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The Generalized Odd Generalized Exponential Gompertz Distribution with Applications

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Abstract: *This study introduces the generalized odd generalized exponential Gompertz distribution. Some mathematical properties were derived, which increase the subject distribution's flexibility by allowing the model to be used for the analysis of a variety of lifetime data types with hazard rates that are monotonic, upside-down, and bathtub-shaped. The new model includes two additional parameters as a mixture of the odd exponential-G family of distributions. The observed Fisher's information matrix was constructed, and the model parameters were estimated using the maximum likelihood method. Through a numerical analysis using actual data, we demonstrate the effectiveness of the suggested model.*

Keywords: *Gompertz distribution; hazard function; moments; maximum likelihood estimation; odd generalized function, T-X family of distributions*

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1.0 Introduction

In the last decades, many researchers have developed several generalized families of continuous distributions that are very flexible, such as: (El-Gohary *et al.* 2013; Tahir *et al.* 2015; El-Damcese *et al.* 2015) and so on. An existing probability model can be extended into a bigger family of distributions by adding extra shape parameter(s), to the baseline distributions, which performs better results for the new distribution that is considerably skewed and has a heavy-tailed probability distribution to fit a real-life data sets. As a result, some generalization techniques were used, and some of the results have come under increased scrutiny in recent years. For examples, see (Cordeiro, 2011; Ramos *et al.* 2013; Bourguignon *et al.* 2014; Nadarajah *et al.* 2015; Afify *et al.* 2016; El-Bassiouny *et al.* 2017; Hamedani *et al.* 2018; Boshi *et al.* 2019; Aminu and Fidelis, 2022; Basu and Kundu, 2023). and many more. These new distributions all share the trait of having extra parameters. According to Saboor *et al.* (2015), the induction of parameter(s) has been effective for examining tail properties as well as for enhancing the goodness-of-fit of the suggested generator family.

In this paper, we introduced three additional parameters as a mixture of the odd exponentiated G family of distributions to come up with a new model from the G-family of distributions

2. 0 The GOGEG Distributions

Given a baseline continuous distribution with parameters $\alpha, \lambda, \beta, \theta$ as

$$F(x) = 1 - H(-\ln G) \tag{2}$$

and the pdf as

$$f(x) = \frac{d}{dx}\left[F(x)\right] = -\frac{d}{dx}\left[R(x)\right]$$

$$1 - F(x) = R(x) = \int_0^{-\ln G} h(x)dx = H(-\ln G) \tag{1}$$

where H and h are the cdf and pdf of the distribution, respectively. From (2.1), we obtain the general formula of cdf for the new family of distributions as

which implies that

$$f(x) = \frac{g}{G} h(-\ln G) \tag{3}$$

Now, suppose that $H(-lnG)$ and $h(-lnG)$ are the cdf and pdf of the GOGEG with the parameters a, b, c, d and are defined as

$$H(-\ln G; a,b,c,d) = \left\{1 - e^{-a\left[e^{\frac{b}{c}\left(e^{c(-\ln G)}-1\right)}-1\right]}\right\}^{d} \tag{4}$$

$$h(-\ln G; a,b,c,d) = abde^{c(-\ln G)} e^{\frac{b}{c}\left(e^{c(-\ln G)}-1\right)} e^{-a\left[e^{\frac{b}{c}\left(e^{c(-\ln G)}-1\right)}-1\right]} \left\{1 - e^{-a\left[e^{\frac{b}{c}\left(e^{c(-\ln G)}-1\right)}-1\right]}\right\}^{d-1} \tag{5}$$

After substituting (4) and (5) in (2) and (3), respectively, we obtain the following new family of continuous distributions based on the interval$(0, \infty)$. The cdf of the new distribution GOGEG is given as

$$F(x) = \left[1 - e^{\frac{b}{c}} \left\{1 - e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d} \tag{6}$$

and its corresponding pdf is given as

$$f(x) = \alpha\beta\theta de^{\lambda x} e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)} e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]} \left\{1 - e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta-1} e^{\frac{b}{c}} \left\{1 - e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta} \left[1 - e^{\frac{b}{c}} \left\{1 - e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d-1} \tag{7}$$

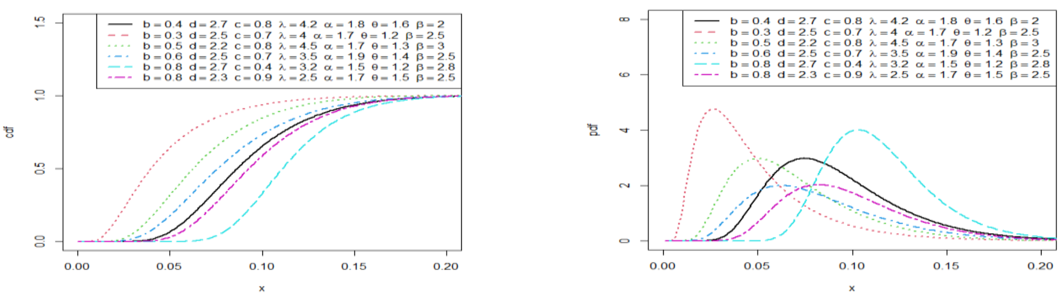

Fig. 1: The CDF and PDF for some values of the parameters of (GOGEG) distribution

The reliability function of GOGEG distribution is given by

$$R(x)=1-\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d} \tag{8}$$

The hazard rate function, $\tau(x)=\frac{f(x)}{R(x)}$, of GOGEG distribution is given by

$$\tau(x)=\frac{\alpha\beta\theta d e^{\lambda x}e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta-1}e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d-1}}{1-\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d}} \tag{9}$$

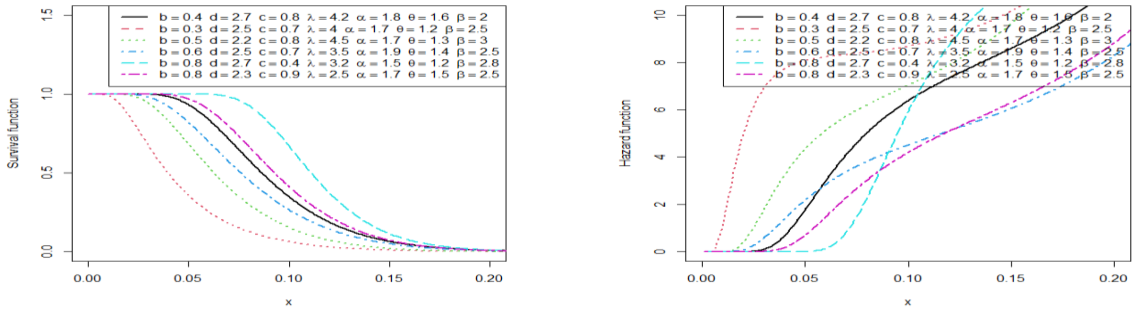

Fig. 2: The Survival function and Hazard function for some values of the parameters of (GOGEG) distribution

2.1 Expansion of density

In this section, we provide an expansion for (7) using the generalized binomial expansion given as

$$f(x)=\alpha bdgG^{\theta-1}\sum_{p-0}^{\infty}\frac{(-1)^{p}}{p!}\left(\frac{b}{c}\right)^{p}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d-1} \tag{10}$$

since

$$(1-z)^{-p}=\sum_{j=0}^{\infty}\frac{\Gamma(p+j)}{j!\Gamma(p)}z^{j} \text{ with } /z/<1\,\&\, p>0$$

$$(1-z)^{\theta}=\sum_{m=0}^{\infty}\frac{(-1)^{m}}{m!}\frac{\Gamma(\theta-1)}{j!\Gamma(m+1)}z^{m} \text{ with } /z/<1\,\&\,\theta>0$$

$$f(x)=\alpha bdgG^{\theta-1}\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\left(\frac{b}{c}\right)^{i}\sum_{m,i=0}^{\infty}\sum_{j=0}^{i}d_{j}^{i}\frac{(-1)^{m+2i-1}}{m!i!}\frac{\Gamma(d)}{\Gamma(d-m)}\left(\frac{bm}{c}\right)^{i}G^{j\theta}$$

Hence, if $d-1>0$ the expansion formula for the pdf of GOGEG distributions is given as

$$f(x)=\sum_{p=0}^{\infty}\sum_{t=0}^{p}\sum_{m,i=0}^{\infty}\sum_{j=0}^{i}d_{t}^{p}d_{j}^{i}\frac{(-1)^{2p+t+m+2i-1}}{p!m!i!}bd\left(\frac{b}{c}\right)^{p+i}m^{i}\frac{\Gamma(d)}{\Gamma(d-m)}gG^{\theta(t+j+1)-1} \tag{11}$$

3. 0 Properties of the GOGEG distribution

In this section, the statistical properties of GOGEG distribution are derived and studied.

3.1 Quantile and median

By inverting equation (6), the quantile function of the GOGEG distribution is derived as:

Preposition 1: The 100 q^{th} quantile is defined as the solution of the following equation, with respect to X_q, where $x_q > 0$ and $0 < q < 1$.

$$X_q = \frac{1}{\lambda} In\left\{1+\frac{c}{b} In\left[1+\frac{\lambda}{\beta} In\left[1-\frac{1}{\alpha} In\left(1-q^{1/d}\right)\right]^{1/\theta}\right]\right\} \tag{12}$$

The median of the GOGEG distributions can be obtained by setting q = 0.5 in (12) as:

$$Med = \frac{1}{\lambda} In\left\{1+\frac{c}{b} In\left[1+\frac{\lambda}{\beta} In\left[1-\frac{1}{\alpha} In\left(1-(0.5)^{1/d}\right)\right]^{1/\theta}\right]\right\} \tag{13}$$

3.2 Moment

Let X be a random variable which follows the GOGEG distribution, then the moment of the distribution is as stated in theorem 1 below.

Preposition 2: If X is a random variable distributed according to GOGEG distribution in (7). Then, the r^{th} moment of the distribution can be derived as follows:

$$\mu^r = \sum_{p=0}^{\infty}\sum_{t=0}^{p}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{p_1=0}^{\infty}\sum_{j_1=0}^{\infty} d_t^p d_j^i d_j^{\theta(t+j+1)-1} \frac{(-1)^{2p+t+m+2i-1}}{p!m!i!}\alpha\beta\theta d\left(\frac{b}{c}\right)^{p+i} m^i \frac{\Gamma(d)}{\Gamma(d-m)}\frac{\Gamma(r+1)}{\Gamma(p_1+1)}\frac{e^{\frac{\beta}{\lambda}}(j_1+1)}{\frac{\beta}{\lambda}(j_1+1)^{-p_1}}\left[\frac{-1}{\lambda(p_1+1)}\right]^{r+1} \tag{14}$$

setting r = 1, we have the mean of x.

Depending on $E(X^r)$; $(r = 1,2,3,4)$, gives the properties of this distribution such as the:

Mean $(\mu = E(X))$, Variance $(V(X) = \sigma^2 = E(X^2) - [E(X)]^2)$,

Coefficient of Skewness $\left(CS = \frac{E(X-\mu)^3}{\sigma^3} = \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3)}{\sigma^3}\right)$ and

Coefficient of Kurtosis $\left(CK = \frac{E(X-\mu)^4}{\sigma^4} = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4)}{\sigma^4}\right)$, can be obtained.

3.3 Shannon Entropy

The Shannon entropy of a random variable X denoted by SH is defined as

$$SH = -\int_0^{\infty} Inf(x) f(x)dx \tag{15}$$

Proof: By taking the natural logarithm to Equation (7), we get

$$Inf(x) = Inbd + Ing - (c+1) InG - \frac{b}{c}\left(G^{-c}-1\right) + (d-1) In\left(1-e^{-a\left[e^{\frac{b}{c}\left(e^{c(-\ln G)}-1\right)}-1\right]}\right) \tag{16}$$

Rewrite SH as

$$SH = -\left\{Inbd + I_1 + (\theta-1) I_2 - \frac{b}{c} I_3 + (d-1) I_4\right\} \tag{17}$$

Now, the Shannon entropy of the GOGEG distribution is given by:

$$SH = -\left\{\begin{array}{l} Inbd + \alpha\beta\theta + \lambda E(X) - \frac{\beta}{\lambda}\left(\sum_{i=0}^{\infty}\frac{\lambda!}{i!}E(X^i)-1\right) + (\theta-1)\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{i}\sum_{t=0}^{\infty} d_p^i \frac{(-1)^{2i-p+1}}{ni!}\left(\frac{n\alpha\beta}{\lambda}\right)^i\left(\frac{p\lambda}{t}\right)^t E(X^t) \\ -\frac{b}{c}\sum_{j}^{\infty}\sum_{j_1}^{\infty}\sum_{j_2}^{j_1}\sum_{j_3}^{\infty} d_{j_2}^{j_1}\frac{(-1)^{2j_1-j_2}}{j_1!}\frac{\Gamma(\theta-1)}{j_1!\Gamma(\theta)}\left(\frac{\alpha\beta j}{\lambda}\right)^{j_1}\left(\frac{j_2\lambda}{j_3}\right)^{j_1} E(X^{j_3}) \\ +(d-1)\sum_{n,t_1=0}^{\infty}\sum_{t_2}^{t_1}\sum_{j}^{\infty}\sum_{j_1}^{\infty}\sum_{j_2}^{j_1}\sum_{j_3}^{\infty} d_{t_2}^{t_1} d_{j_2}^{j_1}\frac{(-1)^{2t_1-t_2+2j_1-j_2+1}}{j_1!}\left(\frac{bn}{c}\right)^{t_1}\frac{\Gamma(\theta t_2+j)}{j_1!\Gamma(\theta t_2)}\left(\frac{\beta j}{\lambda}\right)^{j_1} E(X^{j_3}) \end{array}\right\} \tag{18}$$

4.0 Order Statistics

Suppose $X_1, X_2, ..., X_n$ is a random sample of size n from GOGEG distribution with *cdf* $F\left(x;\alpha,\beta,\theta,\lambda,b,c,d\right)$ and *pdf* $f\left(x;\alpha,\beta,\theta,\lambda,b,c,d\right)$ given by (6) and (7) respectively. Let $X_{1:n} \leq X_{2:n} \leq, ..., X_{n::n}$ denote the order statistics obtained from this sample. The probability density function of $X_{r:n}$ is given by

$$F_{r:n}\left(x;\xi\right) = \frac{n!}{(r-1)!(n-r)!} f\left(x;\xi\right)\left[F\left(x;\xi\right)\right]^{r-1}\left[1-F\left(x;\xi\right)\right]^{n-r} \qquad (19)$$

where $\xi = \left(x;\alpha,\beta,\theta,\lambda,b,c,d\right)$, $f\left(x;\xi\right)$ and $F\left(x;\xi\right)$ are the pdf and cdf of GOGEG distribution. However, since the limiting values of $\left[1-F\left(x;\xi\right)\right]^{n-r}$ as $x \to 0$ and as $x \to \infty$ is between $\left(0,1\right)$, the pdf of the i^{th} order statistics can be written as:

$$f_{r:n}\left(x;\xi\right) = \frac{n!}{(r-1)!(n-r)!}\alpha\beta\theta d e^{\lambda x} e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)} e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta-1} e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d-1}$$

$$\times\left[\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d}\right]^{r-1}\left[1-\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d}\right]^{n-r} \qquad (20)$$

5.0 Estimation of Parameters

In this section, the derivation of unknown parameters $\left(x;\alpha,\beta,\theta,\lambda,b,c,d\right)$ of the GOGEG distribution will be estimated using the Maximum Likelihood Estimators.

5.1 Maximum likelihood estimators

Proposition 3: Let $X_1, X_2, ..., X_n$ be a random sample of size n from GOGEG, then the likelihood function $\mathcal{L}$ of this sample is defined as:

$$L = \prod_{i=1}^{n} f\left(x;\alpha,\beta,\theta,\lambda,b,c,d\right) \qquad (21)$$

Substituting from equation (7) into (21), we get

$$L = \prod_{i=1}^{n}\alpha\beta\theta b d e^{\lambda x} e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)} e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta-1} e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]^{d-1} \qquad (22)$$

The log-likelihood function is given as follows

$$L = nIn(\alpha) + nIn(\beta) + nIn(\theta) + nIn(b) + nIn(d) + \lambda\sum_{i=0}^{n} x + \frac{\beta}{\lambda}\sum_{i=0}^{n}\left(e^{\lambda x}-1\right) - \alpha\sum_{i=0}^{n}\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]$$

$$+(\theta-1)\sum_{i=0}^{n} In\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\} + \frac{b}{c}\sum_{i=0}^{n} In\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta} + (d-1)\sum_{i=0}^{n} In\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]. \qquad (23)$$

To obtain the maximum likelihood estimates of the parameters $\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{b}, \hat{c}, \hat{d}$ we differentiate (23) partially for the parameters $\alpha, \beta, \theta, \lambda, b, c, d$. This gives:

$$\frac{\delta L}{\delta d} = \frac{n}{d} + \sum_{i=0}^{n} In\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right], \qquad (24)$$

$$\frac{\delta L}{\delta\theta}=\frac{n}{\theta}+\sum_{i=0}^{n}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}+\frac{\theta b}{c}\sum_{i=0}^{n}In\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}+\left(d-1\right)\sum_{i=0}^{n}\frac{\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]}{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}}, \quad (25)$$

$$\frac{\delta L}{\delta\alpha}=\frac{n}{\alpha}-\sum_{i=0}^{n}\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]+\left(\theta-1\right)\sum_{i=0}^{n}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}+\frac{b}{c}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}}{e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}}+\left(d-1\right)\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]}{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}}, \quad (26)$$

$$\frac{\delta L}{\delta\beta}=\frac{n}{\beta}+\frac{1}{\lambda}\sum_{i=0}^{n}\left(e^{\lambda x}-1\right)-\frac{\alpha}{\lambda}\sum_{i=0}^{n}\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]+\frac{\alpha\left(\theta-1\right)}{\lambda}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}}{\left(e^{\lambda x}-1\right)}$$

$$+\frac{b}{\lambda c}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}}{e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}}+\frac{\left(d-1\right)}{\lambda}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]}{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}}, \quad (27)$$

$$\frac{\delta L}{\delta\lambda}=\sum_{i=0}^{n}x-\sum_{i=0}^{n}\frac{\beta}{\lambda^{2}}\left(e^{\lambda x}-1\right)+\frac{\beta}{\lambda}xe^{\lambda x}-\frac{\alpha}{\lambda^{2}}\sum_{i=0}^{n}\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]+\frac{\alpha\left(\theta-1\right)}{\lambda^{2}}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}}{\left(e^{\lambda x}-1\right)}$$

$$+\frac{b}{\lambda^{2}c}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}}{e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}}+\frac{\left(d-1\right)}{\lambda^{2}}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]}{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}}, \quad (28)$$

$$\frac{\delta L}{\delta b}=\frac{n}{b}+\frac{\alpha\left(\theta-1\right)}{c}\sum_{i=0}^{n}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}+\frac{\alpha\left(d-1\right)}{c}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]}{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}}, \quad (29)$$

$$\frac{\delta L}{\delta c}=\frac{b}{c^2}\alpha(\theta-1)\sum_{i=0}^{n}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}+\frac{\alpha(d-1)}{c^2}\sum_{i=0}^{n}\frac{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}\left[1-e^{\frac{b}{c}}\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}^{\theta}\right]}{\left\{1-e^{-\alpha\left[e^{\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}-1\right]}\right\}}. \quad (30)$$

Hence, the solution to the above non-linear equations (24) - (30) cannot be solved analytically, numerical solution is applied using Newton-Raphson technique.

6. 0 Results and Discussion

To illustrate the flexibility of the GOGEG distribution introduced in (14). The goodness-of-fit of GOGEG distribution and its four sub-models such as Lindley (L), Exponentiated Exponential (EE), Beta Gompertz (BG), Generalized Gompertz (GG), distributions to a real data set. The model comparison would be based on the minimized log-likelihood estimate and the following information statistics: Akaike information criterion (AIC) and Bayesian information criterion (BIC). The model with the smallest minimized log-likelihood and information statistics value is the best. This data consists of the lifetimes of fifty (50) devices given by Aarset, (1987). The data have a bathtub-shaped failure rate function and is given as follows:

0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Table 1: The models' MLEs and performance requirements based on Aarset data set

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	**AIC**	**BIC**
GOGEG	0.274	0.021	0.346	0.016	0.237	0.870	0.017	182.360	192.341
L	-	-	0.499	-	-	-	-	256.668	257.957
EE	-	-	-	0.370	-	-	-	277.591	281.178
BG	0.221	0.214	-	-	-	-	-	251.344	254.992
GG	-	-	-	0.012		0.038		241.5420	244.7917

In this study, we proposed GOGEG distribution. Table 1 presents the estimated values for each parameter of the GOGEG distribution and that of its four sub-models. The model with smaller values of these statistics is thought to be the best model, as was previously mentioned. Hence, according to Table 1, the GOGEG distribution has the lowest estimates for the AIC and BIC tests. As a result, the OGEE-G distribution offers the best fit for this data and is a very strong competitor to other distributions. Thus, the GOGEG distribution may be selected as the ideal model in this article.

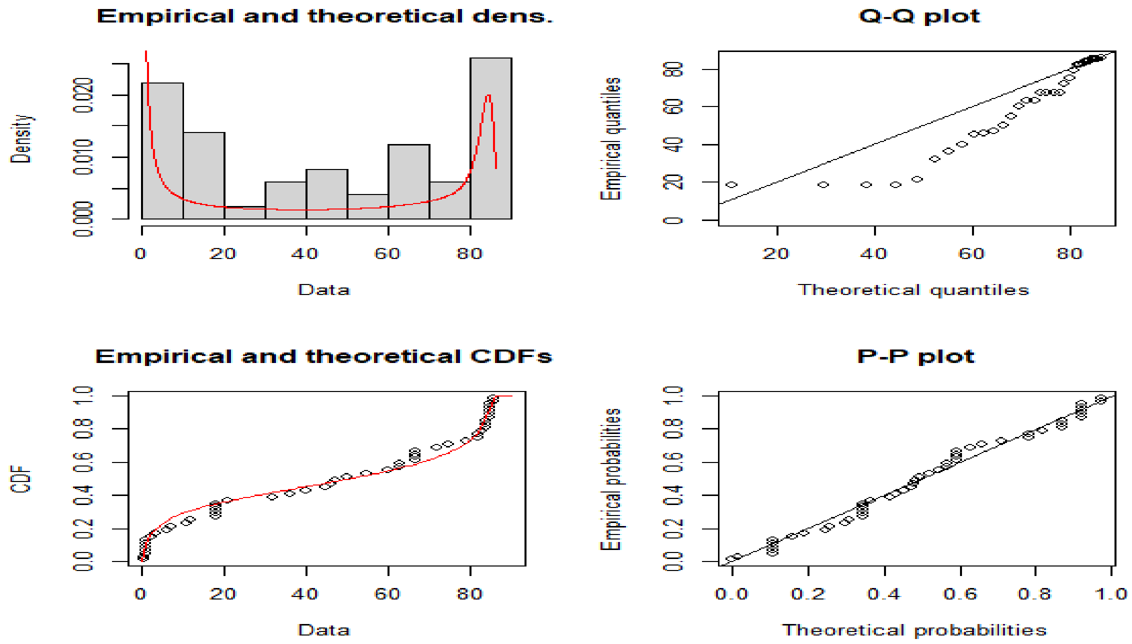

Fig. 3: Density plots for Aarset data set

7. 0 Conclusion

In this study, the Generalized Odd Generalized Exponential Gompertz (GOGEG) distribution, is a new family of continuous distributions, and its many features are examined. This distribution's density expansion and some of its statistical characteristics have been deduced and analysed. Closed forms are used to derive the GOGEG quantile, median, moment, mean, variance, coefficient of skewness, coefficient of kurtosis, and Shannon entropies and pdf of order statistics are derived. Using the maximum likelihood method, we obtained the performance of the distribution. We compare the GOGEG with other well-known distributions. Applications on sets of real data revealed that the GOGEG distribution has the best fits compared to other existing models.

8.0 References

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Compliance with Ethical Standards Declarations

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data Public.

Competing interests

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