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The Modeling of the Worth of an Asset Using a Skew Random Pricing Tree

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Abstract: *The binomial formula given by Cox, Ross and Rubinstein (1979) is a tool for evaluating the call option price. It is well known that the price from the binomial formula converges to the price from the Black-Scholes formula, which was given, by Black, Scholes and Merton (1973) as the number of periods (n) converges to infinity. In this paper, however, a formula for the worth of the expected returns of options and stock according to risk characteristics is derived. The knowledge of the binomial method of option pricing as well as the tree is applied herein in calculating the fair value of options. At each node of the tree, two possible outcomes are considered: an increase in the price of the underlying asset and a decrease in the price of the underlying asset. A sensitivity analysis worth of options is carried out at each node when it is affected by some policies.*

Keywords: *Worth of stock,; Binomial Tree; Option pricing.*

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1.0 Introduction

An option is one of the widely and most commonly used financial products in the market that allows you to trade the price of the market in the future, bond futures, currency futures, commodity equities, and interest rate futures, Lin (2022). European options and American options are the two dominant styles of options in the market. There are two types of options namely the PUT option (here the buyer gains when the price of an asset goes down) and the CALL option (the buyer gains if the price of an asset goes up). These options give the seller or buyer the right but not an obligation to sell or buy the underlying asset.

In trading options, four things are involved: buying, selling a call option, and buying, and selling a put option. Options allow investors to predict or hedge against the risk of an underlying asset.

When there is a relative price performance of two or more assets a Rainbow option is employed where the holder has the right to buy or sell the best or worst of two securities, or options that pay the best or worst of the two assets. Rainbow options, first proposed by Margrabe (1978), are great financial tools to hedge risk if a buyer has multiple assets on hand. Rainbow option exists in different forms;1) the better-off options where the value depends on the largest value or reward among several target assets. Mathematically better off is written as $C = Max(R_1 - R_2 \dots R_4, 0)$,2). Out performance option depends on the difference in the performance of the two underlying assets Mathematically, it can be expressed as $C = Max\,(R_1 - R_2, 0)$

The Black-Scholes model is applied in rainbow options to break down financial assets and derivatives into a set of differential equations, Margrabe (1978). Myron Scholes and Fischer Black, American economists in the year 1973, first proposed this. The B-S model was referred to as the second Revolution on Wall Street. Hull and White (1987) gave an account of the first known instance of an option contract, depending on the nature of the problems they can be modelled for easier solutions using the trees and rainbows option through different equations.

Some randomness exists and can be allowed into stochastic differential equations,. Also, environmental effects are allowed into some of the coefficients of differential equations, then a more realistic mathematical model of the problem can be obtained (Osu and Amadi, 2022).

Investors always invest intending to maximize profit and improve their standards of living. To do this they need to research option pricing to suit their performance.

Moreover, financial derivative securities need to be evaluated. Rainbow options also contain some vanilla options.

The binomial method of option pricing is a well-known mathematical model for calculating the fair value of options. It was created in (1979) by Cox, Ross, and Rubinstein and is based on the assumption that the underlying asset's price follows a binomial distribution over time. A binomial tree is constructed in this method to represent the possible price movements of the underlying asset during the option's lifetime. At each node of the tree, two possible outcomes are considered: an increase in the price of the underlying asset and a decrease in the price of the underlying asset. Chendra *et al.* (2018) presented a modified binomial method for pricing a more common arithmetic Asian option, by taking the average asset between two-time intervals within the period of the option validity. The obtained averages are then used for pricing the option. If the distance between the two time periods is widened by shifting the second period to the right for Asian options with the average strike, the price falls. However, if the distance is increased by shifting the first period to the left, the prices rise. Shvimer and Herbon (2020) empirically tested several binomial models on the S&P500 Index for traceability, proximity to market prices, and profitability, particularly near expiration day.

The binomial tree helps to stimulate the continuous movement of asset prices. Here mean and variance are matched to yield the necessary parameters, which are used to compute the option price backwards from the end of the binomial tree (Mohamad.2011).

The tree works by using the formula for a single period call option; it can be expanded to a two-period call option up to n–a period call option. Then n-period look-alike options using the binomial tree pricing formula is dividing the option effective period T into a small interval as Δt and at each Δt the stock price changes from S to S_u or S_d. In this case, if the possibility of an upward price movement is P then the possibility of the downward movement is $1 - P$. We find the formula by using a risk-neutral pricing principle because the rate of change in the underlying asset follows a normal distribution. The binomial tree option pricing model has already established itself as one of the primary pricing standards for major stock exchanges worldwide.

The notion of option pricing is one of the bases of contemporary finance options are very important especially as they price rights and obligations separately, Hong (2023). It enables the buyer to both prevent negative consequences and profit from positive outcomes.

One needs to properly value options to manage risk effectively as one of the most important uses of options.

Option pricing has been applied to many financial and non-financial fields including the pricing of various derivative securities, and corporate investment decisions. Option pricing helps investors with reasonable option prices and optimal implementation periods to make maximum profits with little investment (Hong, 2023).

The expiration date formula of the call option was derived by Bachelier (2011). He connected heat conduction equations through the Gaussian probability density function and Brownian motion.

Thus one has

$$c(s,t) = SN\left(\frac{s-x}{\theta\sqrt{T}}\right) + \sigma\sqrt{T}n\left(\frac{x-s}{\sigma\sqrt{T}}\right), \qquad (1)$$

where S= Stock price, N= Option strike price
T= option expiration time
N(.) =Probability density function
N=distribution function of the common normal distribution.

Here Bachelier (2011) made use of Brownian motion and the concept of stochastic variables to drive home stock price changes. However, there are some constraints in the application of the model in the sense that he assumed that stock price in an absolute Brownian movement which ignores the value of money and permits a negative number of shares. The Bachelier's option pricing is the mother of financial mathematics.

Stock prices have fixed means and variances and this has led researchers to develop the option pricing formula in Option Prices as an Indicator of Expectations and Preferences (Hong, 2023); thus,

$$C(S,T) = e^{\rho^T}\mathrm{SN}(d_1) - (1-A)XN(d_2) \qquad (2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[ln\left(\frac{S}{x}\right) + \left(\rho + \frac{\sigma^2}{2}\right)T\right], \qquad (3)$$

$$d_1 = d_2 - \sigma\sqrt{T}. \qquad (4)$$

Where ρ is the average growth rate of stock prices and A is the degree of risk aversion. Alexis and Mare (2017) assumed a steady logarithmic distribution for stock returns and thought of including risk insurance hence the model

$$V(s,t) = S\emptyset(d_1) - exp^{-\alpha(T-t)}K\emptyset d_2, \qquad (5)$$

is the same form as the Black-Scholes equation.

It is been suggested in literature that options should have a greater average return because the expected returns of options and stocks are different according to risk characteristics giving rise to the pricing model below:

$$V = e^{(\alpha-\beta)T}SN\left[\frac{\ln(s/k)+(\alpha+\frac{1}{2x^2})}{\sigma\sqrt{T}}T)\right] - e^{-BT}KN\left[\frac{\ln\left(\frac{S}{k}\right)+\left(\alpha-\frac{1}{2\alpha^2}\right)T}{\sigma\sqrt{T}}\right], \qquad (6)$$

when $\alpha = \beta$ the Boness model becomes a special case of Samuelson's model.

The equilibrium value of options in Black-Scholes model is obtained by a portfolio of hedging securities that produce a firm income independent of future stock prices such that the payoff of this portfolio of hedging securities is affected by variations in stock returns.

In Black -Scholes model it is easy for investors to control risks and assess the value of options easily.

The Binomial Tree model is an approach for pricing options created by Cox, Ross and Rubinstein (1979) for valuing American options.

One can use their model with little knowledge of mathematics thereby making it easy to be used.

The binomial tree helps to stimulate the continuous movement of asset prices. Here mean and variance are matched to yield the necessary parameters which are used to compute the option price backwards from the end of the binomial tree. The model assumes that the underlying price is limited to two states, the upstate and the downstate in any given interval of time. Here the growth rate of the stock has no impact on the option's value. What matters is the volatility or how far away the upstate and downstate are.

The formula is in such a way that

$$B_n = e^{-rT} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} max\{S_0 {u_B}^k {d_B}^{m-k} - K, 0\}.(7)$$

where $p = \frac{e^{\frac{rT}{n}} - d_B}{U_B - d_B}$, $u_B = e^{\sigma\sqrt{\frac{T}{n}}}$, $d_B = e^{-\sigma\sqrt{\frac{T}{n}}}$ and $p = \frac{1+R_f-D}{U-D}$.

And the steps are as follows;

(i) Calculate the size and probability of the above and down move

(ii)Calculate the binomial tree and calculate the payoff at all the nodes

(iii) Get the probability of the price going up and going down and then calculate the payoff at the nodes

(iv) Multiply the payoffs with the probability in each state to determine the value of the option for the whole period and then start drawing the binomial tree

(v) And then calculate the expected value of the option for the entire period.

(vi) After that discount the expected value to a period 0 at the risk-free rate to today's value.

The notations are:

π_U = risk-neutral probability of an up move

π_D = risk-neutral probability of a down move

R_f = risk-free rate

U = size of an up move, U = $\frac{1}{D}$ or D = $\frac{1}{U}$.

Tree theory is applied in many areas of mathematics and it forms an interesting part of combinational set theory.

Definition 1. 1: Mathematically a tree is a partially ordered set $T =< T_1 \leq T >$ such that for every $X \epsilon T$, the set $x = \{y \in T: y <_T x\}$is well-ordered by $\leq T$.

It is customary to represent trees in pictures as illustrated below

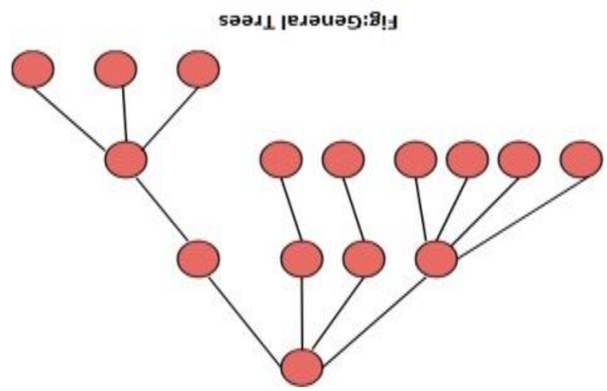

Fig.1: General Tree

Definition 1.2: Let S be a tree, A linearly ordered subset b of S with the property that whenever $x \in b$, then y$<_S$ X implies y $\in b$ is a branch of S. If α is the order type of b under $<_S$, we say that b is an α-branch.

If a branch is not properly contained in any other branch of S then it is maximal. Let θ be an ordinal and λ a cardinal. A tree S is said to be a $(\theta, \lambda) -$ tree if and only if

i. $(\forall \alpha < \theta)(S_\alpha \neq \emptyset)$
ii. $S_\theta = \emptyset$
iii. $(\forall \alpha < \theta)(|S_\alpha| \leq \lambda$

A tree S is said to have unique limits if, whenever $\alpha$is a limit ordinal and x,y $\in S_\alpha$ if ŷ = ẑ then y=z.

A (θ, λ)-tree,S is said to be normal if S has unique limits and each of the following conditions is satisfied.
i. $|S_0|$=1
ii. If $\alpha, \alpha+1 < \theta$ and $x \in S_{\alpha,}$ then there are distinct $y_{1,}y_2 \in S_{\alpha+1}$ such that $x <_s y_1$ and x$<_s y_2$
iii. If $\alpha < \beta < \theta$ and $x \in S_\alpha$ there's a y $\in S_\beta$such that x$<_s y$.

In Mathematics a tree is a connected graph without circuits. In other words, a graph is called a tree if and only if it is connected and has no circuits. When a graph has no circuits and is not connected, it is a forest. When a graph has, only a single vertex it is called a trivial tree (Galai, 1978).

A tree with n-vertices has $(n-1)$ edges. Every connected graph has a subgraph that is a tree. Any graph with n-vertices and less or more than $(n-1)$ edges is not connected and hence not a tree. Any non-trivial tree must have at least one vertex of degree 1.

2. 0 Mathematical formulation

Let

$$\frac{dY}{Y} = dW, \tag{8}$$

so that

$$\int_0^t \frac{dY}{Y} = \int_0^t dW + \int_0^t C(s)ds. \tag{9}$$

Let X be a domain stochastic variable such that $\int_0^t \frac{dY}{Y} = X$, then $X = \int_0^t dW + \int_0^t C(s)ds$.

Then from $\int_0^t \frac{dY}{Y} = X$, one gets the satisfaction of U(x ,t) as Y through

$$\ln Y(t) - \ln Y(0) = X, \Rightarrow \ln Y = X,$$

$\Rightarrow Y = e^X = U(X,t)$.

The domain stochastic differential equation is $\frac{dX}{dt} = \frac{d}{dt}\int_0^t \frac{dW}{ds}$ds + $\frac{d}{dt}\int_0^t C(s)ds = \frac{dW}{dt}$ + C(t).

Or,

$$dX = C(t) + dW. \tag{10}$$

By Ito's lemma

$$dX = Fdt + GdW, \tag{11}$$

then

$$dY = dU = \left(\frac{\partial U}{\partial t} + F\frac{\partial U}{\partial x} + \frac{G^2}{2}\frac{\partial^2 U}{\partial x^2}\right)dt + G\frac{\partial U}{\partial x}dW. \tag{12}$$

For this problem $dX = C(t)dt + dW$ and $U(x,t) = e^Y$

$$F = c\ ,\ G = 1, \frac{\partial U}{\partial t} = 0, \frac{\partial U}{\partial x} = e^x, \frac{\partial^2 U}{\partial x^2} = e^x.$$

Hence

$$dY = e^x\left[\left(F + \frac{G^2}{2}\right)dt + GdW\right].$$

That is $\frac{dY}{Y} = \left(c + \frac{1}{2}\right)dt + dW$

If this must solve equation (12),then $\left(c + \frac{1}{2}\right) = 0$ or$c = -\frac{1}{2}$. Hence

$$X = W - \frac{1}{2}t. \tag{13}$$

And

$$Y = e^{W - \frac{t}{2}}. \tag{14}$$

2.1. The main result

The problem is to find the worth of a stock at each point of the tree of Fig.1.
It is important to observe from Fig.1 that each point satisfies a polar form of a diffusion equation in a spherical coordinate system.
In the spherical coordinate system, the diffusion equation is written as (Datta and Pal, 2018):

$$\frac{\partial w}{\partial t} = \frac{k}{s^2}\frac{\partial}{\partial r}\left(s^2\frac{\partial w}{\partial r}\right) + \frac{1}{s^2\sin\theta}\frac{\partial w}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial \theta}\right) + \frac{1}{s^2\sin^2\theta}\frac{d^2 w}{d\emptyset^2}. \tag{15}$$

Where; w = the worth of the stock, k = diffusion coefficient, $(S, \theta, \emptyset)$ = point in spherical coordinate and $S$is the stock price.
In what follows, we state;
Theorem 2.1: Let equation (15) exist, then the worth of the stock at each point assumed a steady logarithmic distribution for stock returns and thought of including risk insurance is given as;

$$W = \frac{2l}{n\pi}\sum_{n=1}^{\infty}\frac{1}{s}\left[\frac{(-1)^{n+1}}{n}\sin\frac{n\pi s}{l}\, exp\left\{-\left(\frac{n^2c^2\pi^2 t}{l^2}\right)\right\}\right]$$

$$= \frac{2l}{n\pi}\sum_{n=1}^{\infty}\frac{1}{s}\left[\frac{(-1)^{n+1}}{n}\sin\frac{n\pi s}{l}\, exp\left\{-\left(\frac{n^2\pi^2 t}{l^2}\right)\right\}\right] e^{(\alpha-\beta-r)}, \quad (16)$$

where $c^2 = \alpha - \beta - r$ represents the control policies of the insurance and reinsurance company, $r$the growth rate of stock and α, β are the different risk parameters.
Proof:
In the radial direction, a one-dimension form of (15) can be written as:

$$\frac{\partial w(s,t)}{\partial t} = \frac{k\partial^2 w(s,t)}{\partial s^2} + \frac{2k}{s}\frac{\partial w(s,t)}{\partial s} \qquad (17)$$

The second term of the RHS of the above equation (15) is a result of an increase in surface area along the radial direction hence standard LB method cannot be used to solve the equation (15) (Mohamad,2009).
Equation (15) is equivalent to (12) for $F = 0$ and is modified to the form of diffusion equation which is similar to that in the Cartesian coordinate system by substituting the diffusive variable(y) with a new variable of the forms

$$\begin{cases} u(s,t) = sw(s,t) \\ W = \frac{u}{s} \end{cases} \qquad (18)$$

The derivative terms in the above equation can also be written in the new term variable as

$$\frac{\partial w(s,t)}{\partial t} = \frac{1}{s}\frac{\partial u(s,t)}{\partial t} \qquad (19)$$

and

$$\frac{\partial w(s,t)}{\partial s} = \frac{1}{s}\frac{\partial u(s,t)}{ds} - \frac{u(s,t)}{s^2} \qquad (20)$$

$$\frac{\partial^2 w(s,t)}{ds^2} = \frac{1}{s}\frac{\partial^2 u(s,t)}{\partial s^2} - \frac{2}{s^2}\frac{\partial u(s,t)}{\partial s} + \frac{2u(s,t)}{s^3} \qquad (21)$$

Equation (8) can be written as :

$$\frac{1}{s}\frac{\partial u(s,t)}{\partial t} = \frac{k}{s}\frac{\partial^2 u(s,t)}{\partial s^2} - \frac{2k}{s^2}\frac{\partial u(s,t)}{\partial s} + \frac{2ku(s,t)}{s^3} + \frac{2k}{s^2}\frac{\partial u(s,t)}{\partial s} - \frac{2ku(s,t)}{s^3}$$

This simplifies to:

$$\frac{\partial u(s,t)}{dt} = K\frac{\partial^2 u(s,t)}{\partial s^2} \qquad (22)$$

This equation is similar to the diffusion equation in the Cartesian coordinate system and can easily be solved using the standard LB diffusion model. Solution of (22) subject to the boundary conditions

$$\begin{cases} u(s, t = 0) = 0 \\ u(s = 1, t) = 0. \\ u(s, 0) = s \end{cases} \qquad (23)$$

Solving Heat Equations (22) subject to the boundary conditions (23), where $0 < s < l$, using the separation of variable method gives

$$u = \frac{2l}{n\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi s}{l}\, e^{\frac{-n^2c^2\pi^2 t}{l^2}} . \qquad (24)$$

Combining equations (18) and (24) yields equation (16) as required.
A more general form of equation (15) is given by;

$$\partial_t W(S,t) = \frac{c}{S^{d-1}} \partial_r (S^{d-1} \partial_S W(S,t)). \quad (25)$$

Where r is the radius of the spherical coordinate and $\alpha$is a constant with the initial condition $W(S,t\ = 0) = \delta(S - R_a)$, boundary condition $\partial_S W(S,t) I_{x=R_0} = 0$

Proposition 2.1: The solution of equation (25) is equivalently equation (24) (see appendix for poof) given as;

$$W(S,t) = \sum_{n=1}^{\infty} \frac{C_n}{S} \sin\left(\frac{\alpha_n r}{S_0}\right) e^{-c(\frac{\alpha_n}{S_0})^2 t}. \quad (26)$$

Equation (26) together with equation (18) yields

$$u(S,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\alpha_n r}{S_0}\right) e^{-c(\frac{\alpha_n}{S_0})^2 t}$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{\alpha_n r}{S_0}\right) exp\left\{-\left((\alpha - \beta - r)^{\frac{1}{2}} (\frac{\alpha_n}{S_0})^2 t\right)\right\}. \quad (27)$$

Implying that the worth of the stock depends largely on time$t$ rather than price S.

3.0 Solution by Mohand Transform method

Definition 3.1: A function $g : D \rightarrow J$ is called Lipschitz continuous if for each $d_1, d_2 \in D$, there exists a constant a such that; $|g(d_2) - g(d_1)| \leq a|d_2 - d_1|$.

Consider equation (17) with the initial condition; $W = \begin{cases} K^C, & K \geq 0 \\ 0, & K < 0 \end{cases}$, we check to confirm that it is Liptichitzian;

Let $W = K^C$

$$g(W,t) = \frac{\partial W}{\partial t}$$

$$= |g(K_2,t) - g(K_1,t)|$$

$$= \left|K_2 \frac{\partial^2 K_2{}^C}{\partial K_2{}^2} + \frac{2K_2}{S}\frac{\partial K_2{}^C}{\partial K_2} - K_1 \frac{\partial^2 K_1{}^C}{\partial K_1{}^2} - \frac{2K_1}{S}\frac{\partial K_1{}^C}{\partial K_1}\right|$$

$$= \left|K_2 C(C-1) K_2{}^{C-2} + \frac{2K_2 C}{S} K_2{}^{C-1} - C(C-1) K_1 K_1{}^{C-2} - \frac{2C}{S} K_1 K_1{}^{C-1}\right|$$

$$= \left|C(C-1) K_2{}^{C-1} + \frac{2}{S} C K_2{}^C - C(C-1) K_1{}^{C-1} - \frac{2}{S} C K_1{}^C\right|$$

$$= \left|C(C-1)\left\{\left[K_2{}^{C-1} - K_1{}^{C-1}\right] + \frac{1}{S}\left[K_2{}^C - K_1{}^C\right]\right\}\right|$$

$$\leq |C(C-1)| \left|\left[K_2{}^{C-1} - K_1{}^{C-1}\right] + \frac{1}{S}\left[K_2{}^C - k_1{}^C\right]\right|.$$

Therefore it satisfies Lipschitz's condition where the Lipschitz's constant is $C(C-1)$. In what follows, we attempt a solution of equation (17) via the Mohand transform method (MFM).
Using the Mohand transform to solve the equation

$$\frac{\partial W}{\partial t} = k\frac{\partial^2 W}{\partial S^2} + \frac{2K}{S}\frac{\partial W}{\partial S}$$

with initial condition $w = \begin{cases} S^C, & S \geq 0 \\ 0, & S < 0 \end{cases}$.

CASE $W = S^C$

By applying the Mohand transform one gets;

$$\mathrm{M}\left\{\frac{\partial W}{\partial t}\right\} = \mathrm{M}\left\{k\frac{\partial^2 W}{\partial S^2} + \frac{2K}{S}\frac{\partial W}{\partial S}\right\}$$

$$\Longrightarrow vR(W,t) - v^2R(W,0) = \mathcal{M}\left\{K\frac{\partial^2 W}{\partial S^2} + \frac{2K}{S}\frac{\partial W}{\partial S}\right\} \quad (28)$$

At $t = 0$, $R(W,0) = W(S,0) = S^C$ giving rise to

$$vR(W,\tau) = v^2S^C + \mathcal{M}\left\{K\frac{\partial^2 W}{\partial S^2} + \frac{2K}{S}\frac{\partial W}{\partial S}\right\} = v^2S^C + \mathcal{M}\{QW + RW\}. \quad (29)$$

where Q and R are nonlinear operators such that

$$Q = K\frac{\partial^2}{\partial S^2} \text{ and } R = \frac{2K}{S}\frac{\partial}{\partial S}.$$

Representing the solution as an infinite series leads to $W = \sum_{n=0}^{\infty} W_n$ and decomposing the nonlinear operators yields

$QW = \sum_{n=0}^{\infty} A_n$ and$RW = \sum_{n=0}^{\infty} B_n$.

Thus one has

$$\sum_{n=0}^{\infty} R(W_n,\tau) = v^2S^C + \frac{1}{v}\mathcal{M}\{\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n - \sum_{n=0}^{\infty} W_n\}. \quad (30)$$

Taking the inverse Mohand transform of both sides of the equation yields

$$W_{n+1} = \mathcal{M}^{-1}\left\{\frac{1}{V}\mathcal{M}[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n - \sum_{n=0}^{\infty} W_n]\right\}. \quad (31)$$

Decomposing gives

$$A_n = QW = K\frac{\partial^2}{\partial S^2}S^C = KC(C-1)S^{C-2} \quad (32)$$

and

$$B_n = RW = \frac{2K}{S}\frac{\partial}{\partial S}S^C = \frac{2K}{S}CS^{C-1} = 2KCS^{C-2}, \quad (33)$$

so that

$$W_{n+1} = \mathcal{M}^{-1}\left\{\frac{1}{V}\mathcal{M}[\sum_{n=0}^{\infty} KC(C-1)S^{C-2} + \sum_{n=0}^{\infty} 2KCS^{C-2} - \sum_{n=0}^{\infty} S^C]\right\},\ n \geq 0,\ C \geq 3. \quad (34)$$

For a special case where $C = 3$, one gets (as in Azu-Nwosu, 2023)

$$W(S,t) = S^3e^{-\frac{13}{2}t}. \quad (35)$$

combining equation (34) with equation (18) yields

$$u(S,t) = S^4e^{-(\alpha-\beta-r)t}. \quad (36)$$

Equation (36) largely depends on S with time.

3.1 Some numerical examples

Assume the expected volume of the portfolio at a node of a tree (see Fig.1)is NGN20m with the strike price and exercise price at NGN30M and NGN80M respectively at some other nodes of the tree until a steady state prevails. The nodes are changed to NGN40M and NGN60M respectively. Then the worth distribution in the volume of the portfolio at the time,t, is; $u_1(S,t) = 30 + \frac{80-30}{20}S = 30 + \frac{50}{20}S$. The final distribution is $u_2(S) = 40 + \frac{60-40}{20}S = 40 + S$.

To get u in the intermediate period, reckoning time from the instant when the nodes were changed, we assumed

$u = u_1(S,t) + u_2(S)$,

where $u_2(S)$ is the steady state worth distribution in the node (i.e worth after a sufficiently long time) and $u_1(S,t)$ is the transient worth distribution which tends to zero as t increases. Thus $u_2(x) = 40 + S$. Now $u_1(S,t)$ satisfies the one-dimensional heat flow equation (22) with a solution as in (27) or better still (36)

$$u(S,t) = 40 + S + \sum_{n=1}^{\infty} c_n \sin\left(\frac{\alpha_n r}{S_0}\right) exp\left\{-\left((\alpha-\beta-r)^{\frac{1}{2}}\left(\frac{\alpha_n}{S_0}\right)^2 t\right)\right\}. \quad (37)$$

But $u = 40$ when $S = 0$ and $u = 60$ when $S = 20$,$\rightarrow$ $a_n = 0$, $n = \frac{k\pi}{20}$.

We now verify the relationship between the optimal investment worth, dividend, and annual tax rate $a_n$in (16) and (26) respectively, using numerical examples. Fig. 2 below shows the optimal consumption rate for varying $c = (\alpha - \beta - r)$ and $a_n = 20\%$.

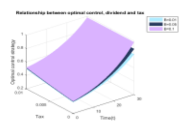

Fig.2: We calculate the expected value $W(S, t)$ with the current price of a stock $S_0 = \$80$ and the expiry is 363day, the size of the up move $\alpha = 1.4$ and the risk free rate $r = 0.06$.

We use a binomial tree, to determine the current worth of the option. $\pi_\alpha = 0.504, \pi_\beta = 0.496$, $\beta = 0.714$, where$\pi_\alpha = \frac{1+Rf-\beta}{\alpha-\beta}$, $\pi_\beta = 1 - \pi_U$are the probabilities of up or down nodes of the tree as in Fig.1.

Fig.3 shows the relationship between the optimal worth control strategy and tax rate with varying risk over time, for all other parameters remain fixed. It is observed that the investment portfolio in the risky asset is positive.

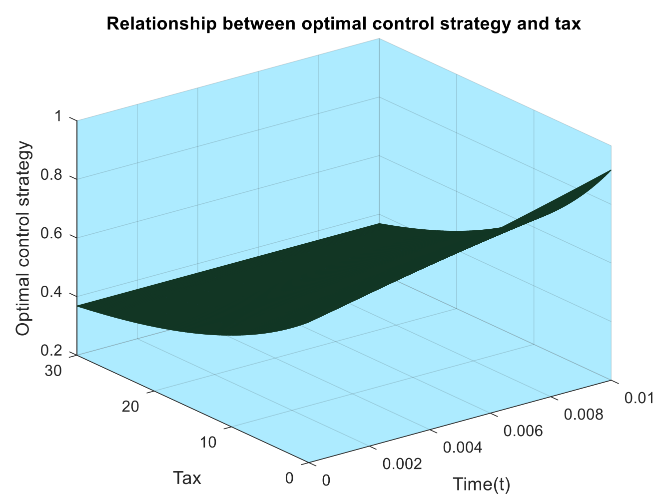

Fig.**3: We calculate the expected value $W(S, t)$ with a current price of a stock $S_0 = \$100$ and the expiry is 363day, the size of the up move $\alpha = 1.6$ and the risk free rate $r = 0.08$.**

We use a binomial tree, to determine the current worth of the option. $\pi_\alpha = 0.455, \pi_\beta = 0.545$, $\beta = 0.625$, where$\pi_\alpha = \frac{1+Rf-\beta}{\alpha-\beta}$, $\pi_\beta = 1 - \pi_U$are the probabilities of up or down nodes of the tree as in Fig.1.

Fig. 4 verifies the relationship between the optimal investment worth, dividend, and annual tax rate $a_n$in (2.20) and (2.19), showing the optimal consumption rate for varying $c = (\alpha - \beta - r)$and $a_n = 40\%$.

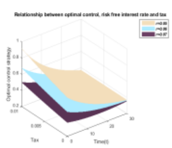

Fig.4: We calculate the expected value $u(S,t)$ with the price of a stock S and the expiry is 363day, the size of the up move $\alpha = 0.28$ and the risk-free rate $r = 0.15$. We use a binomial tree, to determine the current worth of the option. $\pi_\alpha = 0.403, \pi_\beta = 0.634$, $\beta = 0.72$, where$\pi_\alpha = \frac{1+Rf-\beta}{\alpha-\beta}$, $\pi_\beta = 1 - \pi_U$are the probabilities of up or down nodes of the tree as in Fig.1.

3.2 Sensitivity Analysis of$u(S,t)$(3.8) and $W(S,t)$ (3.7)

In this section, we give the sensitivity analyses of equation (3.7) with varying terms of risk parameters and contribution rate as shown in Fig.5, skewed to the right.

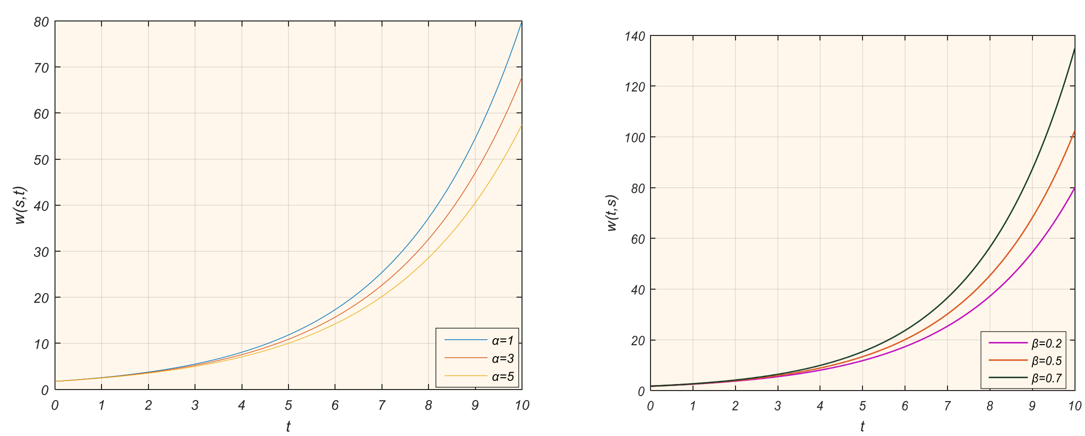

Fig.5: Spatial profile of worth concentration after 365, 1000 and3000days using (3.7).

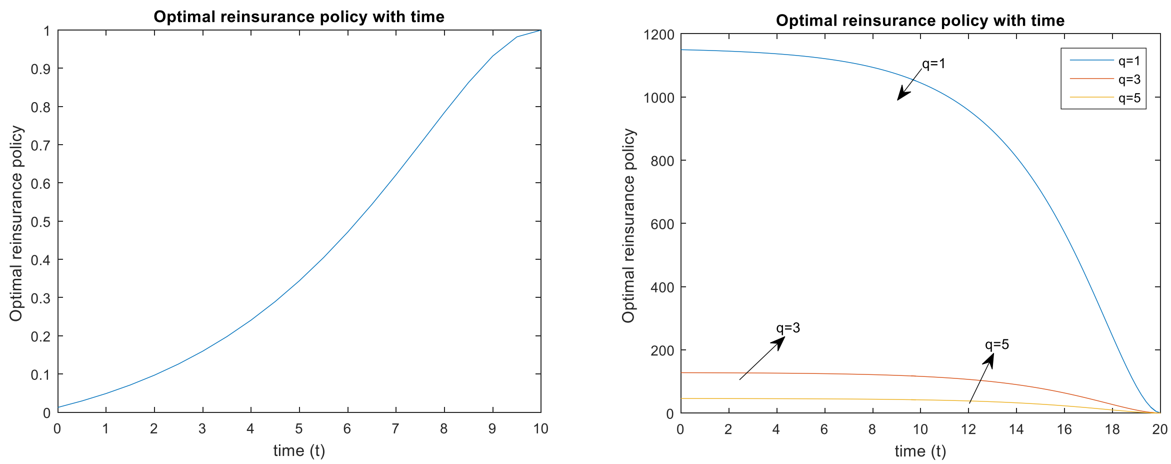

Fig.6: Contribution growth rate and the coefficient of risk using (3.8), with $\alpha = 0.28, \beta = 0.15$ and $r = 0.15$. Note that $u(S,t) = 40 + S(1 + S^3 e^{-(\alpha-\beta-r)t})$ and $u = 40$ when $S = 0$ and $u = 60$ when $S = 20$ as $t \to 0$.

4.0 Conclusion

This work aimed at deriving a formula for worth option payoff to analyze the deposit at each node of a tree with the insight of a binomial tree insurance model, and that was achieved by obtaining the solution of the power option payoff as the initial condition to the heat partial differential equation, which constituted the deposit model. By doing this, ranges of possible worth corresponding to the values of the different policies were obtained. Based on the graphs derived from the results of the study, we observed the establishment of flexibility and leverage by the incorporation of the power option into the proposed deposit insurance model in the sense that an insured entity is given the chance to know what payoff is best for them.

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Compliance with Ethical Standards Declarations

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data Public.

Competing interests

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Authors' contributions

BOO: Coceived and designed the experiment; wrote the paper. Analysed interpreted data. PUD: Performed the experiments, analysed tools and wrote the paper.

Appendix (proof of proposition 1)

An alternative solution of diffusion equation in spherical coordinates (25) without transformation is obtained following the procedure:

Let $\mathrm{F(r,t) = R(r)T(t)}$,Then $\frac{T'}{aT} = \frac{\frac{1}{r}(rR')'}{R} = -\lambda$and $\begin{cases} T' + \lambda aT = 0 \\ (rR')' + \lambda rR = 0 \end{cases}$

The radial equation is $r^2R'' + rR' + \lambda r^2R = 0$ of which the solution is in the form of Bessel functions$R(r) = J_{0(\sqrt{\lambda}r)}$.

The boundary condition requires $R'(r_0) = -\sqrt{\lambda}J_1(\sqrt{\lambda}r_0) = 0$, then $\sqrt{\lambda_n}r_{0=}\alpha_n$, where $\alpha_n = zeroes\ of\ J_1(x)$. Rewriting the solution $R_n = J_0(\frac{\alpha_n r}{r_0})$.

The general solution is then $W(r,t) = \int_1^\infty C_n J_0(\frac{\alpha_n r}{r_0})e^{-\alpha(\frac{\alpha_n}{r_0})^2}t$.

To prove that the eigen-solutions R_n are orthogonal concerning a weighting function: by definition $(rR'_n)' = -\lambda_n rR_n$ and all solutions satisfy $R_n{}'(r_0) = 0$.

Integrating by parts

$$\int_0^{r_0}(rR_n{}')'R_m dr = rR_n{}'R_m\Big]_0^{r_0} - \int_0^{r_0} rR_n{}'R_m{}'dr$$

and the definition

$\int_0^{r_0}[(rR_n{}')'R_m - (rR_m{}')R_n]dr = -(\lambda_n - \lambda_m)$,

which follows that $\int_0^{r_0} rR_nR_m dr = 0$, for $n \neq m$. Finally, the initial condition

$W(r,0) = \delta(r-s) = \sum_{n=1}^{\infty} C_n J_0(\frac{\alpha_n r}{r_0})$,

where $\delta(r)$ is the polar Dirac-Delta and $0 < s < r_0$. Using the proven orthogonality

$\int_0^{r_0} r\delta(r-s)J_0(\frac{\alpha_m r}{r_0}) =$

$C_m \int_0^{r_0}\sum_{n=1}^{\infty} C_n rJ_0(\frac{\alpha_n r}{r_0})J_0(\frac{\alpha_m r}{r_0})dr$,

$J_0\left(\frac{\alpha_m s}{r_0}\right) = C_m \int_0^{r_0} rJ_0{}^2(\frac{\alpha_m r}{r_0})dr.$

In spherical coordinates the Laplacian turns out to be $\nabla^2F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial F}{\partial r})$, but the radial equation is$(r^2R')' + \lambda r^2R = 0$.This has a solution in terms of spherical Bessel functions.Taking only the one that is finite at $r = 0$. $R(r)$ = that is finite at $r = 0$. $R(r) = J_0(\sqrt{\lambda}r) = \frac{\sin(\sqrt{\lambda}r)}{r}$.

By B.C $R'(r_0) = 0 \rightarrow \sqrt{\lambda}r_0\cos(\sqrt{\lambda}r_0)$ - $\sin(\sqrt{\lambda}r_0) = 0$ and $\sqrt{\lambda}r_0 = \tan(\sqrt{\lambda}r_0$

Let α_n be the solutions to the equation $x = tan(x)$ (again ignoring $\alpha_0 = 0$) we have the general solution as required.

In spherical the weighting factor r^2 and not r ,so we have

$C_n = \frac{\int_0^{r_0}F(x,0)R_n(r)r^2dr}{\int_0^{r_0}R_n(r)r^2dr} = \frac{\int_0^{r_0}F(x,0)\sin\left(\frac{\alpha_n r}{r_0}\right)rdr}{\int_0^{r_0}sin^2(\frac{\alpha_n}{r_0}r)dr}$.