exponential potentials were obtained as particular cases. This indicates that it is not necessary to use specialized methods to solve the SE for specific q-deformed exponential potentials. In this regard, to show the usefulness of the proposed method, the authors obtained the bound state solutions of the q-deformed Tietz, Hulthén, Manning-Rosen, Shioberg, quadratic exponential, Wei and Hua among other potential models. The proposal is useful when considering new q-deformed potentials with hypergeometric wavefunctions to be used in quantum chemical applications of diatomic molecules. In the study of (Abdalla et al., 2013), several observations were realized for each potential. However, for all the cases which they studied the oscillations are the usual ones. It is also noted that all of these functions are sensitive to variations in the parameters involved. Their discussion for some particular potential showed that for a large value of the parameter q the system tends to exhibit the energy for an infinite potential well between two points zero and in addition to a free particle. Falaye *et al.* 2012, discussed extensively the Hermiticity for the class of q-deformed potentials. Edet and Okoi, 2019 reported that the energy spectra of the q-deformed Hulthen plus generalized inverse quadratic Yukawa potential (HPGIQYP) was increased in the presence of $q > 0$. But decreased when $q < 0$. Seven special cases of the potential are discussed and the numerical energy eigenvalues are calculated for two values of the deformation parameter in different dimensions. Onate and Ojonubah 2016, applied the thermodynamic functions to study the behaviour of the zinc-blende crystal structure and the results obtained showed fair agreement with reported experimental data for the specific heat capacity. Min-Cang 2013 shows that the deformed hyperbolic Eckart potential is a shape-invariant potential and the bound state energy is independent of the deformation parameter q. Ovando *et al.*,2018 results showed that non-deformed potentials arise from non-q-dependent parameters and rather correspond to a class of q-shifted exponential-type potentials obtained from the Arai's q-deformed hyperbolic functions. As a second result, by using q-dependent parameters, it is possible to obtain true q-deformed exponential-type potentials. As a useful application of these potentials, some examples of q-deformed exponential-type potentials were considered using a proper selection of the involved parameters. Their proposal is general and can be viewed as a unified treatment to the study of q-deformed exponential-type potentials with the advantage that it is not necessary to use a specialized method for solving the Schrödinger equation for each specific potential because many of them are obtained as particular cases of the treated potential. Furthermore, new q-deformed potentials used as interesting alternatives in quantum chemical applications can be derived. Furthermore, the SE is very tedious or almost impossible to solve for physical systems for which $l \neq 0$, except a very few potentials viz; Coulomb, Harmonic, Mie and Pseudopotentials (Sameer & Majid, 2012). So, for the $s-$wave ($l = 0$ case), the SE can be readily solved (Ikhdair 2009). However, in cases where $l \neq 0$ belonging to the p, d, f, g etc. we resort to using an approximation scheme that best approximates the centrifugal term in the SE hence, leading to approximate bound state solutions (Ferreira & Prudente 2013). Moreover, long ago, physicists and chemists have developed different advanced mathematical methods to solve the SE ranging from numerical to analytical techniques. Interestingly, the solutions obtained are the same irrespective of the technique used except for some algebraic configurations. Some of the commonly used techniques in both relativistic and non-relativistic equations for different potentials are the following amongst many others; the Factorization method (Dong 2007), Asymptotic Iteration Method (AIM)(Edet et al., 2022), the Nikiforov-Uvarov (NU) method (Inyang *et al*., 2023c; Ntibi *et al*., 2022; Inyang & Obisung 2022b), among others (Inyang et al., 2020;Ikot et al., 2022; Okorie *et al*., 2020; Tezcan & Sever 2009; Falaye *et al*., 2015;Ma *et al*., 2006). In this work, we shall solve the approximate analytical solutions of Schrödinger equation with q-deformed modified Mobius square potential (MMPS) within the frame work of the Nikiforov-Uvarov method. The paper is structured as follows: A brief review of the Nikiforov–Uvarov method is discussed in Sect. 2. In Sect. 3, we present the solution to the radial part of the Schrodinger equation. In Sect. 4, discussions of numerical results. Finally, a brief conclusion is given in Sect. 5.

The Review of Nikiforov-Uvarov (NU) Method

The NU method (Nikiforov and Uvarov 1988) is, however, the reduction of a second-order linear differential equation to a generalized hyper-geometric type equation. The method gives the solution in terms of special orthogonal functions as well as corresponding energy eigenvalue. With appropriate coordinate transformation, $s = s(r)$, the equation is written as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{2}$$

To obtain an exact solution to eq. (2), we let the wave function be

$$\psi(s) = \phi(s)\chi(s) \tag{3}$$

By substituting eq. (3) into eq. (2), we obtain the hyper-geometric equation

$$\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0 \tag{4}$$

The wave function is given as

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{5}$$

For a fixed n, the hyper-geometric type function $\chi(s)$ is expressed in Rodrigues relation as

$$\chi_n(s) = \frac{B_n(s)}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \tag{6}$$

where B_n is the normalization constant and $\rho(s)$ the weight function which satisfies the condition below;

$$\frac{d}{ds}\left(\sigma(s)\rho(s)\right) = \tau(s)\rho(s) \tag{7}$$

Where also

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{8}$$

For bound solutions, it is required that

$$\frac{d\tau(s)}{ds} < 0 \tag{9}$$

Therefore, the function $\pi(s)$ and the parameter λ required for the NU method are defined as

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{10}$$

$$\lambda = k + \pi'(s) \tag{11}$$

The values k are obtained if the discriminant in the square root of eq. (10) vanish, so the new eigen equation becomes

$$\lambda_n = -\frac{nd\tau(s)}{ds} - \frac{n(n-1)}{2}\frac{d^2\sigma(s)}{ds^2} \tag{12}$$

$$n = 0, 1, 2, -$$

By equating eq. (11) and eq. (12), the energy eigenvalue is obtained

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2.0 Solution of Schrodinger Equation with q-deformed Modified Mobius Square Potential (q-MMSP)

The Schrödinger equation is given by (Inyang et al., 2023):

$$-\frac{\hbar^2}{2m}\nabla^2\psi\left(\vec{r},t\right)+V\left(r,t\right)\psi\left(\vec{r},t\right)=i\hbar\frac{\partial\psi\left(\vec{r},t\right)}{\partial t} \tag{13}$$

where m is the mass of the particle, ∇^2 is the Laplacian operator and $V\left(r,t\right)=V\left(r\right)$ is q-MMSP. Thereby, the Schrödinger equation for the q-MMSP (13) is

$$-\frac{\hbar^2}{2mr^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi\left(r,\theta,\varphi,t\right)+V\psi\left(r,\theta,\varphi,t\right)=i\hbar\frac{\partial\psi\left(r,\theta,\varphi,t\right)}{\partial t} \tag{14}$$

In what follows, let us consider a particular solution to Eq. (14) given in terms of the eigenvalues of the angular momentum operator $\hat{L}^2$ as

$$\psi\left(r,\theta,\varphi,t\right)=e^{-i\frac{Et}{\hbar}}R\left(r\right)Y_{\ell,m}\left(\theta,\varphi\right) \tag{15}$$

where $Y_{\ell,m}\left(\theta,\varphi\right)$ are spherical harmonics and $R\left(r\right)$ is the radial wave function. Then, by substituting Eq. (15) into Eq. (14), we obtain the radial wave equation

$$\frac{d^2R\left(r\right)}{dr^2}+\frac{2}{r}\frac{dR\left(r\right)}{dr}+\left[\frac{2m}{\hbar^2}\left(E+V_1\left(\frac{A+Be^{-2\delta r}}{1-qe^{-2\delta r}}\right)^2\right)-\frac{\ell\left(\ell+1\right)}{r^2}\right]R\left(r\right)=0 \tag{16}$$

Let $R\left(r\right)=\frac{U\left(r\right)}{r}$ (17)

This helps us rewrite eq. (16) as;

$$\frac{d^2U\left(r\right)}{dr^2}+\left[\frac{2m}{\hbar^2}\left(E+V_1\left(\frac{A+Be^{-2\delta r}}{1-qe^{-2\delta r}}\right)^2\right)-\frac{\ell\left(\ell+1\right)}{r^2}\right]U\left(r\right)=0 \tag{18}$$

Expanding the potential term in (18), we rewrite it as follows;

$$\frac{d^2U\left(r\right)}{dr^2}+\left[\frac{2m}{\hbar^2}\left(E+\frac{V_1A^2}{\left(1-qe^{-2\delta r}\right)^2}+\frac{2V_1ABe^{-2\delta r}}{\left(1-qe^{-2\delta r}\right)^2}+\frac{V_1B^2e^{-4\delta r}}{\left(1-qe^{-2\delta r}\right)^2}\right)-\frac{\ell\left(\ell+1\right)}{r^2}\right]U\left(r\right)=0 \tag{19}$$

Using the approximation of (Greene & Aldrich 1976);

$$\frac{1}{r^2}=\frac{4\delta^2e^{-2\delta r}}{\left(1-qe^{-2\delta r}\right)^2} \tag{20}$$

Using eq. (20) , we rewrite (19) as follows;

$$\frac{d^2U\left(r\right)}{dr^2}+\left[\frac{2m}{\hbar^2}\left(E+\frac{V_1A^2}{\left(1-qe^{-2\delta r}\right)^2}+\frac{2V_1ABe^{-2\delta r}}{\left(1-qe^{-2\delta r}\right)^2}+\frac{V_1B^2e^{-4\delta r}}{\left(1-qe^{-2\delta r}\right)^2}\right)-\left(\frac{\ell\left(\ell+1\right)4\delta^2e^{-2\delta r}}{\left(1-qe^{-2\delta r}\right)^2}\right)\right]U\left(r\right)=0 \tag{21}$$

By change of coordinates,

$$s=e^{-2\delta r}. \tag{22}$$

The following transformation takes place

$$\frac{d^2U(s)}{ds^2}+\frac{(1-qs)}{s(1-qs)}\frac{dU(s)}{ds}+\frac{1}{s^2(1-qs)^2}\left[-\left(\varepsilon q^2-\beta_2\right)s^2+\left(2\varepsilon q+\beta_1-\eta\right)s-\left(\varepsilon-\beta_0\right)\right]U(s)=0 \quad (23)$$

where we have used the following dimensionless notations for mathematical simplicity:

$$-\varepsilon=\frac{mE_{n\ell}}{2\hbar^2\delta^2};\beta_0=\frac{mV_1A^2}{2\hbar^2\delta^2};\beta_1=\frac{mV_1AB}{\hbar^2\delta^2};\beta_2=\frac{mV_1B^2}{2\hbar^2\delta^2};\eta=\ell(\ell+1) \quad (24)$$

Comparing (23) with the hypergeometric equation of equation (2), we obtain the following polynomials:

$$\tilde{\tau}=(1-qs),\quad \sigma(s)=s(1-qs),\quad \sigma^2(s)=s^2(1-qs)^2 \quad (25)$$

The polynomial $\pi(s)$ is given by,

$$\pi(s)=-\frac{qs}{2}\pm\sqrt{\left(\frac{q^2}{4}+\varepsilon q^2-\beta_2-kq\right)s^2+\left(k-\left(2\varepsilon q+\beta_1-\eta\right)\right)s+\left(\varepsilon-\beta_0\right)} \quad (26)$$

To find the expression for k, the discriminant of (26) is equated to zero. Thus we obtain,

$$k=-(\eta-\beta_1-2\beta_0q)\pm\sqrt{(\varepsilon-\beta_0)}\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q} \quad (27)$$

The substituting k in $\pi(s)$ in equation (26),

$$\pi(s)=-\frac{qs}{2}\pm\left(\left(\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}+q\sqrt{(\varepsilon-\beta_0)}\right)s-\sqrt{(\varepsilon-\beta_0)}\right) \quad (28)$$

Taking the negative value $\pi(s)$ in equation (28) to obtain,

$$\pi'(s)=-\frac{q}{2}-\left(\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}+q\sqrt{(\varepsilon-\beta_0)}\right) \quad (29)$$

To obtain the polynomial $\tau(s)$, we use $\tau(s)=\tilde{\tau}(s)+2\pi(s)$

$$\tau(s)=1-2qs-2\left(\left(\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}+q\sqrt{(\varepsilon-\beta_0)}\right)s-\sqrt{(\varepsilon-\beta_0)}\right) \quad (30)$$

The derivative of $\tau(s)$ equation (30),

$$\tau'(s)=-2\left(q+\left(\left(\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}+q\sqrt{(\varepsilon-\beta_0)}\right)s-\sqrt{(\varepsilon-\beta_0)}\right)\right)<0 \quad (31)$$

The parameter λ is defined as,

$$\lambda=-(\eta-\beta_1-2\beta_0q)-\sqrt{(\varepsilon-\beta_0)}\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}-\frac{q}{2}-\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q} \quad (32)$$
$$-q\sqrt{(\varepsilon-\beta_0)}$$

λ_n is expressed as,

$$\lambda_n=n^2q+nq+2n\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}-2nq\sqrt{(\varepsilon-\beta_0)}-2n\sqrt{(\varepsilon-\beta_0)} \quad (33)$$

The eigenvalue expression holds if

$$\lambda = \lambda_n . \tag{34}$$

$$\varepsilon = \beta_0 + \frac{1}{4}\left(\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\beta_2}{q^2}-\frac{\beta_1}{q}-\beta_0+\frac{\eta}{q}}\right)^2-\beta_0+\frac{\beta_2}{q^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\beta_2}{q^2}-\frac{\beta_1}{q}-\beta_0+\frac{\eta}{q}}\right)}\right)^2 \tag{35}$$

Substituting equation (24) into (35) and evaluating, we obtain the energy as follows:

$$E_{n\ell} = -V_1A^2 - \frac{\hbar^2\delta^2}{2m}\left(\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{mV_1B^2}{2\hbar^2\delta^2q^2}-\frac{mV_1AB}{\hbar^2\delta^2q}-\frac{mV_1A^2}{2\hbar^2\delta^2}+\frac{\ell(\ell+1)}{q}}\right)^2 - \frac{mV_1A^2}{2\hbar^2\delta^2}+\frac{mV_1B^2}{2\hbar^2\delta^2q^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{mV_1B^2}{2\hbar^2\delta^2q^2}-\frac{mV_1AB}{\hbar^2\delta^2q}-\frac{mV_1A^2}{2\hbar^2\delta^2}+\frac{\ell(\ell+1)}{q}}\right)}\right)^2 \tag{36}$$

To find the eigenfunction, the weight function is first evaluated. From equation (4),

$$\frac{d\phi(s)}{\phi(s)} = \left(\frac{\sqrt{(\varepsilon-\beta_0)}(1-qs) - s\left(\frac{1}{2}+\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}\right)}{s(1-qs)}\right)ds \tag{37}$$

Integrating equation (37), we obtain

$$\phi(s) = s^{\sqrt{(\varepsilon-\beta_0)}}(1-qs)^{\frac{1}{q}\left(\frac{q}{2}+\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}\right)} \tag{38}$$

From equation (7),

$$\frac{d\rho(s)}{\rho(s)} = \left(\frac{2\sqrt{(\varepsilon-\beta_0)}(1-qs) - 2s\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}}{s(1-qs)}\right)ds \tag{39}$$

Integrating equation (39), we obtain

$$\rho(s) = s^{2\sqrt{(\varepsilon-\beta_0)}}(1-qs)^{\frac{2}{q}\sqrt{\frac{q^2}{4}-\beta_2-\beta_1q-\beta_0q^2+\eta q}} \tag{40}$$

Recall $\chi(s)$ is expressed in Rodrigues relation (5) and using (42), we obtain

$$\chi_{n}\left(s\right)=B_{n}\left(s\right)s^{-2\sqrt{(\varepsilon-\beta_0)}}\left(1-qs\right)^{-\frac{2}{q}\sqrt{\frac{q^2}{4}-\beta_2-\beta_1 q-\beta_0 q^2+\eta q}}\frac{d^n}{ds^n}\left[s^{n+2\sqrt{(\varepsilon-\beta_0)}}\left(1-qs\right)^{n+\frac{2}{q}\sqrt{\frac{q^2}{4}-\beta_2-\beta_1 q-\beta_0 q^2+\eta q}}\right] \tag{41}$$

$$=P_n^{\left(2\sqrt{(\varepsilon-\beta_0)},\frac{2}{q}\sqrt{\frac{q^2}{4}-\beta_2-\beta_1 q-\beta_0 q^2+\eta q}\right)}\left(1-2qs\right)$$

In terms of Jacobi Polynomial, the complete wave function of the q-deformed Mobuis Square potential is given as,

$$\psi_{n\ell}\left(s\right)=B_{n\ell}s^{\sqrt{(\varepsilon-\beta_0)}}\left(1-qs\right)^{\frac{1}{q}\left(\frac{q}{2}+\sqrt{\frac{q^2}{4}-\beta_2-\beta_1 q-\beta_0 q^2+\eta q}\right)}P_n^{\left(2\sqrt{(\varepsilon-\beta_0)},\frac{2}{q}\sqrt{\frac{q^2}{4}-\beta_2-\beta_1 q-\beta_0 q^2+\eta q}\right)}\left(1-2qs\right) \tag{42}$$

3.0 Results and Discussion

We investigated the effects of the deformation parameter on the energy eigenvalues of the q-deformed Modified Mobius Square potential.

Table 1, shows the numerical results of energy for 1s state for various values of screening parameter (δ) with $\hbar=\mu=A=1$, $V_1=0.2$ and $B=-2$. It is seen that the energy decreases as the screening and deformation parameters increase.

In the numerical results of energy of the q deformed-MMSP for 2s state for various screening parameter (δ) with $\hbar=\mu=A=1$, $V_1=0.2$ and $B=-2$, it is seen that the energy decreases as the screening and deformation parameters increases. In the numerical results of energy of the q deformed-MMSP for 2p state for various values of screening parameter (δ) with $\hbar=\mu=A=1$, $V_1=0.2$ $B=-2$, it is seen that the energy increases as the deformation and screening parameters increase. It is seen in that the numerical results of Energy for 3s state for various values of screening parameter (δ) with $\hbar=\mu=A=1$, $V_1=0.2$ and $B=-2$, that the energy decreases as the screening and deformation parameters increases. We show numerical results of energy of the q deformed-MMSP for 3p state for various values of screening parameter (δ) with $\hbar=\mu=A=1$, $V_1=0.2$ and $B=-2$. Here, the energy decreases with increasing deformation parameters. Also, the numerical results of energy of the q deformed-MMSP for 3d state for various values of screening parameter (δ) with $\hbar=\mu=A=1$, $V_1=0.2$ and $B=-2$, we see that the energy decreases with increasing screening parameter (δ).The energy tends to increase as the deformation parameter increases but drops as $q\to 1$. In Figure 1, we plot the energy spectra q deformed-MMSP versus V_1 when $q=1$. We see that the energy increases as the potential parameter V_1 increases. In Figure 2, we plot the energy spectra q deformed-MMSP against V_1 when $q=-1$. We notice that the energy decreases as the potential parameter V_1 increases. In Figure 3, we plot the energy spectra of the q deformed-MMSP with the parameter A when $q=1$. It was observed here that the energy decreases with increasing A. In Figure 4, we plot the energy spectra q deformed-MMSP versus A when $q=-1$. We see that the energy increases with rising A. In Figure 5, we show the plot of the energy spectra of q deformed-MMSP versus B when $q=1$. Here, the energy increases with increasing B. In Figure 6, the energy spectra of q deformed-MMSP versus B when $q=-1$ is plotted. It is seen that the energy decreases and comes to a minimum in the neighbourhood of $B=1$ and increases again immediately. In Figure 7, we show here the energy spectra of q deformed-MMSP versus δ when $q=1$. Again, it is seen that the energy decreases as the parameter increases

Table 1: Numerical results of Energy spectra for several quantum state for various values of screening parameter (δ) with $\hbar = \mu = A = 1$, $V_1 = 0.2$ and $B = -2$

State	δ	$q=-0.5$	$q=0.5$	$q=1$
1s	0.025	-0.000213	-0.000590	-0.001563
	0.050	-0.000850	-0.002361	-0.006250
	0.075	-0.001913	-0.005313	-0.014063
	0.100	-0.003400	-0.009444	-0.025000
	0.125	-0.005313	-0.014757	-0.039063
	0.150	-0.007650	-0.021250	-0.056250
2s	0.025	-0.001512	-0.006410	-0.018683
	0.050	-0.006040	-0.025484	-0.070472
	0.075	-0.013563	-0.056757	-0.144571
	0.100	-0.024043	-0.099486	-0.228125
	0.125	-0.037433	-0.152690	-0.310348
	0.150	-0.053668	-0.215200	-0.384486
2p	0.025	0.00058806	0.00163920	0.00347288
	0.050	0.00235896	0.00664460	0.01432650
	0.075	0.00533262	0.01528590	0.03394280
	0.100	0.00954151	0.02803410	0.06484380
	0.125	0.01502970	0.04559670	0.11063400
	0.150	0.02185120	0.06897380	0.17270100
3s	0.025	-0.0041073	-0.0179788	-0.0509571
	0.050	-0.0163670	-0.0706082	-0.1731530
	0.075	-0.0365967	-0.1541860	-0.3087040
	0.100	-0.0645045	-0.2632130	-0.4209770
	0.125	-0.0997071	-0.3912050	-0.5026540
	0.150	-0.1417510	-0.5315370	-0.5608120
3p	0.025	-0.000710	-0.004210	-0.014063
	0.050	-0.002810	-0.016942	-0.056250
	0.075	-0.006211	-0.038510	-0.126563
	0.100	-0.010772	-0.069464	-0.225000
	0.125	-0.016304	-0.110625	-0.351563
	0.150	-0.022585	-0.163153	-0.506250
3d	0.025	0.002192	0.006157	0.013983
	0.050	0.008819	0.025621	0.063501
	0.075	0.020033	0.061630	0.182128
	0.100	0.036070	0.120562	0.501875
	0.125	0.057232	0.213978	-2.300780
	0.150	0.083863	0.363202	-1.126600

.

In Figure 8, we show here the energy spectra of q deformed-MMSP versus δ when $q=-1$. Again, the same trend is noticed. In Figure 9, we show the energy spectra of q deformed-MMSP versus q when $q<0$. The energy decreases as the deformation parameter increases

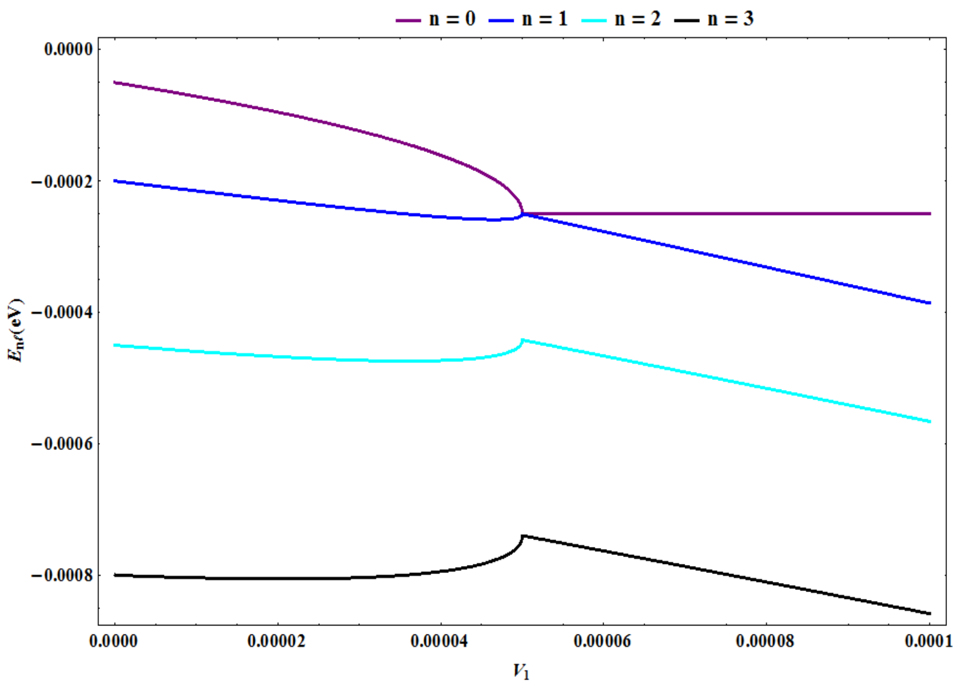

Fig. 1: Energy spectra versus potential strength V_1 when $q=1$.

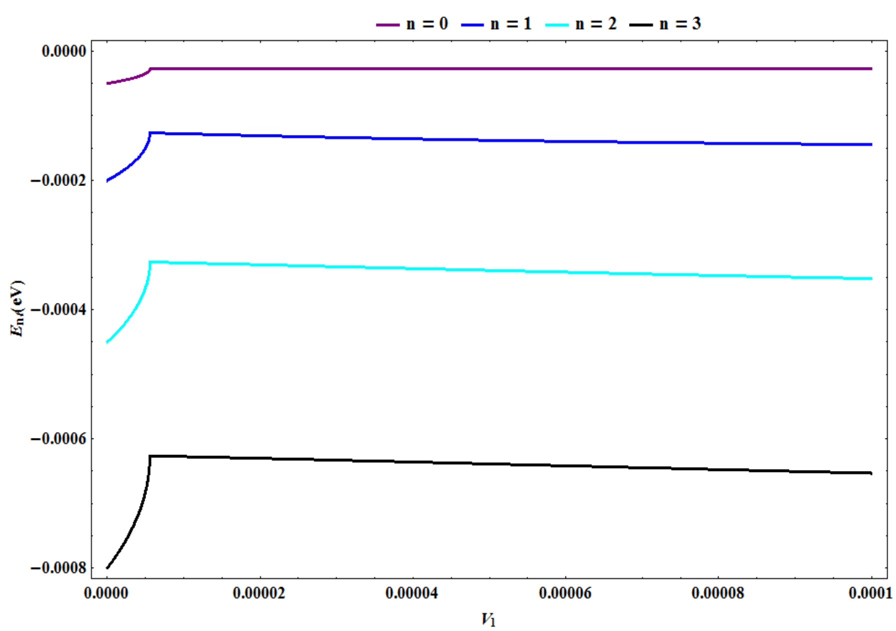

Fig. 2: Energy spectra versus potential strength V_1 when $q=-1$

.

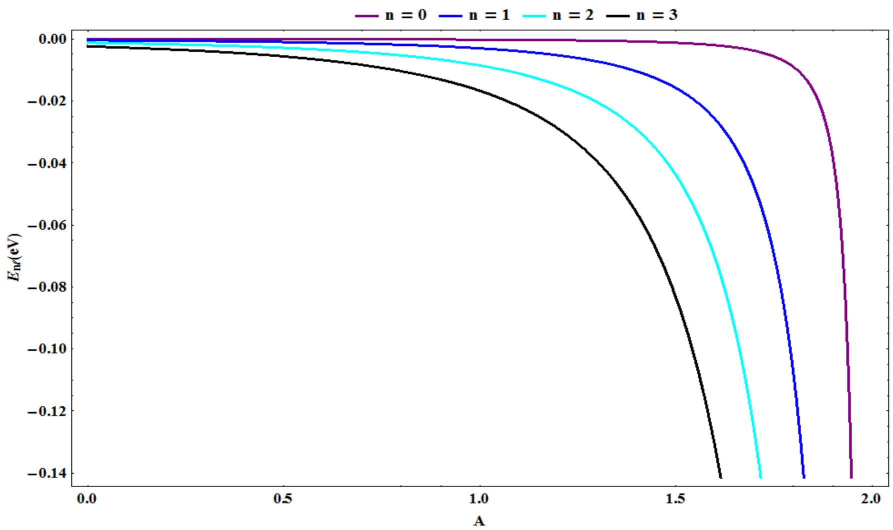

Fig. 3: Energy spectra versus potential strength A **when** $q=1$

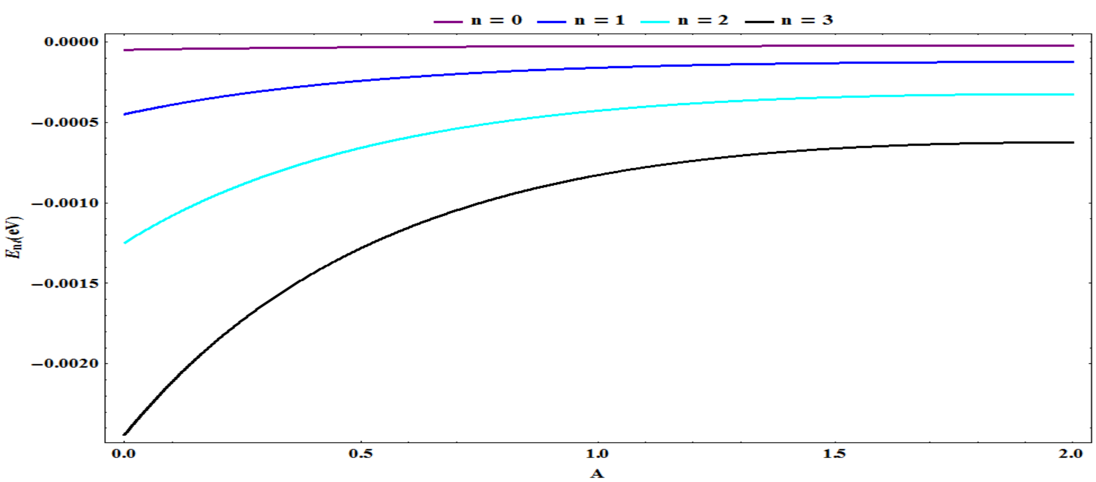

Fig. 4: Energy spectra versus potential strength A **when** $q=-1$

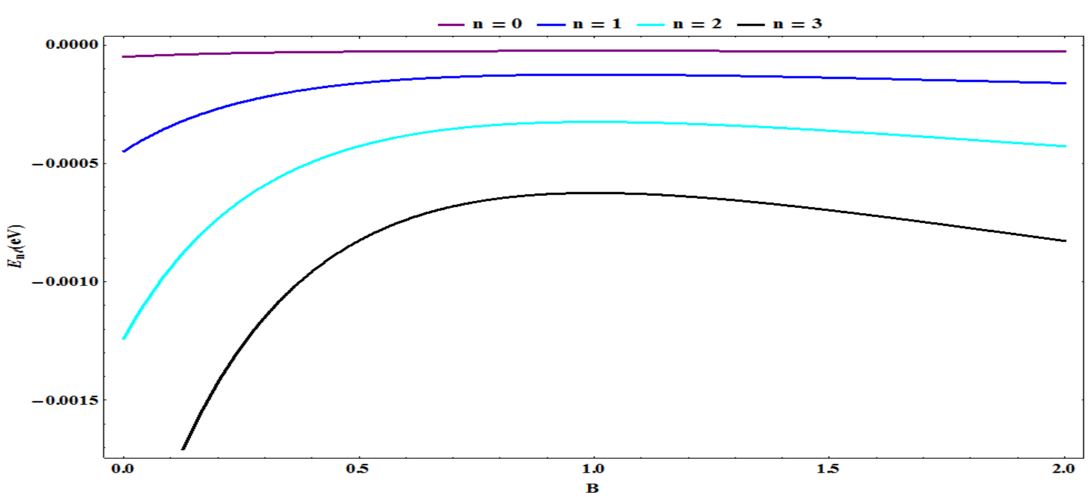

Fig. 5: Energy spectra versus potential strength B **when** $q=1$

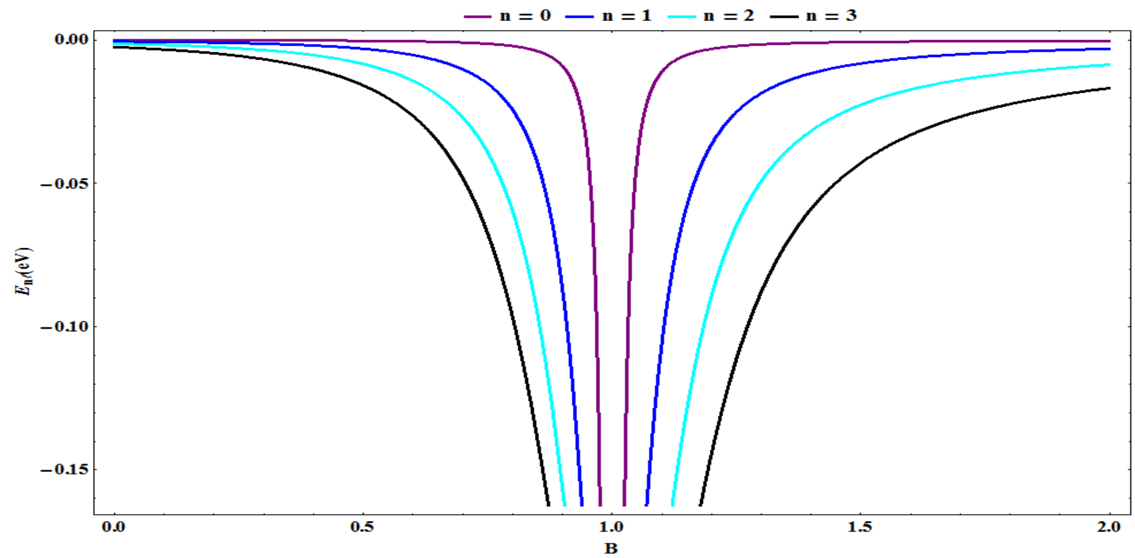

Fig. 6: Energy spectra versus potential strength B **when** $q=-1$

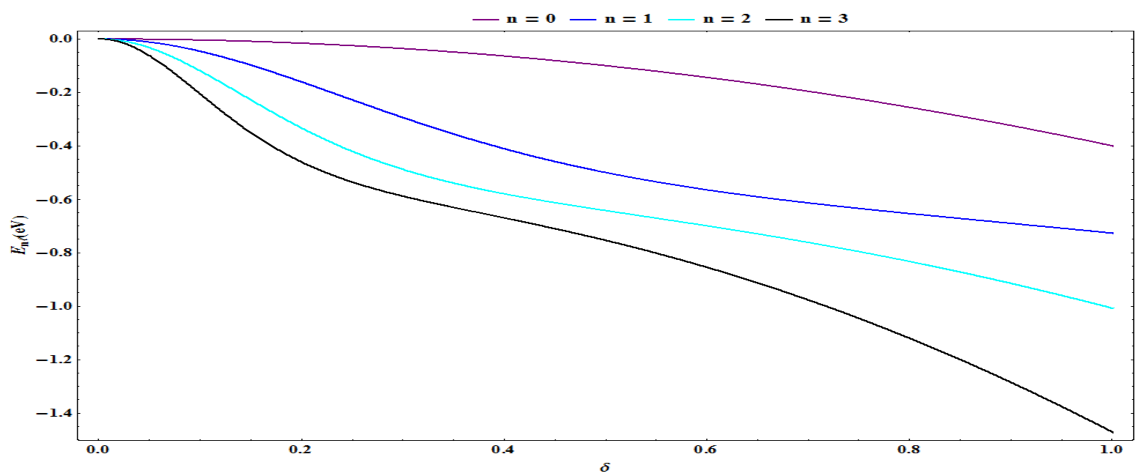

Fig. 7: Energy spectra versus screening parameter δ **when** $q=1$

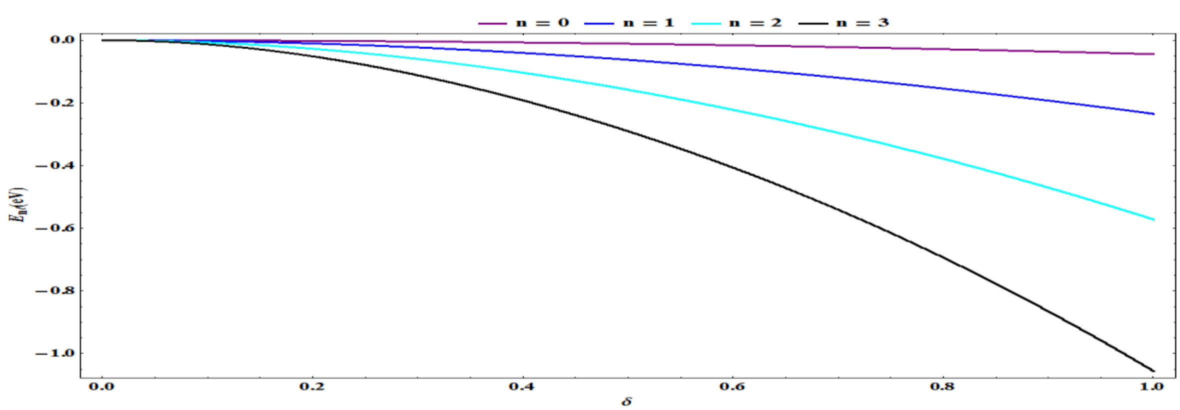

Fig. 8: Energy spectra versus screening parameter δ **when** $q=-1$

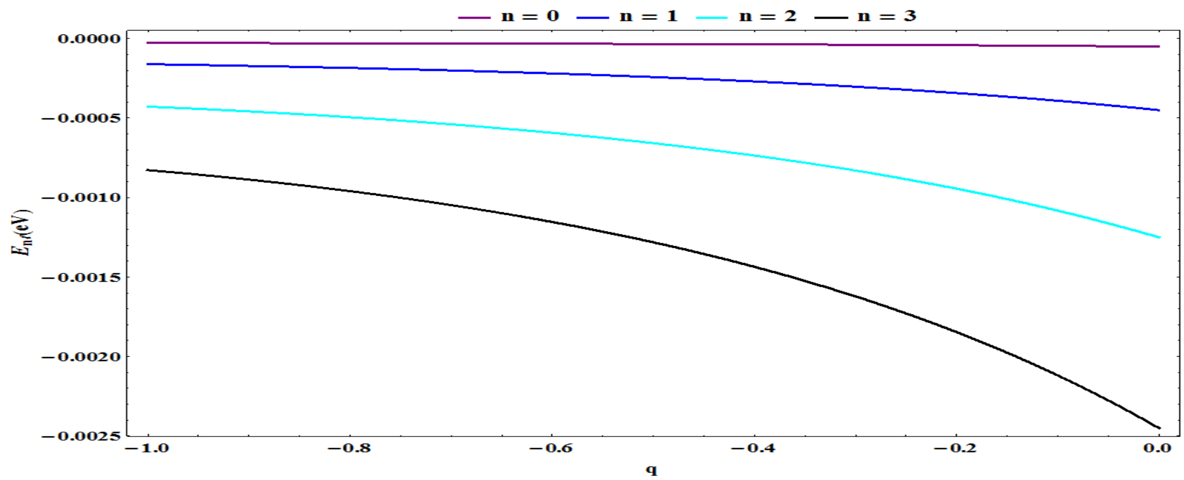

Fig. 9: Energy spectra versus q when $q<0$

4.0 Conclusion

In work, we solve the Schrodinger equation with the q-deformed Modified Mobuis Square potential using the Nikiforov-Uvarov method to obtain the wave function and energy of the system respectively. The energy spectra are numerically and graphically analysed to show the effects of the potential and deformation parameters on the energy spectra of the system. These findings on this system would provide a much broader understanding of molecular interactions.

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Declarations

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data Public.

Competing interests

The authors declared no conflict of interest.

Funding

The authors declared no source of funding

Authors' contributions

All the authors contributed to the conclusion of the manuscript including calculations and interpretation.