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Maximizing an Investment Portfolio for a DC Pension with a Return Clause and Proportional Administrative Charges under Weilbull Force Function

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Abstract: *In this paper, investment in a defined contributory (DC) pension fund system with a return clause of premium and proportional administrative charges is studied under geometric Brownian motion (GBM) and Weilbull mortality force function. To actualize this, an investment portfolio with a risk-free asset and a risky asset which follows the GBM model is considered such that the returned premium is with interest from an investment in a risk-free asset and the Weilbull force function is used to determine the mortality rate of members during accumulation phase. Furthermore, the game-theoretic technique is applied to obtain an optimization problem from the extended Hamilton Jacobi Bellman equation. By using the mean-variance utility and variable separation technique, an investment strategy (IS) is obtained for the risky asset comprising of the risk-free interest rate, instantaneous volatility, administrative charges, the appreciation rate of the risky asset and the mortality force function was obtained together with the efficient frontier which gives the relationship between the investment expectation and the risk involvement in the investment. Furthermore, some numerical simulations were obtained to study the impact of some sensitive parameters of the IS. It was observed that the administrative charges and the mortality rate affect the IS to be adopted. Therefore, an insight into how these parameters behave is very essential in the development of an IS.*

Keywords: Investment strategy, Return clause of premium, Administrative charges, Weibull mortality function.

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1.0 Introduction

The global economic downturn that has negatively affected the world financial market, the stock market in particular via-a-vis investment portfolios has provoked so many reviews of the existing economic models (Witbooi *et al*, 2011; Li *et al*, 2013; Njoku *et al*, 2017, Osu *et al*, 2017; Akpanibah *et al*, 2017; Wang *et al*, 2018; Akpanibah and Osu, 2018; Njoku and Osu, 2019a; Njoku and Osu, 2019b; Osu *et al*, 2019a; Osu *et al*, 2019b; Njoku *et al*, 2019; Osu *et al*, 2020a; Osu *et al*, 2020b; Ini *et al*, 2021; Akpanibah and Ini, 2021; Njoku and Akpanibah, 2022; Njoku *et al*, 2022).

Continuous review of these economic models vis-à-vis the investment strategies has become very necessary since in the DC pension scheme, the tone of the investment returns solely depends on how good the investment strategies by the PFAs are Njoku and Osu, (2019b). More so, because one of the importance of setting up a pension scheme is to help manage the income of retirees Antolin et al, (2010). Currently, there are two types of pension systems in which an employee can be enrolled; the defined benefit system (DB) (Haberman and Sung 1994; JosaFombellida 2001a, 2004b) and the defined contribution (DC) system (Zhang and Rong, 2013; Wu and Zeng, 2015; Sun *et al*, 2016; Akpanibah *et al*., 2019; Lai *et al.*, 2021).

.In a DB pension scheme, members' benefits are predetermined based on age, years in service and salary histories and their benefits after retirement depend mostly on the contributions of the employers. However, in the DC pension scheme, members contribute a percentage of their income into their RSA, as stipulated by the Pension Reform Act of 2006, as amended, and these funds are kept by the pension fund custodians (PFCs) and managed by the PFAs, to maximi the expected returns of its members. Furthermore, in DC plan, contributions are predetermined while the benefits solely depend on investment returns during the accumulation period. However, PFAs must have a good investment model since most of these assets for example; stock and zero coupon bonds in the market are highly risky (Wu and Zeng, 2015). This has led to the study of IS which explains how investment portfolios are managed for optimal expected returns.

There are many papers in the literature about optimal investment management of portfolios whose risky assets are modelled by GBM process with constant volatility as in the Black-Scholes model; these include (Chang *et al*, 2003; Deelstra *et al*, 2003; Cairns *et al*, 2006; Xu *et al*, 2007; Delong *et al*, 2008; Cortois *et al*, 2015; Njoku *et al*, 2017; Akpanibah *et al*, 2020).

. With the extension of the GBM model, another volatility model known as the constant elasticity of variance (CEV) model has been used by some authors to model the stock market prices. These include but not limited to Xiao *et al* (2007) and Gao (2009). Over the years, a good number of literature on the study of OIS for DC plans under stochastic interest rate where the interest rate was of Vasicek and affine structure have been published. They include but are not limited to (Boulier *et al*, 2001; Deelstra *et al*, 2003; Battocchio and .Menoncin, 2004; Zhang and Rong 2013; Njoku *et al*, 2017).

The studies of OIS with return of premium have been carried out in recent years aiming at protecting the rights of members and their families in case of mortality during the accumulation period. This was done by introducing the mortality force functions such as Abraham De Moivre model in (He and Liang, 2013; Sheng and Rong, 2014; Li *et al*, 2017) and Weillbull force function in Chávez, (2016) and Lai *et al.* (2021) into the wealth function. He and Liang (2013) and Sheng and Rong (2014), studied OIS with return of premium under GBM model and Heston volatility model respectively using the mean-variance utility. Li *et al*, (2017) and Wang *et al*, (2018), studied OIS with a return of premium under CEV model and Jump diffusion model respectively using the mean-variance utility. In each of the cases above, the mortality force function used was the Abraham De Moivre model. Chávez, (2016) and Lai *et al.* (2021) studied OIS with a return of premium using Exponential utility and mean-variance utility respectively when the mortality force function followed the Weibull process. Most recently, the OIS with a return of premium with predetermined interest have also been studied by Akpanibah and Osu (2018) and Akpanibah *et al*, (2020); in their work, they assumed the returned premiums are with predetermined interest from investment on risk-free asset.

So far, from the knowledge of the literature, no work combines the return clause of premium with interest during the accumulation phase and administrative charges under Weilbull mortality force function and mean-variance utility to obtain OIS. Hence, the main contribution of this very work is the introduction of a return clause of premium with predetermined interest rate and administrative charges into the work of Njoku and Akpanibah (2022) and He and Liang (2013). Also, the mortality force function used here is the Weilbull model different from the Abraham De Moivre model under GBM model. Furthermore, we maximize the members' portfolios by solving for an efficient and robust investment strategy for the risky asset and the efficient frontier.

2.0 Pension Wealth Formulation with Return Clause and Administrative Charges

Let $D_t(t)$ represents the price of the risk free asset and its price process follows the following dynamics

$$\frac{dD_t(t)}{D_t(t)} = rdt, D_t(0) = 1 \tag{1}$$

where $r > 0$ is the predetermined interest rate of the risk free asset.

Similarly, the pension fund administrator may also be willing to invest in a risky asset (stock) modelled by the geometric Brownian motion whose price process is given as follows

$$\frac{d\mathcal{S}_t(t)}{\mathcal{S}_t(t)} = \mu dt + \vartheta dM_t,\ \mathcal{S}_t(0) = s_0 \tag{2}$$

where $\mu$is the expected appreciation rate of $\mathcal{S}_t(t)$, ϑ is the volatility of the stock market price and M_t is the Brownian motion generating the available information in the market represented by $\mathcal{F}_t$ called the filtration in a complete probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$, where Ω, is a real space and $\mathcal{P}$ a probability measure.

Next, we consider $P(t)$ to be the fraction of the accumulated wealth to be invested in risky asset and $1 - P(t)$, the fraction to be invested in a risk-free asset.

Let a be the monthly contributions at a given time by the pension member, φ_0 the initial age during the accumulation phase, T the accumulation phase period,and $\varphi_0 + T$ is the terminal age of the member. R_{i,φ_0+t} is the mortality rate from time t to $t + i$, ta is the accumulated contributions at time t, taR_{i,φ_0+t} is the returned contributions to the dead members' families within the accumulation period and

$(1 - P(t))\mathcal{X}(t)\frac{D_{t+i}}{D_t}R_{i,\varphi_0+t}$ is returned interest from risk free assets during the accumulation period.

Corresponding to investment strategy $P(t)$ and the accumulation phase period $[t, t + i]$, the differential form associated with the fund size based on He and Liang (2013) is given as:

$$\mathcal{X}(t+i) = \begin{bmatrix} \mathcal{X}(t)\left(\left(1 - P(t)\right)\frac{D_{t+i}}{D_t} + P(t)\frac{\mathcal{S}_{t+i}}{\mathcal{S}_t}\right) + ai \\ -(1 - P(t))\mathcal{X}(t)\frac{D_{t+i}}{D_t}R_{i,\varphi_0+t} - taR_{i,\varphi_0+t} \end{bmatrix}\left(\frac{1}{1 - R_{i,\varphi_0+t}}\right) \tag{3}$$

$$\mathcal{X}(t+i) = \begin{bmatrix} \mathcal{X}(t)\begin{pmatrix} 1 + \left(1 - P(t)\right)\left(\frac{D_{t+i}}{D_t} - \frac{D_t}{D_t}\right)\left(1 - iR_{\varphi_0+t}\right) \\ +P(t)\left(\frac{\mathcal{S}_{t+i}}{\mathcal{S}_t} - \frac{\mathcal{S}_t}{\mathcal{S}_t}\right) \end{pmatrix} \\ +ai - (1 - P(t))\mathcal{X}(t)R_{i,\varphi_0+t} - taR_{i,\varphi_0+t} \end{bmatrix}\left(1 + \frac{R_{i,\varphi_0+t}}{1 - R_{i,\varphi_0+t}}\right) \tag{4}$$

$$\mathcal{X}(t+i) - \mathcal{X}(t) = \begin{bmatrix} \mathcal{X}(t)\begin{pmatrix} \left(1 - P(t)\right)\left(\frac{D_{t+i}}{D_t} - \frac{D_t}{D_t}\right)\left(1 - iR_{\varphi_0+t}\right) \\ +P(t)\left(\frac{\mathcal{S}_{t+i}}{\mathcal{S}_t} - \frac{\mathcal{S}_t}{\mathcal{S}_t}\right) \end{pmatrix} \\ +ai - (1 - P(t))\mathcal{X}(t)R_{i,\varphi_0+t} - taR_{i,\varphi_0+t} \end{bmatrix}\left(1 + \frac{R_{i,\varphi_0+t}}{1 - R_{i,\varphi_0+t}}\right) \tag{5}$$

$$\begin{cases} R_{i,\varphi_0+t} = 1 - \text{Exp}\{-\int_0^i A(\varphi_0+t+e)de\} = A(\varphi_0+t)i + O(i), \\ \frac{R_{i,\varphi_0+t}}{1-R_{i,\varphi_0+t}} = A(\varphi_0+t)i + O(i) \\ i \to 0, R_{i,\varphi_0+t} = A(\varphi_0+t)dt, \frac{R_{i,\varphi_0+t}}{1-R_{i,\varphi_0+t}} = A(\varphi_0+t)dt, \\ ai \to adt, \frac{D_{t+i}}{D_t} - \frac{D_t}{D_t} \to \frac{dD_t(t)}{V_t(t)}, \frac{\mathcal{S}_{t+i}-\mathcal{S}_t}{\mathcal{S}_t} \to \frac{d\mathcal{S}_t(t)}{\mathcal{S}_t(t)} \end{cases} \tag{6}$$

Substituting (2.6) into (2.5) we have

$$d\mathcal{X}(t) = \begin{bmatrix} \mathcal{X}(t)\begin{pmatrix} P(t)\frac{d\mathcal{S}_t(t)}{\mathcal{S}_t(t)} \\ +(1-A(\varphi_0+t)dt)(1-P(t))\frac{dD_t(t)}{D_t(t)} \end{pmatrix} \\ +adt - taA(\varphi_0+t)dt \\ -(1-P(t))\mathcal{X}(t)A(\varphi_0+t)dt \end{bmatrix}(1+A(\varphi_0+t)dt) \tag{7}$$

Where $A(t)$is the force function and ϑ is the maximal age of the life table and are related as follows according to equation (3) in He and Liang (2013).

$$A(t) = kt^n \quad 0 \le t < T$$

This implies that

$$A(\varphi_0+t) = k(\varphi_0+t)^n \tag{8}$$

Substituting (1), (2) and (8) into (7), we have

$$d\mathcal{X}(t) = \begin{bmatrix} \begin{Bmatrix} \mathcal{X}(t)\begin{pmatrix} P(t)(\mu - r + k(\varphi_0+t)^n) \\ +r \end{pmatrix} \\ +a(1-tk(\varphi_0+t)^n) \end{Bmatrix} dt + \mathcal{X}(t)P(t)\vartheta dM_t \\ \mathcal{X}(0) = x_0 \end{bmatrix} \tag{9}$$

Let ρ be a fee which is dependent on the value of the assets under the management of the pension fund administrators (PFA). According to Lai *et al*, (2021), the fee which is termed charge on balance is a percentage of the value of assets. Under this assumption, our optimal problem becomes

$$d\mathcal{X}(t) = \begin{bmatrix} \begin{Bmatrix} \mathcal{X}(t)\begin{pmatrix} P(t)(\mu - r + k(\varphi_0+t)^n) \\ +(r-\rho) \end{pmatrix} \\ +a(1-tk(\varphi_0+t)^n) \end{Bmatrix} dt + \mathcal{X}(t)P(t)\vartheta dM_t \\ \mathcal{X}(0) = x_0 \end{bmatrix} \tag{10}$$

3. 0 Maximization of Pension Wealth and Investment Strategy

In this section, the wealth function involving the return clause and the administrative charges in (10) will be maximized subject to mean-variance utility function as stated in Bjork and Murgoci (2010), given as

$$U(t,x) = \sup_P\left\{E_{t,x}\mathcal{X}^P(T) - Var_{t,x}\mathcal{X}^P(T)\right\} \quad , \tag{11}$$

Following the game theoretic approach in (He and Liang, 2013; Sheng and Rong, 2014; Li *et al*, 2017), the mean variance utility in (11) is similar to the following Markovian time inconsistent stochastic optimal control problem with value function $U(t,x)$

$$\begin{cases} V(t,x,P) = E_{t,x}[\mathcal{X}^P(T)] - \frac{\gamma}{2} Var_{t,x}[\mathcal{X}^P(T)] \\ = (E_{t,x}[\mathcal{X}^P(T)] - \frac{\gamma}{2}\left(E_{t,x}[\mathcal{X}^P(T)^2\right] - (E_{t,x}[\mathcal{X}^P(T)])^2)) \\ U(t,x) = \sup_P V(t,x,P) \end{cases} \tag{12}$$

Following He and Liang, (2013), the optimal investment strategyP^* satisfies:

$$U(t,x) = \sup_P V(t,x,P^*) \tag{13}$$

where q is the risk-aversion coefficient of the pension fund manager

Let $u^P(t,x) = E_{t,x}[\mathcal{X}^P(T)]$,$v^P(t,x) = E_{t,x}[\mathcal{X}^P(T)^2]$, then

$$U(t,x) \;\; = \sup_P e\big(t,x,\, u^P(t,x), v^P(t,x)\big),$$

where

$$e(t,x,u,v) = u - \frac{\gamma}{2}(u - v^2) \tag{14}$$

Theorem 3.1 (verification theorem) If there exists three real functions $E, f, G{:}[0,T] \times \Re \rightarrow \Re$ satisfying
the following EHJB equations:

$$\begin{cases} \sup_P \left\{ \begin{matrix} E_t - e_t + (E_x - e_x)\left[x\begin{pmatrix} P(t)(\mu - r + k(\varphi_0 + t)^n) \\ +(r-\rho) \end{pmatrix} \\ +a(1 - tk(\varphi_0 + t)^n) \right] \\ +\frac{1}{2}P^2x^2\vartheta^2(E_{xx} - \mathcal{N}_{xx}) \end{matrix} \right\} = 0 \\ E(T,x) = \; e(T,x,x^2) \end{cases} \tag{15}$$

where:

$$\mathcal{N}_{xx} = qF_x^2, \tag{16}$$

$$\begin{cases} \left\{ F_t + F_x\left[\begin{matrix} x\begin{pmatrix} P(t)(\mu - r + k(\varphi_0 + t)^n) \\ +(r-\rho) \end{pmatrix} \\ +a(1 - tk(\varphi_0 + t)^n) \end{matrix} \right] + \frac{1}{2}P^2x^2\vartheta^2F_{xx} \right\} = 0 \\ F(T,x) = \; x \end{cases} \tag{17}$$

$$\begin{cases} \left\{ G_t + G_x\left[\begin{matrix} x\begin{pmatrix} P(t)(\mu - r + k(\varphi_0 + t)^n) \\ +(r-\rho) \end{pmatrix} \\ +a(1 - tk(\varphi_0 + t)^n) \end{matrix} \right] + \frac{1}{2}P^2x^2\vartheta^2G_{xx} \right\} = 0 \\ G(T,x) = \; x^2 \end{cases} \tag{18}$$

Then$U(t,x) = E(t,x), u^{P^*} = F(t,x), v^{P^*} = G(t,x)$for the optimal investment strategy

P^*.

Proof: The details of the proof can be found in (He and Liang, 2009; Liang and Huang, 2011; Zeng and Li, 2011).

Proposition 1

The optimal investment strategy for the stock market price is given as

$$P(t)^* = \frac{(\mu - r + k(\varphi_0 + t)^n)e^{(r-\rho)(t-T)}}{\gamma x \vartheta^2}$$

Proof

Also, Recall from (14),

$f(t,x,u,v) = u - \frac{q}{2}(v - u^2)$

$$e_t = e_x = e_{xx} = e_{xu} = e_{xv} = e_{uv} = e_{vv} = 0,\ e_u = 1 + \gamma u,\ e_{uu} = \gamma,\ e_v = -\frac{\gamma}{2} \tag{19}$$

Substituting (19) into (15) and differentiating it with respect to $P(t)$, we have

$$P(t)^* = -\left[\frac{(\mu - r + k(\varphi_0 + t)^n)E_x}{(E_{xx} - \gamma F_x^2)x\vartheta^2}\right] \tag{20}$$

Substituting (20) into (15) and (17), we have

$$\begin{bmatrix} E_t + E_x \begin{bmatrix} (r-\rho)x \\ +a(1 - tk(\varphi_0 + t)^n) \end{bmatrix} \\ -\frac{(E_x)^2}{2(E_{xx} - \gamma F_x^2)}\left(\frac{(\mu - r + k(\varphi_0 + t)^n)^2}{\vartheta^2}\right) \end{bmatrix} = 0 \tag{21}$$

$$\begin{bmatrix} F_t + F_x \begin{bmatrix} (r-\rho)x \\ +a(1 - tk(\varphi_0 + t)^n) \end{bmatrix} \\ -\frac{E_x F_x}{(E_{xx} - \gamma F_x^2)}\left(\frac{(\mu - r + k(\varphi_0 + t)^n)^2}{\vartheta^2}\right) + \frac{F_{xx}}{2}\left[\frac{(\mu - r)^2(\mathcal{P}_x)^2}{(E_{xx} - \gamma F_x^2)\vartheta^2}\right] \end{bmatrix} = 0 \tag{22}$$

To solve equation (21) and (22), we suppose a solution for $E(t,x)$and $F(t,x)$as follows:

$$\begin{cases} E(t,x) = x l_1(t) + \frac{1}{\gamma} m_1(t),\ l_1(T) = 1, m_1(T) = 0, \\ E_t = l_{1t} x + \frac{m_{1t}}{\gamma}, E_x = l_1,\ E_{xx} = 0 \end{cases} \tag{23}$$

$$\begin{cases} F(t,x) = x l_2(t) + \frac{1}{\gamma} m_2(t),\ l_2(T) = 1, m_2(T) = 0, \\ F_t = l_{2t} x + \frac{m_{2t}}{\gamma}, F_x = l_2,\ F_{xx} = 0 \end{cases} \tag{24}$$

Substituting (23) and (24) into (21) and (22), we have:

$$\begin{cases} \begin{Bmatrix} l_{1t} x + \frac{m_{1t}}{\gamma} + l_1(t) \begin{bmatrix} (r-\rho)x \\ +a(1 - tk(\varphi_0 + t)^n) \end{bmatrix} \\ + \frac{l_1^2}{2\gamma l_2^2}\left(\frac{(\mu - r + k(\varphi_0 + t)^n)^2}{\vartheta^2}\right) \end{Bmatrix} = 0 \\ l_1(T) = 1, m_1(T) = 0, \end{cases} \tag{25}$$

$$\begin{cases}\left\{\begin{array}{c} l_{2t}x+\frac{m_{2t}}{\gamma}+l_2(t)\begin{bmatrix}(r-\rho)x\\ +a(1-tk(\varphi_0+t)^n)\end{bmatrix} \\ +\frac{l_1}{\gamma l_2}\left(\frac{(\mu-r+k(\varphi_0+t)^n)^2}{\vartheta^2}\right)\end{array}\right\}=0 \\ l_2(T)=1, m_2(T)=0\end{cases} \tag{26}$$

Simplifying (25) and (26), we have

$$\begin{cases} x(l_{1t}+(r-\rho)l_1)=0 \\ \frac{1}{\gamma}\left(m_{1t}+a(1-kt(\varphi_0+t)^n)l_1+\frac{l_1^2}{2\gamma l_2^2}\left(\frac{(\mu-r+k(\varphi_0+t)^n)^2}{\vartheta^2}\right)\right)=0\end{cases} \tag{27}$$

$$\begin{cases} x(l_{2t}+(r-\rho)l_2)=0 \\ \frac{1}{\gamma}\left(m_{2t}+a(1-kt(\varphi_0+t)^n)l_2+\frac{l_1}{\gamma l_2}\left(\frac{(\mu-r+k(\varphi_0+t)^n)^2}{\vartheta^2}\right)\right)=0\end{cases} \tag{28}$$

Since$x\neq 0$ and $\frac{1}{\gamma}\neq 0$, (27) and (28) can be written as;

$$\begin{cases} l_{1t}+(r-\rho)l_1=0 \\ l_1(T)=1\end{cases} \tag{29}$$

$$\begin{cases} m_{1t}+a(1-kt(\varphi_0+t)^n)l_1+\frac{l_1^2}{2\gamma l_2^2}\left(\frac{(\mu-r+k(\varphi_0+t)^n)^2}{\vartheta^2}\right)=0 \\ m_1(T)=0\end{cases} \tag{30}$$

$$\begin{cases} (l_{2t}+(r-\rho)l_2)=0 \\ l_2(T)=1\end{cases} \tag{31}$$

$$\begin{cases} m_{2t}+a(1-kt(\varphi_0+t)^n)l_2+\frac{l_1}{\gamma l_2}\left(\frac{(\mu-r+k(\varphi_0+t)^n)^2}{\vartheta^2}\right) \\ m_2(T)=0\end{cases} \tag{32}$$

Solving (29) – (32), we have the following solutions

$$l_1(t)=EXP[(r-\rho)(T-t)] \tag{33}$$

$$m_1(t)=\left(\begin{array}{c} a\int_t^T(1-kt(\varphi_0+\tau)^n)e^{(r-\rho)(T-\tau)}d\tau \\ +\frac{1}{2\gamma\vartheta^2}\begin{bmatrix}(\mu-r)^2(T-t)+\frac{2k(\mu-r)}{n+1}((\varphi_0+T)^{n+1}-(\varphi_0+t)^{n+1}) \\ +\frac{k^2}{2n+1}((\varphi_0+T)^{2n+1}-(\varphi_0+t)^{2n+1})\end{bmatrix}\end{array}\right) \tag{34}$$

$$l_2(t)=EXP[(r-\rho)(T-t)] \tag{35}$$

$$m_2(t)=\left(\begin{array}{c} a\int_t^T(1-kt(\varphi_0+\tau)^n)e^{(r-\rho)(T-\tau)}d\tau \\ +\frac{1}{\gamma\vartheta^2}\begin{bmatrix}(\mu-r)^2(T-t)+\frac{2k(\mu-r)}{n+1}((\varphi_0+T)^{n+1}-(\varphi_0+t)^{n+1}) \\ +\frac{k^2}{2n+1}((\varphi_0+T)^{2n+1}-(\varphi_0+t)^{2n+1})\end{bmatrix}\end{array}\right) \tag{36}$$

Next, we put (33) and (34) into (23). Similarly, we put (35) and (36) into (24), we will have

$$E(t,x) = \left(\begin{array}{c} xe^{(r-\rho)(T-t)} \\ + \left(\begin{array}{c} a\int_t^T (1-kt(\varphi_0+\tau)^n)e^{(r-\rho)(T-\tau)}d\tau \\ +\frac{1}{2\gamma\vartheta^2}\left[\begin{array}{c} (\mu-r)^2(T-t)+\frac{2k(\mu-r)}{n+1}((\varphi_0+T)^{n+1}-(\varphi_0+t)^{n+1}) \\ +\frac{k^2}{2n+1}((\varphi_0+T)^{2n+1}-(\varphi_0+t)^{2n+1}) \end{array} \right] \end{array} \right) \end{array} \right) \tag{37}$$

$$F(t,x) = \left(\begin{array}{c} xe^{(r-\rho)(T-t)} \\ + \left(\begin{array}{c} a\int_t^T (1-kt(\varphi_0+\tau)^n)e^{(r-\rho)(T-\tau)}d\tau \\ +\frac{1}{\gamma\vartheta^2}\left[\begin{array}{c} (\mu-r)^2(T-t)+\frac{2k(\mu-r)}{n+1}((\varphi_0+T)^{n+1}-(\varphi_0+t)^{n+1}) \\ +\frac{k^2}{2n+1}((\varphi_0+T)^{2n+1}-(\varphi_0+t)^{2n+1}) \end{array} \right] \end{array} \right) \end{array} \right) \tag{38}$$

Substituting E_x, E_{xx}, F_x, into (20) we have$P(t)^*$ which complete the proof.

Proposition 2

The efficient frontier of the pension fund is given as

$E_{t,x}[\mathcal{X}^P(t)] =$

$$\left(\begin{array}{c} xe^{(r-\rho)(T-t)} \\ + \left(\begin{array}{c} a\int_t^T (1-kt(\varphi_0+\tau)^n)e^{(r-\rho)(T-\tau)}d\tau \\ +\vartheta\sqrt{\left[\begin{array}{c} (\mu-r)^2(T-t)+\frac{2k(\mu-r)}{n+1}\begin{pmatrix} (\varphi_0+T)^{n+1} \\ -(\varphi_0+t)^{n+1} \end{pmatrix} \\ +\frac{k^2}{2n+1}\begin{pmatrix} (\varphi_0+T)^{2n+1} \\ -(\varphi_0+t)^{2n+1} \end{pmatrix} \end{array} \right] Var_{t,x}[\mathcal{X}^{P^*}(t)]} \end{array} \right) \end{array} \right) \tag{39}$$

Proof

Recall that $\mathcal{X}^{P^*}$

$$Var_{t,x}[\mathcal{X}^{P^*}(t)] = E_{t,x}[\mathcal{X}^{P^*}(t)^2] - (E_{t,x}[\mathcal{X}^{P^*}(t)])^2$$

$$Var_{t,x}[\mathcal{X}^{P^*}(t)] = \frac{2}{\gamma}(F(t,x)-E(t,x)) \tag{40}$$

Substituting (37) and (38) into (40), we have

$$Var_{t,x}[\mathcal{X}^{P^*}(t)] = \frac{1}{\vartheta^2\gamma^2}\left[\begin{array}{c} (\mu-r)^2(T-t)+\frac{2k(\mu-r)}{n+1}((\varphi_0+T)^{n+1}-(\varphi_0+t)^{n+1}) \\ +\frac{k^2}{2n+1}((\varphi_0+T)^{2n+1}-(\varphi_0+t)^{2n+1}) \end{array} \right] \tag{41}$$

$$\frac{1}{\gamma} = \vartheta\left(\sqrt{\frac{Var_{t,x}[\mathcal{X}^{P^*}(t)]}{\left[\begin{array}{c}(\mu-r)^2(T-t)+\frac{2k(\mu-r)}{n+1}\begin{pmatrix}(\varphi_0+T)^{n+1}\\-(\varphi_0+t)^{n+1}\end{pmatrix}\\+\frac{k^2}{2n+1}\begin{pmatrix}(\varphi_0+T)^{2n+1}\\-(\varphi_0+t)^{2n+1}\end{pmatrix}\end{array}\right]}}\right) \tag{42}$$

$$E_{t,x}[\mathcal{X}^{P^*}(t)] = F(t,x) \tag{43}$$

Substituting (38) into (43), we have

$$E_{t,x}[\mathcal{X}^{P^*}(t)] = \left(\begin{array}{c} xe^{(r-\rho)(T-t)} \\ +\left(\begin{array}{c} a\int_t^T(1-kt(\varphi_0+\tau)^n)e^{(r-\rho)(T-\tau)}d\tau \\ +\frac{1}{\gamma\vartheta^2}\left[\begin{array}{c}(\mu-r)^2(T-t)+\frac{2k(\mu-r)}{n+1}\begin{pmatrix}(\varphi_0+T)^{n+1}\\-(\varphi_0+t)^{n+1}\end{pmatrix}\\+\frac{k^2}{2n+1}\begin{pmatrix}(\varphi_0+T)^{2n+1}\\-(\varphi_0+t)^{2n+1}\end{pmatrix}\end{array}\right]\end{array}\right)\end{array}\right) \tag{44}$$

Substitute (42) in (44), we have (39).

4. 0 Numerical Simulations and Discussion

In this section, we present some numerical simulations to illustrate the impact of some sensitive parameters on the optimal strategy. To achieve this, proposition 1 and 2 were used as our focus equations with the following parameters: $\mu = 0.2$, $r = 0.1$, $x = 1$, $\vartheta = 0.6$, $\rho = 0.05$, $\varphi_0 = 20$, $T = 40$, $k = 0.01$, $n = 0.001$, $\gamma = 0.05$

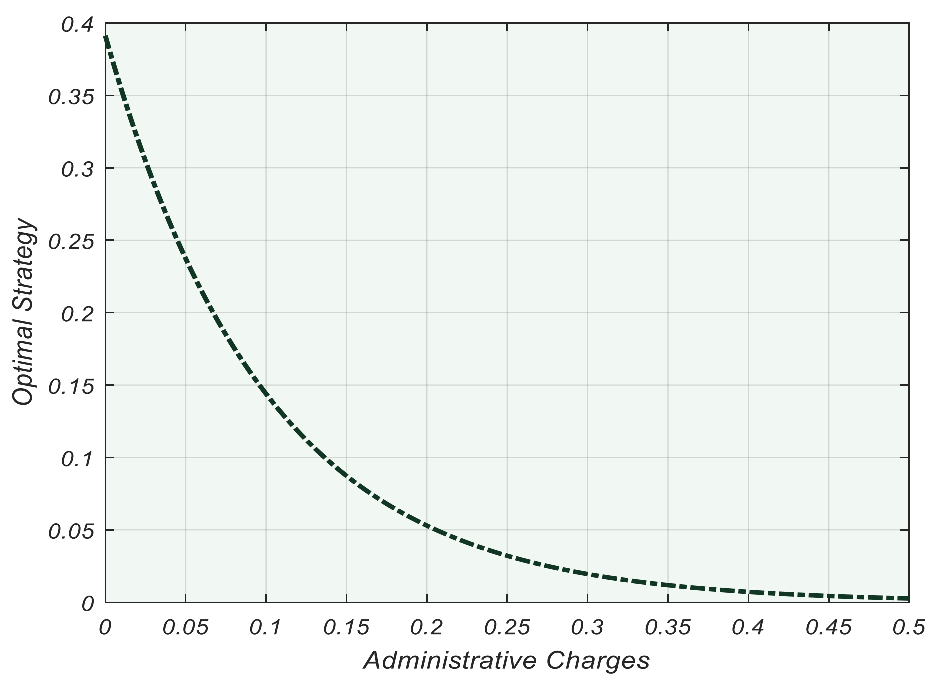

Fig. 1 Impact of proportional administrative charges on investment strategy

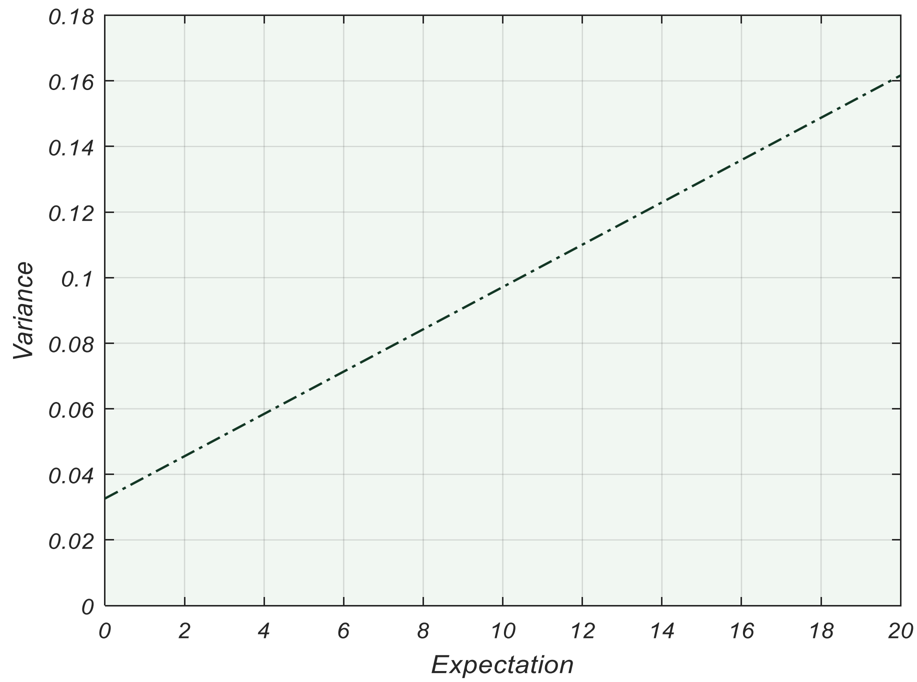

Fig. 2 Relationship between variance and expectation

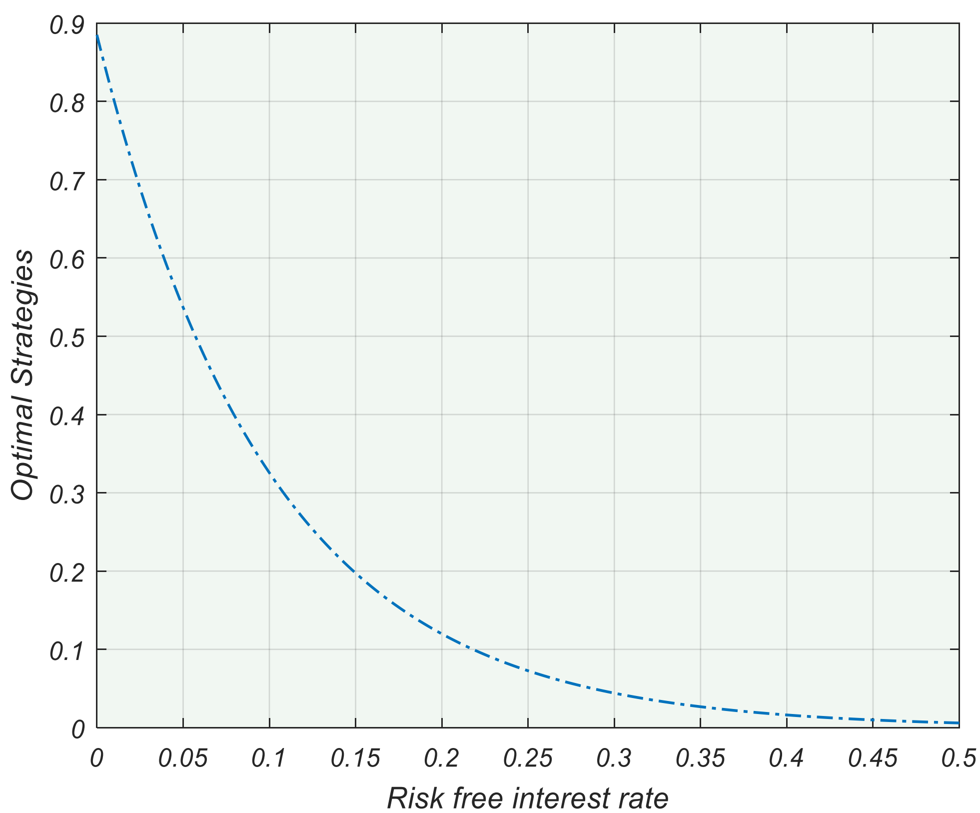

Fig 3 Impact of risk free interest rate of investment strategy

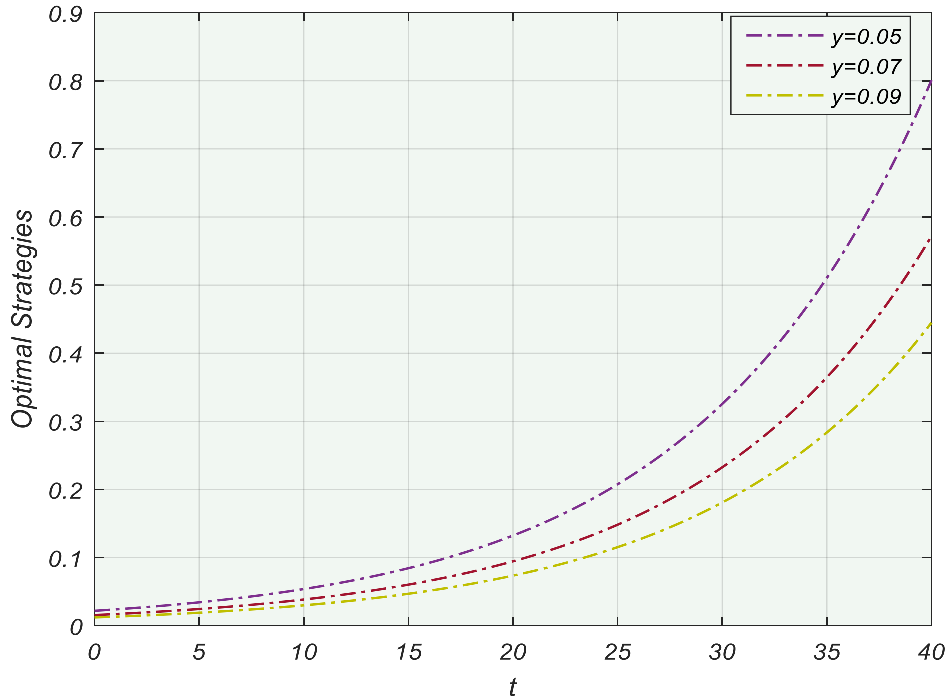

Fig. 4 Impact of risk averse coefficient on investment strategy

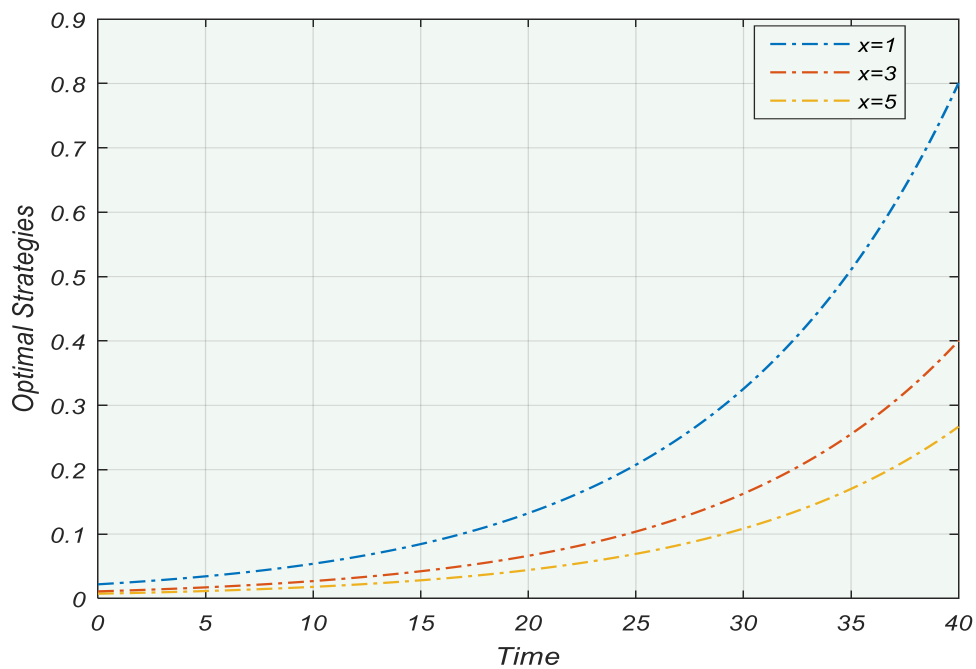

Fig 5 Impact of Initial fund size on investment strategy

Fig.1, presents the relationship between the optimal investment strategy and the proportional administrative charges. It was observed that the optimal investment strategy developed by the fund administratorsis a decreasing function of the proportional administrative charges. The implication of the graph in Fig.1 is that, the higher the administrative charges on investment of the risky asset, the more likelihood for the members of the scheme to be discouraged from investing more in risky assets and may invest more if otherwise

From Fig. 2, the graph presents the relationship between the variance and expectation at the end of the accumulation phase. It describes an efficient frontier which gives a relationship between the expectation and the variance. It is observed that the risk involved in investing in the risky assets is an increasing function of the members' expectations; the consequence of this is that members with a higher proportion of investment in risky assets have higher probabilities of having more returns at the end of the investment period and vice versa.

Also, Fig. 3 presents a graph of the optimal investment strategy against the risk-free interest rate. It is observed that the optimal investment strategy is a decreasing function of the risk-free interest rate. This simply indicates that members will likely want to invest in risky assets when the interest rate from the risk-free asset is not attractive. However, if the risk-free interest rate is attractive enough, PPM members may be advised by their fund administrators to invest more in the risk-free asset, thereby reducing their investment in the risky asset.

Fig. 4, presents the impact of the risk-averse coefficient on the investment strategy and we observed that the optimal control strategy for the risky asset is inversely proportional to the risk aversion coefficient parameter. What we deduced from the graph in Fig. 4 is that members with a higher risk aversion coefficient may invest a lesser percentage of their wealth in the risky asset (stock) while members with a lower risk aversion coefficient may invest a higher percentage of their wealth in the risky assets while reducing investment in the risk-free asset.

Fig. 5, presents the impact of the initial fund size on the optimal investment strategy. It was observed that the optimal investment strategy for the stock market price is a decreasing function of the initial fund size parameter of the pension member. The implication of Fig. 5 is that if the initial fund size at the time of investment is much, members may be discouraged from taking more risks thereby reducing the proportion of their wealth to be invested in the risky asset and may invest more if otherwise.

5.0 Conclusion

In conclusion, this paper investigated how investment in a defined contributory (DC) pension fund system with a return clause of premium and proportional administrative charges are determined and the factors to be considered while venturing into investment in a risky asset. It also helps in determining the proportion of members' wealth required to be invested in the two assets under consideration during the accumulation period considering the mortality risk involved. To achieve this, an investment portfolio with a risk-free asset and a risky asset which follows the GMB model was considered and the returned premium was with interest from an investment in a risk-free asset. Also, the Weibull force function was used to describe the mortality risk of members during the accumulation phase. Furthermore, the game-theoretic technique is applied to obtain an optimization problem from the extended Hamilton Jacobi Bellman equation. By using the mean-variance utility and variable separation technique, an investment strategy (IS) is obtained for the risky asset comprising of the risk-free interest rate, instantaneous volatility, administrative charges, appreciation rate of the risky asset and the mortality force function together with the efficient frontier which gives the relationship between the investment expectation and the risk involvement in the investment. Furthermore, some numerical simulations were obtained to study the impact of some sensitive parameters of the IS. It was observed that the administrative charges and the mortality rate affect the IS to be adopted. Therefore, an insight into how these parameters behave is essential in the development of an IS.

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Declarations

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable.

Availability of data and materials

The publisher has the right to make the data public.

Competing interests

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All aspects of the work was carried out by the author.