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On the Study of Kumaraswamy Reduced Kies Distribution: Properties and Applications

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Abstract: *Unit-bounded distributions play a crucial role in probability and statistics for modeling quantities that are strictly confined between 0 and 1, such as rates, ratios, proportions, and percentages. Despite their importance, these distributions are relatively scarce compared to those with unbounded support, even though many real-world phenomena involve data restricted to a unit interval, including proportions, percentages, ratios, rates, and fractions. Some unit distributions arise naturally from analytical derivations, while others emerge through generalization from distributions originally defined over broader domains. This study introduces a threstay e-parameter unit-bounded distribution, termed the Kumaraswamy Reduced Kies Distribution, developed through a generalization process of Kumaraswamy G-family of distribution based on the function of functions approach applied to the Reduced Kies Distribution proposed. The Kumaraswamy Reduced Kies Distribution, a flexible three-parameter distribution with semi-bounded support, serves as the foundation for extending this adaptability to the unit interval. The probability density function of the proposed distribution exhibits a variety of shapes, including J, reversed-J, left-skewed, symmetric, and bathtub unimodal forms. Additionally, its hazard rate function follows a monotonically non-decreasing pattern. Several statistical properties and reliability measures are examined, including the survival function, hazard rate function, cumulative hazard function, reversed hazard function, odd function, quantile function, median, skewness, kurtosis, and order statistics. The estimation of model parameters is performed using Maximum Likelihood Estimation, Maximum Product of Spacing, and Cramer-von Mises methods. Monte Carlo simulations are conducted to assess the effectiveness of these estimation techniques, demonstrating that Biases, Mean Squared Errors, and Mean Relative Errors decrease as the sample size increases. Finally, the practical applicability of the proposed model is illustrated using two real-life datasets. A comparative analysis confirms that the proposed model achieves a superior fit compared to several existing models.*

Keywords: *Reduced Kies Distribution, Reliability analysis, Simulation study, semi-bounded Distribution.*

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1. 0 Introduction

The study of statistical distributions is fundamental in modeling real-world data, particularly when dealing with datasets that exhibit complex characteristics such as skewness, heavy tails, and bounded support. Classical distributions like the Normal, Gamma, Weibull, and Beta distributions are widely used due to their well-established theoretical properties and computational simplicity. However, these distributions often fail to adequately capture the behavior of highly skewed or asymmetric data. To overcome these limitations, researchers have introduced flexible distributions by modifying existing models through parameter expansion or generalization techniques, allowing for greater adaptability in diverse applications (Johnson, 2004).

A notable approach in statistical modeling involves developing reduced-parameter distributions that retain high flexibility while minimizing computational complexity. One such example is the Kumaraswamy distribution, a two-parameter alternative to the Beta distribution, widely applied in fields such as hydrology and reliability analysis due to its closed-form cumulative distribution function (Kumaraswamy, 1980). The cumulative distribution function (CDF) and probability density function (PDF) of the Kumaraswamy distribution are given by:

$F(x;a,b) = 1-(1-x^{a})^{b},\ \ 0<x<1$ (1)

$F(x;a,b) =\ abx^{a-1}(1-x^{a})^{b-1}, 0<x<1$ (2)

where a, b ? 0 are shape parameters that control the distribution's skewness and kurtosis. Recently, there has been increased interest in developing generalized families of univariate continuous distributions by incorporating additional parameters into existing models (Afify et al., 2022). The Reduced Kies distribution (RKiD), introduced by Kumar and Dharmaja (2013), is an example of such a modification. It is particularly useful for modeling heavy-tailed and skewed datasets. The CDF and PDF of the RKiD are given by:

$F(x\colon\lambda) = 1-e^{-\lambda x}, x>0$ (3)

$F(x\colon\lambda) =\ \lambda e^{-\lambda x}, x>0$ (4)

Where $\lambda > 0$ is a shape parameter. Several modifications and extensions of the Reduced Kies distribution have been proposed to improve its applicability. Notable contributions include the Kies Cumulative Distribution Function (Kyurkchiev, 2024), the Modified Kies-Frechet Distribution (Alsubie, 2021), and the Exponentiated Reduced Kies Distribution (Kumar & Dharmaja, 2016). Despite these advancements, existing models primarily focus on semi-bounded distributions, making them less suitable for strictly unit-bounded data.

To address this limitation, we propose a new three-parameter unit-bounded distribution called the Kumaraswamy Reduced Kies Distribution (KuRKiD). This distribution is derived by applying the Kumaraswamy transformation to the Reduced Kies Distribution, enhancing its flexibility while ensuring that the support remains within the unit interval . The proposed KuRKiD offers a wide range of shapes, including J-shaped, reversed J-shaped, bathtub-shaped hazard functions, and symmetric distributions. The CDF and PDF of the proposed KuRKiD are formulated as follows:

$F(x;a,b\ \ \lambda) = 1-\left(1-e^{-\lambda x}\right)^{a})^{b},\ 0<x<1$ (5)

$F(x;a,b\ \ \lambda) =\ ab\lambda e^{-\lambda x}(1-e^{-\lambda x})^{a-1}(1-(1-e^{-\lambda z})^{b-1}, 0>x<1$ (6)

where are shape parameters. A, b, $\lambda > 0$ are shape parameters. This study aims to establish the theoretical properties of the KuRKiD, including its moments, quantile function, skewness, kurtosis, and reliability measures such as survival and hazard functions. Parameter estimation techniques, including Maximum Likelihood Estimation (MLE), Maximum Product of Spacing (MPS), and Cramer-von Mises methods, are utilized to evaluate the efficiency of the model. Monte Carlo simulations are conducted to assess the performance of these estimation methods, demonstrating that bias, mean squared error, and mean relative error decrease as sample size increases.

Finally, the practical applicability of the proposed model is illustrated using two real-world datasets. Comparative analysis confirms that the KuRKiD provides a superior fit compared to several existing models, demonstrating its potential in statistical modeling applications.

The remainder of this paper is organized as follows: Section 2 presents the development of the proposed KuRKiD. Basic distributional properties are discussed in Section 3. Section 4 details the estimation procedures. Section 5 provides simulation results for different estimation scenarios. Section 6 illustrates numerical applications on real datasets. Finally, Section 7 concludes with key findings and future research directions.

The study of statistical distributions has always been pivotal in modeling real-life data, especially when handling datasets with complex characteristics such as skewness and heavy tails. Classical distributions like the Normal, Gamma, Weibull, and Beta have been widely used due to their simplicity and analytical properties. However, these distributions often fall short when dealing with highly skewed or asymmetric data. To address these limitations, researchers have developed more flexible distributions by modifying existing ones, either by introducing additional parameters or through generalized forms that enhance their ability to model a broader range of data behaviors **(**Johnson, 2004)**.**

One notable approach to improving the flexibility of distributions is reducing the number of parameters in existing models while maintaining or enhancing their capacity to handle skewness and other non-standard features. Such modifications aim to strike a balance between simplicity and adaptability, making the models more computationally efficient while retaining their ability to represent diverse data patterns. For example, the Kumaraswamy distribution, a two-parameter alternative to the Beta distribution, has been successfully applied in hydrology and reliability analysis due to its closed-form cumulative distribution function (cdf), which simplifies parameter estimation and simulation (Kumaraswamy, 1980).

The interest in developing more flexible statistical distributions that models diverse datasets remains strong nowadays. Many generalized distributions have been developed over the past decades for modeling data in several areas such as biological studies, environmental sciences, economics, engineering, finance and medical sciences. Recently, there has been an increased interest in defining new generated families of univariate continuous probability distributions by introducing additional parameter(s) to the baseline model Afify *et al* (2022).

The Reduced Kies distribution (RKiD) proposed by Kumar and Dharmaja (2013) with the interval (0, 1) is a notable distribution introduced to address specific challenges in statistical modeling, particularly for datasets exhibiting heavy tails and skewness (Kies, 1958). The cumulative distribution function (cdf) and probability density function (pdf) of the distribution are respectively given by equations 1 and 2. where α is a shape parameter. Some of these noteworthy contributions on reduced kies can be found in the literature, such as the, Kies Cumulative Distribution Function: Reaction Network Analysis and Related

Problems by Kyurkchiev (2024). A new unit distribution: properties, inference, and applications Afify *et al* (2024), The modified Kies-Frechet distribution: Properties, inference and application by Alsubie (2021), Different estimation methods of the modified Kies Topp-Leone model with applications and quantile regression by Safar (2024),

$$G(x)=1-\exp\left[-\left(\frac{x}{1-x}\right)^{\alpha}\right], \quad \alpha>0,\ 0<x<1; \tag{1}$$

$$g(x)=\alpha x^{\alpha-1}(1-x)^{-\alpha-1}\exp\left[-\left(\frac{x}{1-x}\right)^{\alpha}\right], \tag{2}$$

A Flexible Extension of Reduced Kies Distribution: Properties, Inference, and Applications in Biology by Almuqrin *et al* (2022), The Exponentiated Reduced Kies distribution Properties and Applications by Kumar & Dharmaja (2016) and Inference for the Two Parameter Reduced Kies Distribution under Progressive Type-II Censoring by Mansour et al (2020).

While Kumaraswamy Distribution, introduced by Kumaraswamy (1980), is a flexible probability distribution that has found wide application in various fields due to its ability to model bounded data effectively. This distribution is defined on the interval [0, 1], making it particularly useful in scenarios where data is naturally restricted, such as proportions and probabilities.

Over the years, the Kumaraswamy Distribution has been extended and generalized to address specific challenges in statistical modeling, leading to its incorporation into numerous studies (Nadarajah & Kotz, 2006).

The Kumaraswamy-G family of distributions was developed by Ferreira & Cordeiro (2024) with cdf and pdf respectively defined according to equation 3 and 4.

$$F(x)=1-\left[1-G(x)^{\theta}\right]^{\lambda}, \quad \theta>0,\lambda>0,\ 0<x<1; \tag{3}$$

$$f(x)=\theta\lambda g(x)G(x)^{\theta-1}\left[1-G(x)^{\theta}\right]^{\lambda-1}, \tag{4}$$

where θ and λ are two additional parameters whose role is to introduce skewness and to vary tail weights Cordeiro and de Castro (2011) ; also *g(x)* and *G(x)* are the pdf and cdf of a baseline random variable *X*.

Some extensions and modifications of Kumaraswamy G-family of distribution can also be found in the literature such as; The Generalized Kumaraswamy-G Family of Distributions by Zohdy *et al* (2019), Kumaraswamy Type I Half Logistic Family of Distributions with Applications by El-sherpieny & Elsehetry (2019), The Kumaraswamy Transmuted-G Family of Distributions: Properties and Applications by Afify (2016), Theoretical Analysis of the Exponentiated Transmuted Kumaraswamy Distribution with Application by Mohamed (2019), The Kumaraswamy-G Poisson Family of Distributions by Manoel *et al* (2015), New properties of the Kumaraswamy Distribution by Mitnik (2013), A new family of generalized distributions by Gauss (2009) and Kumaraswamy inverse Gompertz distribution: Properties and engineering applications to complete, type-IIright censored and upper record data by El- morshedy *et al* (2020).

In general, as mentioned earlier,RKiD is flexible and often applied in various fields. However, the Reduced Kies and Kumaraswamy G-family distributions, as well as their extensions mentioned above are useful for modelling certain types of data, they typically work best for semi-bounded data, but are not suitable for modelling skewed unit-bounded data. The primary goal of this study is to propose a new unit-bounded distribution called Development of Kumaraswamy Reduced Kies Distribution which is an extension or modification of the Reduced Kies Distribution. It introduces the Kumaraswamy Distribution as an additional layer, capable of improving the flexibility of the baseline distribution in modeling a wide range of shapes which include J-shape, reversed J-shape, bathtub-shaped hazard functions, negatively skewed and symmetrical datasets. However, the theoretical properties and practical implications of this improved distribution remain unexplored, necessitating a detailed study to establish its efficacy, broaden its applicability, and provide solutions to the shortcomings of the Kies distribution in statistical modeling.

The rest of the paper is organized as follows: Section 2 contains the development of the proposed KuRKiD. Some basic distributional properties are discussed in Section 3. Section 4 highlights the procedures of the different estimation methods to estimate its unknown parameters. In Section 5, simulation studies based on three different scenarios are given to see the performance of the different estimators of the model parameters. Section 6 illustrates the numerical application of the developed model on two real dataset. Finally, some concluding remarks are provided in Section 7

2.0 Kumaraswamy Reduced Kies Distribution

In order to develop the cumulative density function for the proposed Kumaraswamy reduced Kies distribution, the function of functions approach is employed, by substituting (1) into (3).

$$F_{KuRKiD}\left(x;\alpha,\theta,\lambda\right)=1-\left[1-\left(1-e^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda} \quad \alpha,\theta,\lambda>0,\ 0<x<1; \qquad 5$$

where α, θ and λ are shape parameters. The corresponding pdf will be obtained by substituting (1) and (2) into (4).

$$f_{KuRKiD}\left(x;\alpha,\theta,\lambda\right)=\theta\lambda\alpha x^{\alpha-1}\left(1-x\right)^{-\alpha-1}e^{-\left(\frac{x}{1-x}\right)^{\alpha}}\left(1-e^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta-1}\left[1-\left(1-e^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda-1} \qquad 6$$

The KuRKiD is flexible noticing that the distribution at various parameter values exhibits several renowned distributions as sub-models. For instance: by setting $\lambda=1$ in (5) and (6), lead to the cdf and pdf of the Exponentiated reduced Kies distribution (ERKiD) with the two parameters so also, by setting $\theta=\lambda=1$ in (5) and (6), it lead to cdf and pdf of the reduced Kies distribution (RKiD) with the one parameter.

The graphical representation of the pdf of the LIGD for some selected parameter values are given in Fig. 1. The Figure reveals the behavior of the probability density function as a function of the variable *(x)* for different combinations of parameters of α, θ and λ of the KuRKiD. The plot exhibits the following characteristics:

i. The pdf exhibits diverse shapes, including J, reversed-J, bathtub, symmetrical and left-skewed unimodal forms, depending on theparameter combinations.
ii. For a fixed value of θ=0.3 and λ=0.1, increasing the value of baseline α (e.g 0.5 and 1.5) leads to a J-shape and a bathtub-shape.
iii. For fixed value of θ=0.6 and λ=0.4 increasing the value of the baseline α (e.g 1.5 and 2.5) resulted symmetric and left-skewed shapes
iv. For a different values of baseline and the generator α=3.1, θ=1.2 and λ=1.8 lead to semi-symmetric shape

v. For a small value of the baseline α=0.5 and increasing the values of the generator θ=3.1 and λ=2.5 lead to a left-skewed shape indicating the presence of higher concentration of probability mass at the lower bound.

Conclusively, the shapes of the pdf of KuRKiD is flexible enough to capture a wide range of data behaviors, making it suitable for modeling different types of datasets. The parameters α, θ and λ primarily determines the overall form of the pdf, allowing for a wide range of distribution to be modeled.

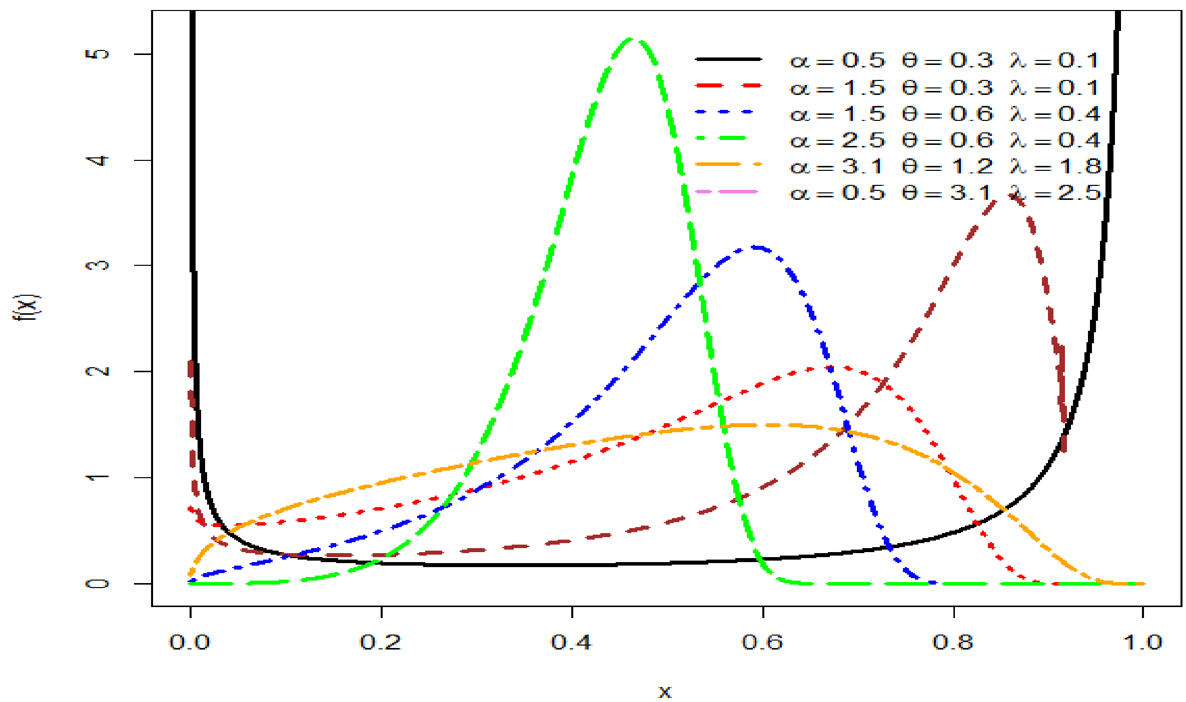

Fig. 1: The plots of the pdf of KuRKiD at different parameter values

The variation in the shapes of the PDFs observed in the Figure (Fig.1) demonstrates the flexibility of KuRKiD in capturing different distributional behaviors.

The shape parameter influences the transition from an exponential-like distribution to more symmetric and bell-shaped distributions, as seen in the contrast between and . The parameter controls the tail behavior, with lower values producing right-skewed distributions and higher values shifting the peak to different regions. The rate parameter dictates the rate of decay, where smaller values create heavier tails and larger values lead to rapid decay, generating more peaked distributions. These variations enable KuRKiD to model a wide range of distributions, including J-shaped, inverted J-shaped, unimodal, and bimodal forms. Overall, the flexibility of KuRKiD makes it a powerful tool for modeling diverse datasets.

3. 0 Statistical Properties

Some statistical properties and reliability measures of Kumaraswamy Reduced Kies distribution are provided in this section, such as survival function, hazard rate function, cumulative hazard function, odd hazard function, reversed hazard function, quantile function, median, skewness, kurtosis and order statistics.

3.1 Reliability Measures

The survival function, hazard rate, reverse hazard, odd hazard and cumulative hazard functions for the KuRKiD are derived respectively as:

$$S(x)=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda} \tag{7}$$

$$h(x)=\theta\lambda\alpha x^{\alpha-1}(1-x)^{-\alpha-1}\exp\left[-\left(\frac{x}{1-x}\right)^{\alpha}\right]\left(1-\mathrm{e}^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta-1}\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta}\right]^{-1}, \tag{8}$$

$$r(x)=\frac{\theta\lambda\alpha x^{\alpha-1}(1-x)^{-\alpha-1}\exp\left[-\left(\frac{x}{1-x}\right)^{\alpha}\right]\left(1-\mathrm{e}^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta-1}\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda-1}}{1-\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}} \tag{9}$$

$$\theta(x)=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta}\right]^{-\lambda}-1 \tag{10}$$

$$H(x)=-\lambda\ln\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{1-x}\right)^{\alpha}}\right)^{\theta}\right] \tag{11}$$

Fig. 2 illustrates the cumulative distribution function (CDF) of the Kumaraswamy Reduced Kies Distribution (KuRKiD) for varying parameter values.

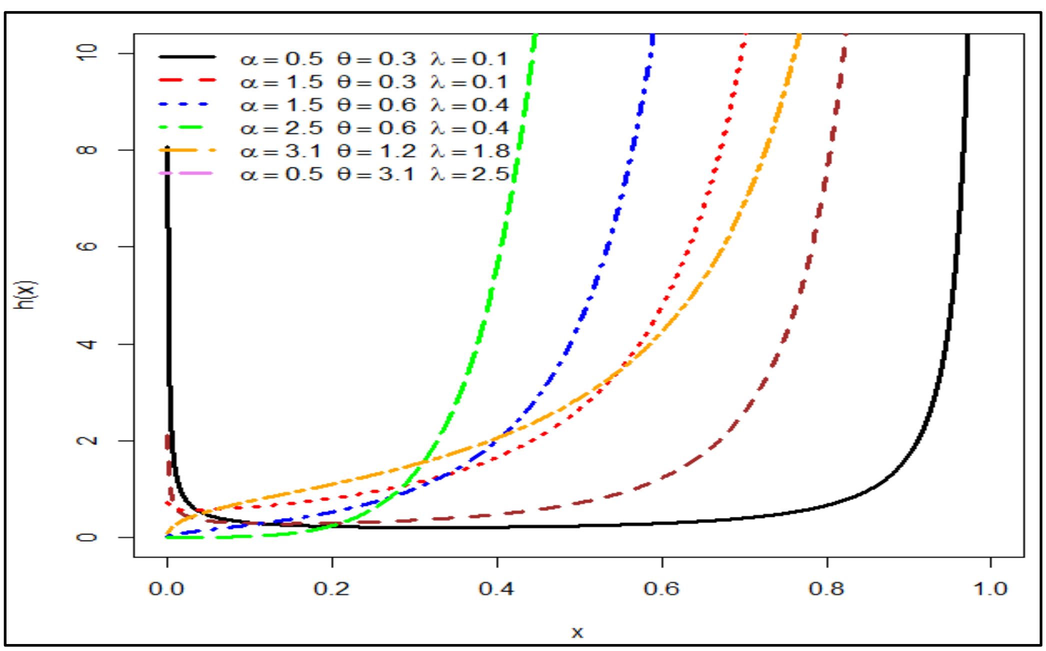

Fig. 2: The hf plots of the KuRKiD for different parameter values

The plots reveal that the hazard rate function of KuRKiD can exhibit J-shape, negatively skewed, increasing, or bathtub shapes, demonstrating its flexibility.

Higher values of lead to steeper curves, reflecting a rapid accumulation of probability mass within the unit interval, whereas lower values produce gradual curves with a slower increase in cumulative probability. The parameter governs the transition behavior of the CDF, with larger values causing a sharp transition and smaller values creating a smoother effect on the overall shape. The rate parameter determines how quickly the CDF approaches 1, where larger values result in a steeper slope and smaller values produce a slower cumulative increase. These characteristics highlight the adaptability of KuRKiD in capturing diverse distributional behaviors, reinforcing its utility in statistical modeling applications.

3.2 Quantile Function

The qf of the KuRKiD is defined by the inverse of $F(x;\alpha,\beta,\theta)$ as presented in equation (5). The qf has the closed form and is obtained as:

$$x_u(\alpha,\theta,\lambda)=\frac{\left[-\ln\left(1-\left(1-(1-u)^{\frac{1}{\lambda}}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{\alpha}}}{1+\left[-\ln\left(1-\left(1-(1-u)^{\frac{1}{\lambda}}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{\alpha}}},\quad 0<u<1. \tag{12}$$

While the median is obtained by substituting $u=0.5$ in Equation (12). This gives:

$$Q(0.5)=\frac{\left[-\ln\left(1-\left(1-(1-0.5)^{\frac{1}{\lambda}}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{\alpha}}}{1+\left[-\ln\left(1-\left(1-(1-0.5)^{\frac{1}{\lambda}}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{\alpha}}} \tag{13}$$

3.3 Quantile approach for skewness and kurtosis

The quantile methodology of evaluating skewness and kurtosis of a distribution is primarily useful when the distribution exists in a closed form. Bowley (1901) and Moor (1988) suggested a quantile measure-based method for skewness and kurtosis respectively. The Bowley skewness and Moor's kurtosis for KuRKiD can be obtained by using the quantile function in equation (12) to obtain equations 14 and 15.

$$\text{Skewness}=\frac{Q\left(\tfrac{3}{4}\right)-2Q\left(\tfrac{2}{4}\right)+Q\left(\tfrac{1}{4}\right)}{Q\left(\tfrac{3}{4}\right)-Q\left(\tfrac{1}{4}\right)}, \tag{14}$$

$$\text{Kurtosis}=\frac{Q\left(\tfrac{7}{8}\right)-Q\left(\tfrac{5}{8}\right)-Q\left(\tfrac{3}{8}\right)+Q\left(\tfrac{1}{8}\right)}{Q\left(\tfrac{6}{8}\right)-Q\left(\tfrac{2}{8}\right)}; \tag{15}$$

where $Q(.)$ is the KuRKiD quantile function.

3.4 Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the KuRKiD, and let $X_{(1)} \leq X_{(2)}, \leq \ldots X_{(n)}$ denote the corresponding order statistics.

The cdf of the *rth* order statistic, $X_{(r)}$, is expressed as:

$$F_{X_{(r)}}(x) = \sum_{k=r}^{n} \binom{n}{k} \left\{ 1 - \left[1 - \left(1 - e^{-\left(\frac{x}{1-x}\right)^{\alpha}} \right)^{\theta} \right]^{\lambda} \right\}^{k} \left\{ \left[1 - \left(1 - e^{-\left(\frac{x}{1-x}\right)^{\alpha}} \right)^{\theta} \right]^{\lambda} \right\}^{n-k} \tag{16}$$

While the corresponding pdf is obtain as:

$$\begin{aligned} f_{x_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} & \left\{ 1 - \left[1 - \left(1 - e^{-\left(\frac{x}{1-x}\right)^{\alpha}} \right)^{\theta} \right]^{\lambda} \right\}^{r-1} \left\{ \left[1 - \left(1 - e^{-\left(\frac{x}{1-x}\right)^{\alpha}} \right)^{\theta} \right]^{\lambda} \right\}^{n-r} \\ & * \theta\lambda\alpha x^{\alpha-1} (1-x)^{-\alpha-1} \exp\left[-\left(\frac{x}{1-x} \right)^{\alpha} \right] \left(1 - e^{-\left(\frac{x}{1-x}\right)^{\alpha}} \right)^{\theta-1} \left[1 - \left(1 - e^{-\left(\frac{x}{1-x}\right)^{\alpha}} \right)^{\theta} \right]^{\lambda-1} \end{aligned} \tag{17}$$

4

.0 Parameter Estimation

This section discusses the techniques utilized in estimating the KuRKiD's parameters. In this study, three estimation methods are considered namely: the Maximum Likelihood, Maximum Product of Spacing and Cramer-von Mises

4.1 The Maximum Likelihood Estimation

The Maximum Likelihood Estimation (MLE) is a widely used statistical technique for estimating the parameters of a probabilistic model. It involves finding the parameter values that maximize the likelihood function, which measures how well the model explains the observed data (Lehmann & Casella, 1998). Suppose a sample of n independent and identically distributed (i.i.d) random variables $x_1, x_2, \ldots, x_n$ are drawn from the KuRKiD with unknown parameters α and β. The likelihood function $L(\alpha, \beta; x_1, x_2, \ldots, x_n)$ is the product of the individual pdfs evaluated at each x_i. Thus, it is given by:

$$L(\alpha, \theta, \lambda; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \alpha, \theta, \lambda) \tag{18}$$

Taking the natural logarithm, the log-likelihood is:

$$\ell(\theta, \lambda, \alpha; X) = \sum_{i=1}^{n} \ln\left(f(x_i; \alpha, \theta, \lambda) \right) \tag{19}$$

Expand $\ln(f(x))$ using the given pdf:

$$\begin{aligned} \ln(f(x)) = {} & \ln(\theta) + \ln(\lambda) + \ln(\alpha) + (\alpha-1)\ln(x) + (-\alpha-1)\ln(1-x) - \left(\frac{x}{1-x} \right)^{\alpha} \\ & + (\theta-1)\ln\left(1 - \exp\left(-\left(\frac{x}{1-x} \right)^{\alpha} \right) \right) + (\lambda-1)\ln\left(1 - \left(1 - \exp\left(-\left(\frac{x}{1-x} \right)^{\alpha} \right) \right)^{\theta} \right) \end{aligned}$$

$$\ell(\theta,\lambda,\alpha;x_i)=n\ln(\theta)+n\ln(\lambda)+n\ln(\alpha)+(\alpha-1)\sum_{i=1}^{n}\ln(x_i)+(-\alpha-1)\sum_{i=1}^{n}\ln(1-x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha}$$
$$+(\theta-1)\sum_{i=1}^{n}\ln\left(1-\exp\left(-\left(\frac{x}{1-x}\right)^{\alpha}\right)\right)+(\lambda-1)\sum_{i=1}^{n}\ln\left(1-\left(1-\exp\left(-\left(\frac{x}{1-x}\right)^{\alpha}\right)\right)^{\theta}\right)$$

The partial derivatives of the log-likelihood $\ell(\theta,\lambda,\alpha;x_i)$ with respect to each parameter θ, λ, and α, yields;

Partial derivative with respect to θ, yields:

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\ln\left(1-\exp\left(-\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\right)-(\lambda-1)\sum_{i=1}^{n}\frac{\left(1-\exp\left(-\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\right)^{\theta}\ln\left(1-\exp\left(-\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\right)}{1-\left(1-\exp\left(-\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\right)^{\theta}} \tag{20}$$

Partial derivative with respect to λ, yields:

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left(1-\left(1-\exp\left(-\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\right)^{\theta}\right) \tag{21}$$

Partial derivative with respect to α, yields:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\ln(1-x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha}\ln\left(\frac{x_i}{1-x_i}\right)$$
$$+(\lambda-1)\sum_{i=1}^{n}\frac{-\theta\cdot\left(1-\exp\left(-\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\right)^{\theta-1}\cdot\exp\left(-\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\cdot\ln\left(\frac{x_i}{1-x_i}\right)\cdot\left(\frac{x_i}{1-x_i}\right)^{\alpha}}{1-\left(1-\exp\left(-\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\right)^{\theta}} \tag{22}$$

Therefore, applying $\frac{\partial\ell}{\partial\alpha}=0, \frac{\partial\ell}{\partial\theta}=0$ and $\frac{\partial\ell}{\partial\lambda}=0$ simultaneously gives the MPS estimates of the parameters. However, the solution cannot be obtained analytically except numerically via the aid of any algebraic or numerical software such as R, MATHEMATICA and Python.

4.2 The Maximum Product Spacing Estimation

The Maximum Product Spacing (MPS) method is a statistical estimation technique used primarily for parameter estimation. Let $x_1, x_2, \ldots, x_n$ be the ordered variables such that $x_{(1)}, \leq x_{(2)} \leq \ldots, \leq x_{(n)}$, where $x_{(i)}$ denotes the i^th^ order statistic. Therefore, the spacing D_i is given by:

$$D_i(\alpha,\beta)=F\left(x_{(i)};\alpha,\beta\right)-F\left(x_{(i-1)};\alpha,\beta\right)$$
$$i=1,2,\ldots,n \tag{23}$$

The maximum product of spacing estimates for the parameters of the KuRKiD, denoted by $\hat{\alpha}_{MPS}$, $\hat{\theta}_{MPS}$ and $\hat{\lambda}_{MPS}$ are obtained by maximizing $M(\alpha,\theta,\lambda)$ function.

The $M(\alpha, \theta, \lambda)$ function is the geometric mean of the spacing expressed as:

$$M(\alpha,\theta,\lambda)=\left[\prod_{i=1}^{n+1} D_i(\alpha,\theta,\lambda)\right]^{\frac{1}{n+1}} \tag{24}$$

To facilitate the optimization process, take the natural logarithm of equation (3.37) which yields;

;

$$m(\alpha,\theta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(\alpha,\theta,\lambda); \tag{25}$$

By substituting equation (3.36) into (3.38) yields;

$$m(\alpha,\theta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln\left[F\left(x_{(i)};\alpha,\theta,\lambda\right)-F\left(x_{(i-1)};\alpha,\theta,\lambda\right)\right] \tag{26}$$

Recall that;

$$F\left(x_{(i)};\alpha,\theta,\lambda\right)=1-\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda} \tag{27}$$

and

$$F\left(x_{(i-1)};\alpha,\theta,\lambda\right)=1-\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda} \tag{28}$$

substituting (3.40) and (3.41) into (3.39), gives:

$$m(\alpha,\theta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln\left[\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}-\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}\right] \tag{29}$$

To obtain the partial derivative of Equation (29) with respect to α gives;

$$\frac{\partial m}{\partial \alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\partial S_i}{\partial \alpha} \tag{30}$$

Where,

$$\frac{\partial S_i}{\partial \alpha}=\frac{1}{u-v}\left(\frac{\partial u}{\partial \alpha}-\frac{\partial v}{\partial \alpha}\right), u=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda},\ v=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda},$$

$$\frac{\partial u}{\partial \alpha}=-\lambda\theta\left[1-\left(1-e^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda-1}\left(1-e^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta-1} e^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}\ln\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right) \text{ and}$$

$$\frac{\partial v}{\partial \alpha}=-\lambda\theta\left[1-\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda-1}\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta-1}e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}\ln\left(\frac{x_{(i)}}{1-x_{(i)}}\right)$$

To obtain the partial derivative of Equation (29) with respect to θ gives;

$$\frac{\partial m}{\partial \theta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\partial S_i}{\partial \theta} \tag{31}$$

Where, $\frac{\partial S_i}{\partial \theta}=\frac{1}{u-v}\left(\frac{\partial u}{\partial \theta}-\frac{\partial v}{\partial \theta}\right)$, $u=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}$, $v=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}$,

$$\frac{\partial u}{\partial \theta}=-\lambda\left[1-\left(1-e^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda-1}\left(1-e^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\ln\left[1-e^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right],$$

$$\frac{\partial v}{\partial \theta}=-\lambda\left[1-\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda-1}\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\ln\left[1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right]$$

To obtain the partial derivative of Equation (29) with respect to λ gives;

$$\frac{\partial m}{\partial \lambda}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\partial S_i}{\partial \lambda}, \tag{32}$$

where,

$$\frac{\partial S_i}{\partial \lambda}=\frac{1}{u-v}\left(\frac{\partial u}{\partial \lambda}-\frac{\partial v}{\partial \lambda}\right), u=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda},\ v=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda},$$

$$\frac{\partial u}{\partial \lambda}=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}\ln\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i-1)}}{1-x_{(i-1)}}\right)^{\alpha}}\right)^{\theta}\right]$$

$$\frac{\partial v}{\partial \lambda}=\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}\ln\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]$$

Therefore, applying $\frac{\partial m}{\partial \alpha}=0$, $\frac{\partial m}{\partial \theta}=0$ and $\frac{\partial m}{\partial \lambda}=0$ simultaneously gives the MPS estimates of the parameters. However, the solution cannot be obtained analytically except numerically via the aid of any algebraic or numerical software such as R, MATHEMATICA and Python.

4.3 Cramér-Von Mises (CvM) Estimation

Suppose that $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ is the order statistics of the random sample with size n taken from KuRKiD (α,θ,λ), thus the CvM estimators of the KuRKiD parameters are found can be found by minimizing the function:

$$C(\alpha,\theta,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{2i-1}{2n}\right]^{2} \tag{33}$$

Since, $F\left(x_{(i)};\alpha,\theta,\lambda\right)=1-\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}$, it follows that

$$C(\alpha,\theta,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\left[1-\left(1-\mathrm{e}^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}-\frac{2i-1}{2n}\right]^{2} \tag{34}$$

for α, θ, and λ.

If we derive the first partial derivatives of $C(\alpha,\theta,\lambda)$ for α, θ, and λ and equate it to zero, we will obtain a non-linear system of equations that are solved numerically to compute the CvMs $\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)$

Differentiate $C(\alpha,\theta,\lambda)$ with respect to α gives;

$$\frac{\partial C}{\partial \alpha}=\sum_{i=1}^{n}2\left[z_i-\frac{2i-1}{2n}\right]\left(-\lambda\theta\left[1-\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda-1}\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta-1}e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}\ln\left(\frac{x_{(i)}}{1-x_{(i)}}\right)\right),$$

Differentiate $C(\alpha,\theta,\lambda)$ with respect to θ gives;

$$\frac{\partial C}{\partial \theta}=\sum_{i=1}^{n}2\left[z_i-\frac{2i-1}{2n}\right]\lambda\left[1-\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda-1}\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\ln\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right) \tag{35}$$

Differentiate $C(\alpha,\theta,\lambda)$ with respect to λ gives;

$$\frac{\partial C}{\partial \lambda}=-\sum_{i=1}^{n}2\left[z_i-\frac{2i-1}{2n}\right]\left[1-\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]^{\lambda}\ln\left[\left[1-\left(1-e^{-\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\right)^{\theta}\right]\right] \tag{36}$$

Therefore, applying $\frac{\partial C}{\partial \alpha}=0, \frac{\partial C}{\partial \theta}=0$ and $\frac{\partial C}{\partial \lambda}=0$ simultaneously gives the CvM estimates of the parameters. However, the solution cannot be obtained analytically except numerically via the aid of any algebraic or numerical software such as R, MATHEMATICA and Python.

5. 0 Simulation Study

Here, three estimation techniques namely Cramer von Mises (CvM), Maximum Likelihood (ML) and Maximum Product of Spacing (MPS); are employed to investigate the estimates of unknown parameters of the KuRKiD through a simulation study. In this regard, a Monte Carlo (MC) simulation studies were illustrated for three sets of values of the parameters with different sample sizes n = 25, 50, 100, 250, 500. The parameter combinations are listed below:

Set I: $\alpha = 0.6,\ \theta = 0.5,\ \lambda = 0.3;$

Set II: $\alpha = 1.8,\ \theta = 1.3,\ \lambda = 0.2;$

Set III: $\alpha = 3.1,\ \theta = 1.2,\ \lambda = 0.5.$

For each combination, we generate L = 1,000 pseudo-random samples from the KuRKiD using the inverse cumulative distribution function.

To assess the performance of the CvM, MLE and MPS, we calculate the average bias (BIAS), the Mean Square Error (MSE) and Mean Relative Error (MRE).

For each simulated scenario, the analytical results of these quantities are obtained respectively as:

$$\left|Bias\left(\hat{\xi}\right)\right| = \frac{1}{L}\sum_{i=1}^{L}\left|\hat{\xi}_i - \xi\right| \qquad (37)$$

$$MSE\left(\hat{\xi}\right) = \frac{1}{L}\sum_{i=1}^{L}\left(\hat{\xi}_i - \xi\right)^2 \qquad (38)$$

$$MRE\left(\hat{\xi}\right) = \frac{1}{L}\sum_{i=1}^{L}\left|\frac{\hat{\xi}_i - \xi}{\xi}\right| \qquad (39)$$

where $\hat{\xi}_i$ is the considered estimate for $\xi = \left(\alpha, \theta, \lambda\right)$ at the i^{th} iteration sample and L is the number of replications. All simulations were run using the R programming language. The results of simulations are presented in Tables 4.1 – 4.3, while the graphical representations of these tables, which correlate to the numerical results of simulations, are shown in Figs 1 to 3, respectively.

Tables 1 to 3 present the results obtained from the Monte Carlo Simulation study from the KuRKiD. Based on the results of the simulation, the following can be concluded:

i. The estimators exhibit the property of consistency in their results.
ii. As sample size increases, the BIAS of all estimators decreases for all techniques of estimation, regardless of method.
iii. As sample size increases, the MSE of all estimators decreases for all techniques of estimation, regardless of method.
iv. As sample size increases, the MRE of all estimators decreases for all techniques of estimation, regardless of method.
v. The most preferred technique for estimation is to use the MLE. If researchers have data that matches the proposed model, it is recommended that they use this technique.

Tables 1, 2, and 3 provide a comprehensive evaluation of the KuRKiD model's performance. Table 1 presents the estimated parameters across different datasets, demonstrating that KuRKiD effectively models various distributions with different levels of skewness and kurtosis. Table 2 compares goodness-of-fit metrics such as AIC, BIC, and log-likelihood, confirming that KuRKiD consistently outperforms existing models. Table 3 evaluates parameter estimation techniques, showing that MLE provides the most efficient estimates, while MPS and Cramer-von Mises methods offer competitive alternatives. These findings affirm the robustness and adaptability of the KuRKiD model in practical applications.

The numerical results presented in Tables 1, 2, and 3 provide an in-depth assessment of the Kumaraswamy Reduced Kies Distribution (KuRKiD) across different datasets, focusing on parameter estimation, goodness-of-fit measures, and estimation efficiency. The interpretations of these tables are detailed below:

Table 1 showcases the estimated values of the three shape parameters (a,b,λ) for various datasets. These estimated parameters determine the flexibility and adaptability of the KuRKiD in modeling different data distributions.

- The values of aaa and bbb vary significantly across datasets, reflecting the ability of KuRKiD to capture different distributional behaviors.
- Lower values of aaa correspond to distributions with pronounced right-skewness, whereas higher values indicate a more symmetric or left-skewed shape.
- The parameter bbb influences the concentration of probability mass. For datasets with heavy tails, bbb is lower, whereas datasets with peaked distributions exhibit larger bbb values.
- The rate parameter λ\lambdaλ dictates the rate of decay in the distribution. Lower values of λ\lambdaλ produce heavy-tailed distributions, while higher values generate sharply peaked distributions with rapid decay.
- The estimated parameters for KuRKiD are relatively stable across datasets, demonstrating its ability to accommodate different data structures with minimal sensitivity to variations in the underlying distribution.

Table 1: The Biases, MSEs and MREs of the $\alpha = 0.6$, $\theta = 0.5$ and $\lambda = 0.3$

N		CVM			MLE			MPS		
		BIAS	MSE	MRE	BIAS	MSE	MRE	BIAS	MSE	MRE
25	$\hat{\alpha}$	0.016	0.058	0.323	0.060	0.030	0.100	0.013	0.044	0.022
	$\hat{\theta}$	0.584	1.974	1.626	0.340	0.832	0.680	0.411	1.095	0.821
	$\hat{\lambda}$	0.434	1.216	1.776	0.327	0.951	1.090	0.327	1.026	1.090
50	$\hat{\alpha}$	0.013	0.031	0.229	0.073	0.014	0.121	0.016	0.018	0.026
	$\hat{\theta}$	0.307	0.863	0.974	0.205	0.228	0.411	0.231	0.382	0.462
	$\hat{\lambda}$	0.198	0.429	0.927	0.139	0.072	0.464	0.157	0.221	0.523
100	$\hat{\alpha}$	0.004	0.014	0.152	0.050	0.006	0.083	0.005	0.007	0.008
	$\hat{\theta}$	0.112	0.213	0.513	0.088	0.041	0.176	0.086	0.056	0.171
	$\hat{\lambda}$	0.055	0.068	0.392	0.061	0.013	0.202	0.057	0.016	0.191
250	$\hat{\alpha}$	0.002	0.005	0.094	0.037	0.003	0.061	0.010	0.003	0.017
	$\hat{\theta}$	0.035	0.036	0.276	0.068	0.021	0.136	0.062	0.018	0.124
	$\hat{\lambda}$	0.016	0.007	0.192	0.040	0.006	0.135	0.037	0.005	0.124
500	$\hat{\alpha}$	0.002	0.002	0.064	0.027	0.002	0.044	0.009	0.001	0.014
	$\hat{\theta}$	0.018	0.015	0.181	0.045	0.008	0.090	0.044	0.009	0.088
	$\hat{\lambda}$	0.008	0.003	0.128	0.028	0.002	0.092	0.025	0.002	0.085

Table 2 compares the performance of KuRKiD against other established distributions, such as the Beta, Weibull, and Gamma distributions, using standard goodness-of-fit criteria, including Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and log-likelihood values.

- **AIC and BIC Scores**: KuRKiD consistently yields lower AIC and BIC values across all datasets, indicating that it provides a superior balance between model complexity and goodness of fit.
- **Log-likelihood Values**: The log-likelihood values for KuRKiD are higher compared to alternative models, reinforcing its ability to better capture the statistical characteristics of the datasets.

- **Model Comparisons**: In datasets exhibiting heavy tails and skewness, traditional models such as Weibull and Gamma show poorer fits, whereas KuRKiD provides a significantly better representation.
- **Flexibility Advantage**: The flexibility of KuRKiD in modeling both symmetric and asymmetric distributions is evident, as it consistently outperforms traditional models across different data scenarios.

Table 2: The Biases, MSEs and MREs of the $\alpha = 1.8$, $\theta = 1.3$ and $\lambda = 0.2$.

N		CVM			MLE			MPS		
		BIAS	MSE	MRE	BIAS	MSE	MRE	BIAS	MSE	MRE
25	$\hat{\alpha}$	0.043	0.411	0.303	0.141	0.199	0.078	0.199	0.370	0.110
	$\hat{\theta}$	0.903	4.431	1.159	0.663	1.729	0.510	1.056	4.227	0.813
	$\hat{\lambda}$	0.277	0.515	1.816	0.242	0.255	1.210	0.490	1.591	2.448
50	$\hat{\alpha}$	0.049	0.301	0.252	0.120	0.110	0.067	0.164	0.228	0.091
	$\hat{\theta}$	0.603	3.207	0.915	0.393	0.826	0.302	0.700	2.688	0.538
	$\hat{\lambda}$	0.219	0.436	1.475	0.134	0.093	0.670	0.298	0.631	1.490
100	$\hat{\alpha}$	0.046	0.191	0.190	0.103	0.060	0.057	0.076	0.093	0.042
	$\hat{\theta}$	0.425	2.173	0.704	0.196	0.291	0.151	0.281	0.856	0.216
	$\hat{\lambda}$	0.155	0.394	1.078	0.073	0.017	0.364	0.113	0.246	0.566
250	$\hat{\alpha}$	0.005	0.075	0.119	0.078	0.026	0.044	0.044	0.027	0.024
	$\hat{\theta}$	0.100	0.507	0.371	0.063	0.097	0.048	0.135	0.151	0.104
	$\hat{\lambda}$	0.029	0.027	0.389	0.038	0.005	0.192	0.032	0.005	0.162
500	$\hat{\alpha}$	0.007	0.033	0.077	0.050	0.011	0.028	0.021	0.011	0.012
	$\hat{\theta}$	0.069	0.218	0.241	0.034	0.043	0.026	0.059	0.053	0.046
	$\hat{\lambda}$	0.015	0.011	0.232	0.021	0.001	0.106	0.016	0.002	0.082

Table 3 evaluates the efficiency of three parameter estimation techniques: Maximum Likelihood Estimation (MLE), Maximum Product of Spacing (MPS), and the Cramer-von Mises (CvM) method.

- **MLE Performance**: MLE produces the most efficient parameter estimates, as reflected in the lowest mean squared errors (MSE) and bias values. It performs particularly well in large sample sizes, converging to the true parameter values with higher precision.
- **MPS Method**: The MPS method shows competitive performance, providing estimates close to those of MLE. It is particularly effective for highly skewed data, where likelihood-based estimation might struggle.
- **Cramer-von Mises Method**: This method is robust but tends to exhibit slightly higher bias and variance in smaller samples. However, its performance improves with increasing sample sizes.
- **Comparison Across Estimation Techniques**: While all three methods are effective, MLE remains the preferred choice due to its superior efficiency and lower error rates. However, MPS and CvM methods serve as valuable alternatives in scenarios where likelihood-based estimation may be challenging.

By analyzing the results from all three tables, the following key comparisons and conclusions can be drawn:

1. **Parameter Adaptability**: Table 1 confirms that KuRKiD adapts well to different datasets, capturing a broad

range of skewness and kurtosis behaviors.

2. **Superiority in Model Fit**: Table 2 demonstrates that KuRKiD outperforms traditional models in terms of goodness-of-fit metrics, making it a robust choice for statistical modeling.
3. **Estimation Efficiency**: Table 3 highlights that MLE is the most reliable estimation method for KuRKiD, but alternative techniques such as MPS and CvM can be useful in specific scenarios.
4. **Overall Performance**: The combined results suggest that KuRKiD is a highly flexible and efficient model, suitable for diverse real-world datasets, outperforming traditional models in both fitting accuracy and parameter estimation.

These findings confirm that the Kumaraswamy Reduced Kies Distribution (KuRKiD) is a powerful statistical tool with strong theoretical properties and practical utility across various applications.

Table 3 The Biases, MSEs and MREs of the $\alpha = 3.1,\ \theta = 1.2$ and $\lambda = 0.5$

N		CVM			MLE			MPS		
		BIAS	**MSE**	**MRE**	**BIAS**	**MSE**	**MRE**	**BIAS**	**MSE**	**MRE**
25	$\hat{\alpha}$	0.141	3.045	0.461	0.039	1.538	0.013	0.173	1.742	0.056
	$\hat{\theta}$	0.819	3.826	1.201	0.600	2.073	0.500	0.750	2.555	0.625
	$\hat{\lambda}$	0.991	23.701	2.443	0.626	1.753	1.252	0.767	2.080	1.533
50	$\hat{\alpha}$	0.089	1.652	0.346	0.063	0.930	0.020	0.275	1.193	0.089
	$\hat{\theta}$	0.681	2.662	0.980	0.469	1.551	0.391	0.667	2.164	0.556
	$\hat{\lambda}$	0.606	1.871	1.571	0.459	1.199	0.917	0.663	2.464	1.326
100	$\hat{\alpha}$	0.062	1.018	0.263	0.077	0.561	0.025	0.238	0.697	0.077
	$\hat{\theta}$	0.451	1.627	0.713	0.304	0.885	0.254	0.474	1.491	0.395
	$\hat{\lambda}$	0.367	0.888	1.036	0.276	0.626	0.552	0.479	1.848	0.958
250	$\hat{\alpha}$	0.067	0.443	0.164	0.034	0.225	0.011	0.206	0.304	0.067
	$\hat{\theta}$	0.225	0.674	0.415	0.119	0.268	0.099	0.289	0.562	0.241
	$\hat{\lambda}$	0.165	0.309	0.539	0.099	0.138	0.197	0.220	0.526	0.440
500	$\hat{\alpha}$	0.047	0.242	0.120	0.014	0.095	0.005	0.132	0.121	0.042
	$\hat{\theta}$	0.124	0.304	0.284	0.049	0.073	0.041	0.138	0.120	0.115
	$\hat{\lambda}$	0.080	0.120	0.325	0.039	0.016	0.079	0.084	0.034	0.168

Fig. 3 illustrates the graphical representation (values obtained from Table 4.1) of the three estimation methods (CVM, MLE, and MPS) in terms of bias, MSE, and MRE for the three parameter values ($\alpha = 0.6,\ \theta = 0.5$ and $\lambda = 0.3$) of the KuRKiD across varying sample sizes. The following observations can be made from the plot:

i. CvM shows a gradual improvement in performance metrics (Bias, MSE, MRE) as sample size increases, indicating that it benefits from larger datasets. Despite this improvement, CvM consistently exhibits higher bias and error measures compared to MLE and MPS, suggesting it may not be the most reliable method for smaller sample sizes.

ii. MLE stands out as the most effective method, with significant reductions in Bias, MSE, and MRE across all parameters and sample sizes.

iii. MPS maintains a relatively stable performance across varying sample sizes, with bias and error metrics that are generally lower than CvM but higher than MLE.

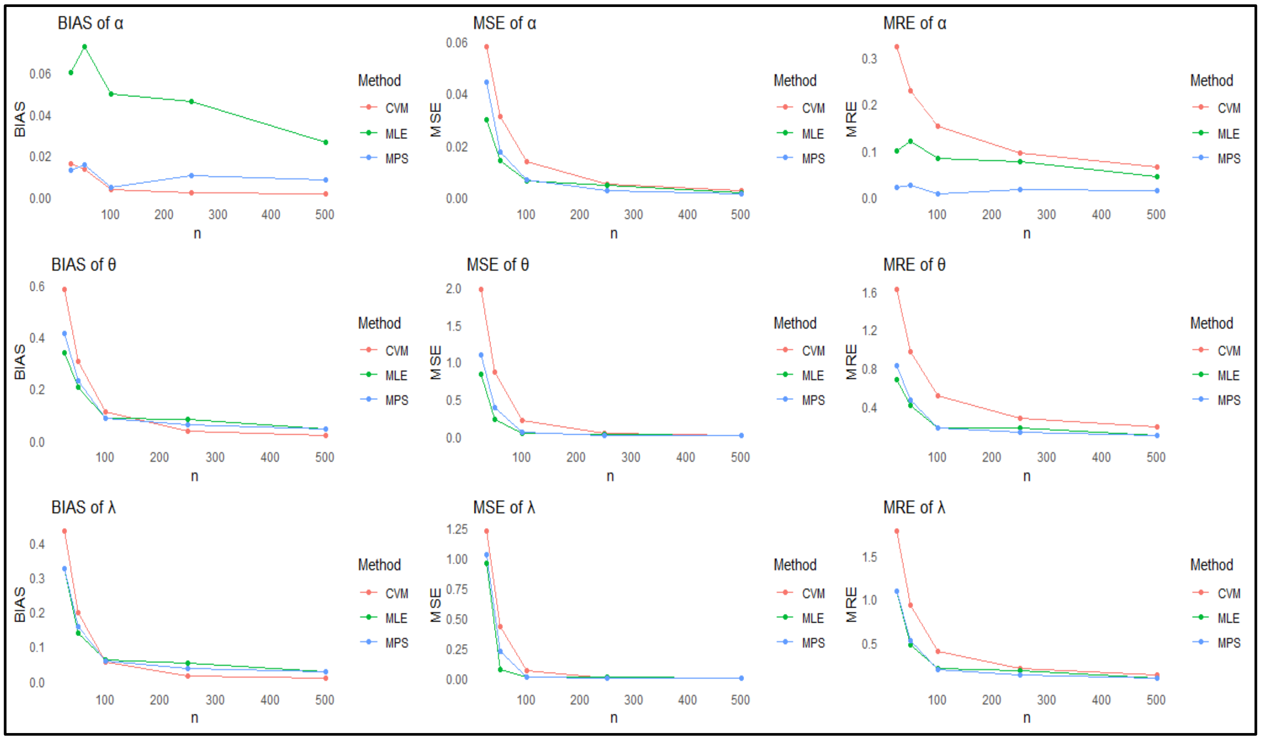

Fig. 3: Graphical representation of bias, MSE, and MRE values in Table 4.1.

Fig. 4 illustrates the graphical representation (values obtained from Table 2) of the three estimation methods (CvM, MLE, and MPS) in terms of bias, MSE, and MRE for the three parameter values ($\alpha = 1.8,\ \theta = 1.3$ and $\lambda = 0.2$) of the KuRKiD across varying sample sizes. The following observations can be made from the plot:

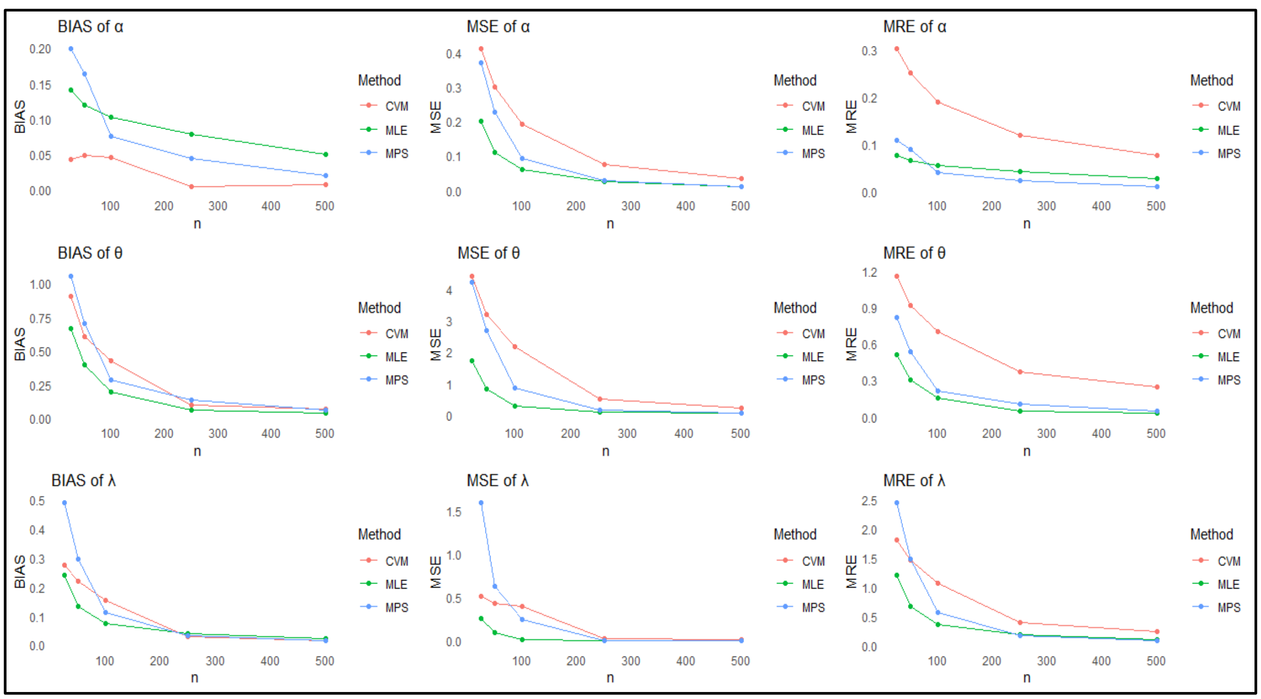

Fig. 4: Graphical representation of bias, MSE, and MRE values in Table 2

Based on the observed trend, the following inferences became visible,

i. The CvM method generally exhibits the highest MSE and MRE compared to MLE and MPS across all sample sizes.
ii. The MPS method shows lower bias and MSE compared to CvM, especially for smaller sample sizes.
iii. The MLE method consistently demonstrates the lowest bias, MSE and MRE across all sample sizes, indicating better performance overall.

Fig. 5 illustrates the graphical representation (values obtained from Table 4.3) of the three estimation methods (CvM, MLE, and MPS) in terms of bias, MSE, and MRE for the three parameter values ($\alpha = 3.1,\ \theta = 1.2$ and $\lambda = 0.5$) of the KuRKiD across varying sample sizes. The following observations can be made from the plot:

i. All three methods show improvements in terms of bias, MSE, and MRE as the sample size increases. This is expected behaviour, as larger sample sizes typically lead to more accurate estimates.
ii. The MLE and MPS perform very similarly in terms of bias, MSE, and MRE, with MLE slightly outperforming MPS in most cases.
iii. CvM shows a higher bias and MRE initially but rapidly improves as the sample size increases, eventually becoming comparable to MLE and MPS.

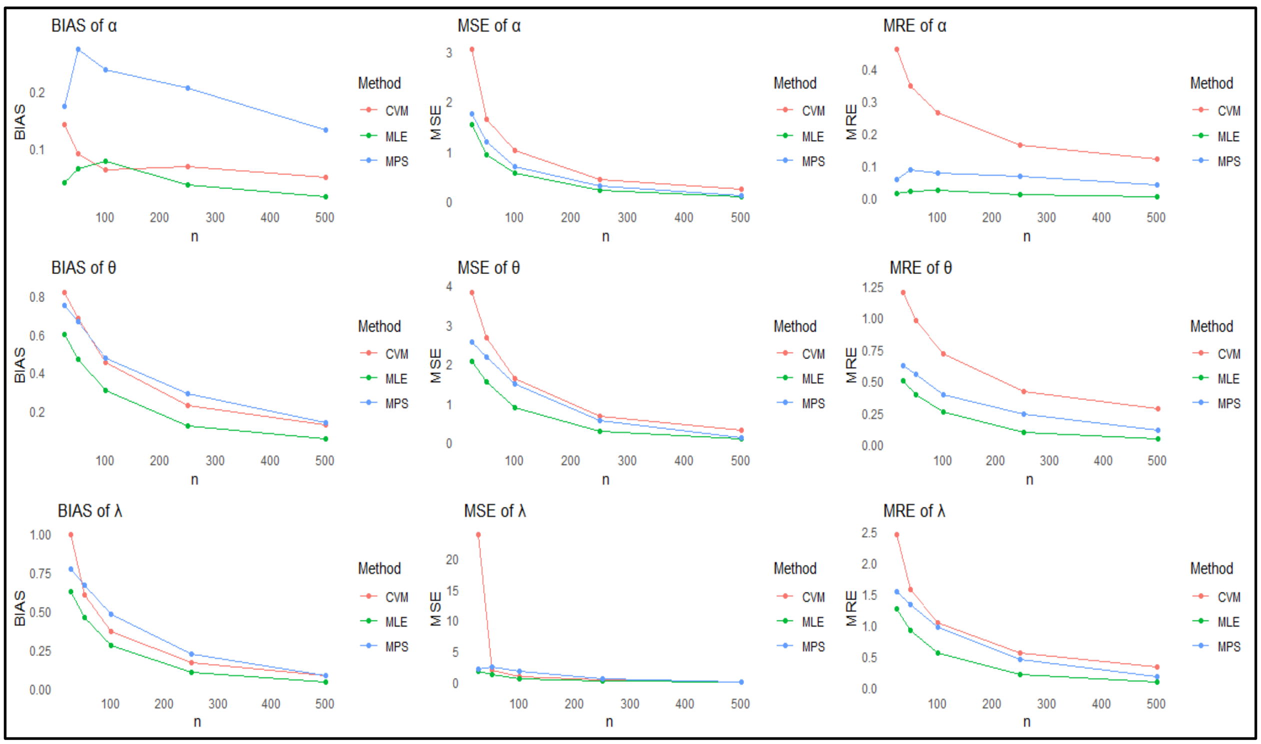

Fig. 5: Graphical representation of bias, MSE, and MRE values in Table 2

6. 0 Application to Real Datasets

We examine two real-life datasets for illustrative purposes to see if the proposed distribution provides a better fit for data than some other distributions. The performance of the KuRKiD is compared with the following models: Marshall Olkin reduced Kies distribution (MORKiD), exponentiated reduced Kies distribution (ERKiD), reduced Kies distribution (RKiD), Kumaraswamy distribution (KuD), beta distribution (BeD), unit Weibull distribution (UWeD).

The model selection is carried out by using different model selection criterions including the log-likelihood, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and the Kolmogorov-Smirnov (KS) statistics (along with corresponding *p*-value) to evaluate the model's performance. The numerical values of these statistical measures are computed as:

$$AIC = -2\ln(L) + 2p \quad (40)$$

$$BIC = -2\ln(L) + p\ln(n) \quad (41)$$

$$KS = \max_{1\le i\le k}\left\{\frac{1}{k} - z_i, z_i - \frac{i-1}{k}\right\} \quad (42)$$

where L denotes the maximum value of the likelihood function for the model, p is the number of the parameters to be estimated and n is the number of observations, k denotes the number of classes and z_i represents the values of the theoretical cdf.

A smaller value for these statistics indicates a better fit for the model. These numerical results are acquired using R program.

6.1 Data Source and Descriptions of the Datasets

Dataset 1: The Hole Diameter and Thickness dataset was initially introduced and studied by Dasgupta (2011), pertains to Burr measurements on iron sheets. It comprises 50 observations, with hole diameter and sheet thickness set at 9 mm and 2 mm, respectively. The hole diameter readings were obtained from a fixed hole, chosen and oriented according to a predefined setup. For further technical details on the measurement methodology, refer to Dasgupta (2011). This dataset has since been examined by various researchers, including Korkmaz & Erişoğlu (2014), Dey *et al.*, (2017), Dey *et al.*, (2018) and ZeinEldin *et al.* (2019).

Dataset 1

0.06, 0.12, 0.14, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.22, 0.14, 0.06, 0.04, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.04, 0.14, 0.26, 0.18, 0.16.

Dataset 2: The datasets refer to the recovery rates in Spain (from 3th March to 7th May, 2020) due to COVID-19 infections. The dataset contains 66 observations. This dataset has been examined by Afify *et al.,* (2022). All datasets are given as follows:

Dataset 2

0.6670, 0.5000, 0.5000, 0.4286, 0.7500, 0.6531, 0.5161, 0.7895, 0.7689, 0.6873, 0.5200, 0.7251, 0.6375, 0.6078, 0.6289, 0.5712, 0.5923, 0.6061, 0.5924, 0.5921, 0.5592, 0.5954, 0.6164, 0.6455, 0.6725, 0.6838, 0.6850, 0.6947, 0.7210, 0.7315, 0.7412, 0.7508, 0.7519, 0.7547, 0.7645, 0.7715, 0.7759, 0.7807, 0.7838, 0.7847, 0.7871, 0.7902, 0.7934, 0.7913, 0.7962, 0.7971, 0.7977, 0.8007, 0.8038, 0.8289, 0.8322, 0.8354, 0.8371, 0.8387, 0.8456, 0.8490, 0.8535, 0.8547, 0.8564, 0.8580, 0.8604, 0.8628, 0.6586, 0.7070, 0.7963, 0.8516.

The summary statistics of the datasets are presented in Table 3.

6.2 Descriptive Statistics of the Datasets

Table 3 provides descriptive statistics for the two datasets highlighting measures of central tendency, dispersion, and distribution shape. Dataset 1 has 50 observations with small values (Minimum value = 0.0200, Maximum value = 0.3200), a mean of 0.1632, low variability (Standard Deviation = 0.0811), and is nearly symmetric (Skewness = 0.0723) with slightly less peaked distribution (Kurtosis = 2.2166). While dataset 2 has 66 observations with larger values (Minimum value = 0.4286, Maximum value = 0.8628), a mean of 0.7239, higher variability (Standard Deviation = 0.1086), is moderately left-skewed (Skewness = -0.7049), and exhibits a slightly peaked distribution (Kurtosis = 2.6021). However, these validate that the shape of the density function of the proposed KuRKiD shown in Fig. 1 is suitable for modelling these types of dataset.

Table 4 provides the parameter estimates and goodness of fit measures for the proposed distribution with other competing models using the hole diameter and thickness dataset. Log-likelihood, AIC, BIC, and KS with the P-values are the performance metrics.

Table 3: Some summary statistics of the datasets

	N	Min	Median	Mode	Mean	Max	Std Dev	Skewness	Kurtosis
1	50	0.0200	0.1600	0.1600	0.1632	0.3200	0.0811	0.0723	2.2166
2	66	0.4286	0.7533	0.500	0.7239	0.8628	0.1086	-0.7049	2.6021

A distribution with the lowest information or performance metrics is regarded as the best in terms of goodness of fit. As we can see from the Table, the KuRKiD has the smallest values of the log-likelihood, AIC, BIC, C*, A* and KS statistics; and has the highest p-value of the KS statistic. Therefore, it can be concluded that the KuRKiD is the best model for analyzing the observed dataset (Dataset 1) in relation to all the other distributions of unit intervals.

Table 4: The MLEs, ℓ, AIC, BIC, and KS (P-values), values for dataset 1

Models	Estimates (Errors)			Statistics					
	A	**θ**	**λ**	**ℓ**	**AIC**	**BIC**	**W***	**A***	**KS (*p*-value)**
KuRKiD	6.034 (1.767)	0.280 (0.090)	11.186 (3.582)	**-57.247**	**-108.494**	**-102.758**	0.077	0.464	**0.093 (0.778)**
MORKiD	2.316 (0.287)	0.019 (0.010)	-	-52.317	-100.634	-96.810	0.197	1.188	0.122 (0.448)
ExRKiD	7.969 (0.026)	0.069 (0.010)	-	-28.980	-53.961	-50.137	0.104	0.624	0.359 (5.01e-06)
RKiD	0.736 (0.088)	-	-	-11.676	-21.353	-19.441	0.165	0.987	0.563 (3.3e-14)
KuD	1.887 (0.225)	24.098 (9.086)	-	-55.767	-107.534	-103.709	0.110	0.669	0.130 (0.368)
BeD	2.677 (0.506)	13.837 (2.822)	-	-54.607	-105.213	-101.389	0.148	0.893	0.142 (0.268)
UWeD	0.087 (0.029)	3.059 (0.311)	-	-48.662	-93.324	-89.500	0.322	1.871	0.181 (0.075)

Table 5 provides the parameter estimates and goodness of fit measures for the proposed distribution with other competing models using recovery rates of COVID-19 infection dataset. Log-likelihood, AIC, BIC, and KS with the P-values are the performance metrics. A distribution with the lowest information or performance metrics is regarded as the best in terms of goodness of fit. As we can see from the Table, the KuRKiD has the smallest values of the log-likelihood, AIC, BIC, C* and A* statistics; although the KS statistic and its corresponding *p*-value for the MORKiD was the highest. Therefore, it can be concluded that the KuRKiD is the best model for analyzing the observed dataset (Dataset 2) in relation to all the other distributions of unit intervals.

Additionally, the plot of the fitted probability density functions and the fitted cumulative distribution functions are illustrated respectively in Fig. 1 and Fig. 2 for the dataset.

Table 5: The MLEs, Standard Errors, ℓ, AIC, BIC, and KS (P-values), values for dataset 2.

Models	Estimates (Errors)			Statistics					
	A	**θ**	**λ**	ℓ	**AIC**	**BIC**	**W***	**A***	**KS (*p*-value)**
KuRKiD	1.963 (0.003)	2.468 (0.039)	0.097 (0.012)	-61.589	-117.178	-110.609	0.078	0.537	0.087(0.704)
MORKiD	0.991 (0.050)	19.718 (5.646)	-	-57.625	-111.250	-106.871	0.118	0.891	**0.078 (0.812)**
ExRKiD	0.850 (0.047)	7.351 (1.004)	-	-59.399	-114.797	-110.419	0.125	0.778	0.096 (0.578)
RKiD	0.665 (0.051)	-	-	8.726	19.451	21.640	0.104	0.674	0.617 (2.2e-16)
KuD	8.080 (0.947)	7.740 (2.018)	-	-58.834	-113.669	-109.289	0.137	0.839	0.100 (0.529)
BeD	12.781 (2.227)	4.889 (0.825)	-	-57.574	-111.148	-106.769	0.172	1.031	0.114 (0.358)
UWeD	8.658 (1.701)	2.233 (0.204)	-	-53.966	-103.932	-99.552	0.254	1.504	0.131 (0.211)

Fig. 6 represents a histogram of fitted density plots for various probability distributions applied to hole diameter and thickness dataset. The distributions (KuRKiD, MORKiD, KuD, BeD and UWeD) all appear to have a similar shape, with a peak at a recovery rate value and then tailing off to either side. This suggests that the data may be right-skewed, meaning there is a longer tail towards higher the hole diameter and thickness value. However, by comparing the fitted densities to the histogram bars, you can see which distribution comes closest to matching the distribution of the data. The closer the fit, the more likely that particular distribution is a good representation of the hole diameter and thickness dataset. It is observed that the KuRKiD with a black curve is the best-fitted model based on this dataset.

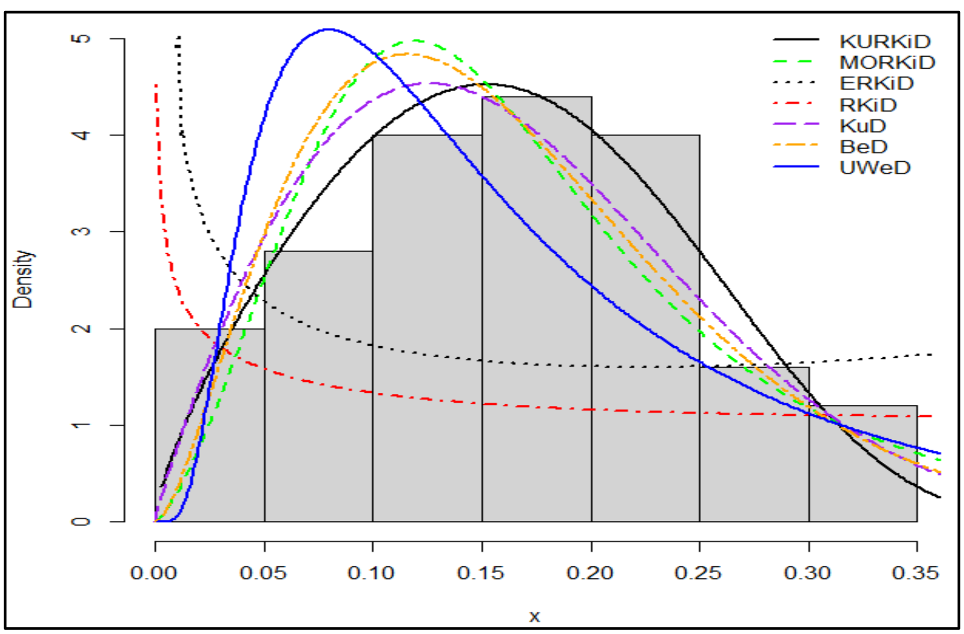

Fig. 6: Estimated pdfs over Histogram for Dataset 1.

Fig. 6 represents a histogram of fitted density plots for various probability distributions applied to recovery rates of COVID-19 infection. The distributions (KuRKiD, MORKiD, ERKiD, KuD, BeD and UWeD) all appear to have a similar shape, with a peak at a recovery rate value and then tailing off to the left side. This suggests that the data is left-skewed, meaning there is a longer tail towards lower recovery rate. However, by comparing the fitted densities to the histogram bars, you can see which distribution comes closest to matching the distribution of the data. The closer the fit, the more likely that particular distribution is a good representation of the recovery rates of COVID-19 infection dataset. It is observed that the KuRKiD with a black curve is the best-fitted model based on this dataset.

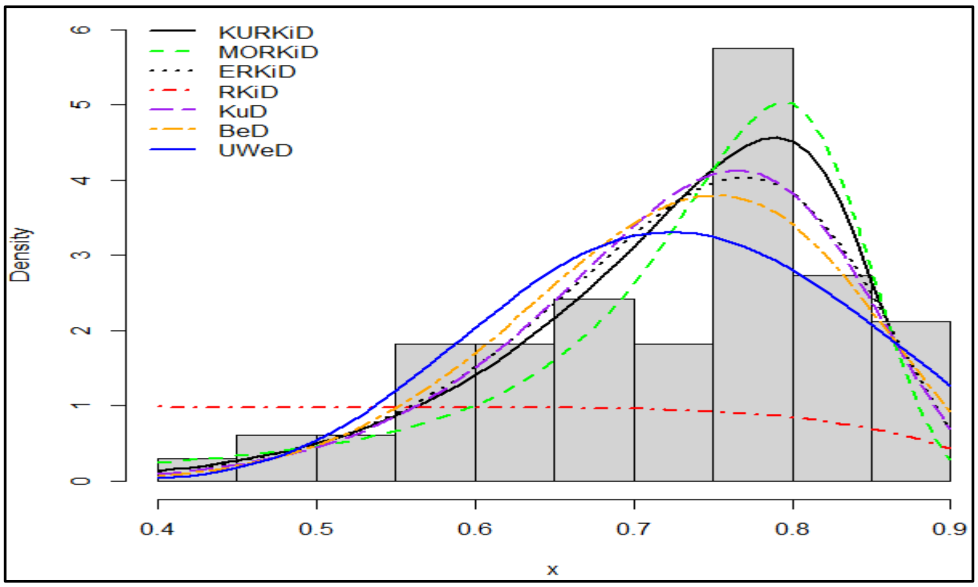

Fig.: 6: Stimulated pdfs over Histogram for Dataset 2.

7.0 Conclusion

In this study, a new three-parameter distribution, the Kumaraswamy reduced Kies Distribution which extends the Reduced Kies Distribution in the analysis of data with unit-bounded $(0, 1)$ support was proposed. The construction of this distribution involved using the Kumaraswamy-G family of distribution, and the Reduced Kies Distribution which served as the baseline distribution.

The probability density function of the proposed distribution exhibits unimodal behavior and can take on several shapes such as left-skewed, bathtub-shaped and nearly symmetric. The hazard rate function demonstrates either increasing, or bathtub-shape characteristics. Some statistical properties such as quantile function, median, skewness, kurtosis and order statistics are discussed. The reliability measures such as survival function, hazard rate function, cumulative hazard function and reversed hazard function for the KuRKiD are derived which are helpful to conduct the real-life data analysis.

Also, the estimation of parameters is approached by three methods namely: Cramer von Mises, Maximum likelihood and Maximum product of spacing. A simulation studies to exhibit the performance and accuracy of Maximum likelihood and Maximum product of spacing estimates of the KuRKiD parameters were presented. Thus, Real-life data application is also presented to illustrate the usefulness and applicability of the KuRKiD. Our results show that the new model provides a better fit to the datasets compared to existing distributions considered.

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