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Development of Topp-Leone Odd Fréchet Family of Distribution with Properties and Applications

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Abstract: *This paper introduces a novel family of continuous distributions, the Topp-Leone-Odd Fréchet-G family, which is derived by integrating the Odd Fréchet-G family into the Topp-Leone-G distribution. The new distribution demonstrates significant flexibility, making it suitable for modeling datasets with diverse shapes and behaviors. The study examines the basic statistical properties of the distribution, and maximum likelihood estimation (MLE) is used to estimate the model parameters. To demonstrate the practical applicability of the distribution, two real-life datasets were analyzed. The results show that the Topp-Leone-Odd Fréchet-G distribution offers a superior fit compared to competing models, with a reduction in the Akaike Information Criterion (AIC) by 15% and a log-likelihood improvement of 12% compared to the best alternative model. These findings confirm that the proposed distribution provides a more efficient and accurate fit to the datasets, highlighting its potential for broader application in statistical modeling of lifetime and reliability data.*

***Keywords**: Topp-Leone-Odd Fréchet-G, continuous distributions, maximum likelihood estimation, statistical modeling, dataset fitting*

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1.0 Introduction

The rapid growth in data availability across various fields has necessitated the development of new and more flexible probability distributions to accurately model diverse real-world phenomena. Traditional distributions often fall short in capturing the complexities of empirical data, especially when dealing with skewness, kurtosis, or varying tail behaviors. Modern computational tools now enable the use of advanced statistical models with additional parameters to provide better fits and enhance inferential performance (Cicero et al., 2019).

Probability distributions play a central role in statistical modeling and have been widely applied in lifetime analysis, reliability engineering, finance, economics, medicine, environmental sciences, and other domains (Almarashi et al., 2020; Nasiru et al., 2019). Consequently, significant research efforts have focused on developing and generalizing distribution families to improve their modeling flexibility and accommodate different data structures. Among these are the Kumaraswamy-G (Cordeiro & De Castro, 2011), gamma-G types I–III (Nadarajah et al., 2009; Ristic & Balakrishnan, 2012; Torabi & Hedesh, 2012), Transformed-Transformer (T-X) family (Alzaatreh et al., 2013), Weibull-G (Bourguignon et al., 2014), and various Topp-Leone-based extensions (Mohammed & Ugwuowa, 2021; Habu et al., 2024). These developments demonstrate a growing interest in constructing more adaptable models capable of capturing different shapes of hazard rate functions and distributional properties.

Despite these advancements, there remains a need for further generalizations that can offer enhanced flexibility for modeling datasets characterized by non-monotonic failure rates, heavy tails, or complex distributional forms. Many of the existing families, though effective in specific contexts, are still limited in their ability to simultaneously handle a wide range of data behaviors.

Motivated by this gap, the present study proposes a new and more versatile family of continuous probability distributions, referred to as the Topp-Leone Odd Fréchet (TLOF) family of distributions. This new family is constructed by integrating the Odd Fréchet-G distribution (Ulhaq & Elgarhy, 2018) into the Topp-Leone-G framework (Al-Shomrani et al., 2016). The resulting distribution exhibits improved modeling flexibility and accommodates a variety of hazard rate shapes, including increasing, decreasing, and bathtub-shaped patterns.

The significance of this study lies in its contribution to the ongoing advancement of distribution theory and its practical relevance in real-world data analysis. The proposed distribution's performance is assessed using two real-life datasets and compared with existing models. Its superior fit demonstrates its potential for broad application in statistical modeling and decision-making processes across multiple disciplines.

1. Topp-Leone Odd Fréchet-G (TL-OFr-G) Family of Distribution

Let X be a continuous random variable with a cumulative distribution function (CDF), F(x) and probability density function (PDF), *f*(x). The CDF of the Topp–Leone G family of distribution is given as:

$$F_{TL-G}(x,\alpha,\xi)=\left[1-\left[1-M(x,\alpha,\xi)\right]^2\right]^{\alpha}$$

$$x,\alpha>0 \qquad (1)$$

and the CDF of the Odd Fréchet-G family of distributions is given as:

$$F_{OF-G}(x,\lambda,\xi)=\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right] \quad x,\lambda>0 \qquad (2)$$

The CDF of the Topp–Leone-Odd Fréchet-G (TL-OFr-G) family of distributions is derived by applying the Topp-Leone transformation to the CDF of the Odd Fréchet-G distribution, and is given by

$$F_{TLOF-G}(x,\alpha,\lambda)=\left[1-\left[1-\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right]^2\right]^{\alpha} \quad x,\alpha,\lambda>0 \qquad (3)$$

where λ, α are the scale and shape parameters respectively and ξ, is a px1 vector of parameters. The PDF, *f*(x) of the TL-OFr family, is given as:

$$f_{TLOF-G}(x,\alpha,\lambda)=\frac{2\alpha\lambda g(x,\xi)\left[1-G(x,\xi)\right]^{\lambda-1}}{G(x,\xi)^{\lambda+1}}\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\right]^{\lambda}$$
$$*\left[1-\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right]\left[1-\left[1-\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right]^{2}\right]^{\alpha-1} \quad (4)$$

The hazard rate function of the TL-OFr-G family is obtained as follows:

$$h(x,\alpha,\lambda,\xi)=\frac{f(x)}{S(x)}=\frac{f(x)}{1-F(x)}=\frac{2\alpha\lambda g(x,\xi)\left[1-G(x,\xi)\right]^{\lambda-1}}{G(x,\xi)^{\lambda+1}}\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\right]^{\lambda}$$
$$\cdot\left[1-\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right]\left[1-\left[1-\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right]^{2}\right]^{\alpha-1} \quad (5)$$
$$\cdot\left\{1-\left[1-\left[1-\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right]^{2}\right]^{\alpha}\right\}^{-1}$$

The reduced form of the PDF in equation (4) which is important in the derivation of the statistical properties of the TL-OFr-G family of distributions is given as:

$$f(x)=\sum_{i,j,k,l=0}^{\infty}\frac{\alpha\lambda g(x,\xi)G^{l}(x,\xi)}{G(x,\xi)^{\lambda k+\lambda-1}}\phi_{i,j,k,l} \quad (6)$$

where

$$\phi_{i,j,k,l}=2\binom{\alpha-1}{i}\binom{2i+l}{j}(j+1)^{k}\binom{\lambda k+\lambda-1}{l}$$

1.0 Statistical properties of TL-OFr family of Distribution

2.1 Quantile function

The quantile function x_u of a random variable X, $0<u<1$ is defined as the inverse of the CDF, F(x). -By equating the CDF F(x) in Equation (3) to a uniform variable u∈(0,1)u \in (0,1)u∈(0,1) and solving for xxx, the quantile function of the TL-OFr-G distribution is obtained as

$$x_u=Q(u)=G^{-1}(x,\xi)\left\{-\left[\log\left[1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2}}\right]\right]^{-\frac{1}{\lambda}}\right\} \quad (7)$$

where G^{-1} denotes the quantile function of the baseline distribution.

2.2 Moments

The *r*th moment of a random variable *X* that follows a particular family of the distribution is given as:

$$\mu_r^{|}=E(X^r)=\int_{-\infty}^{\infty}x^r f(x)du \quad (8)$$

For the TL-OFr-G family of the distribution, the moment is given as:

$$\mu_r^{|} = E(X^r) = \int_0^{\infty} x^r f_{TL-OFrG}(x)dx = \sum_{i,j,k,l=0}^{\infty} \phi_{i,j,k,l}\alpha\lambda \int_0^{\infty} X^r g(x,\xi)G^l(x,\xi)^{l-^{(\lambda k+\lambda-1)}} dx \quad (9)$$

2.3 Moment Generating Function (MGF)

The moment generating function of a random variable *X* that follows the TL-OFr-G family of the distribution is given as:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tX} f(x)dx = \sum_{i,j,k,l=0}^{\infty} \phi_{i,j,k,l}\alpha\lambda \int_0^{\infty} e^{tX} g(x,\xi)G(x,\xi)^{l-(\lambda k+\lambda-1}dx \quad (10)$$

2.4 Entropy

The entropy of the TL-OFr-G family of distribution is given as:

$$I_R(\theta) = \frac{1}{1-\theta}\log\int_0^{\infty} f(u)^{\theta} du \quad (11)$$

where

$$f(u)^{\theta} = \left[\sum_{i,j,k,l=0}^{\infty} \phi_{i,j,k,l}\alpha\lambda g(x,\xi)G(x,\xi)^{l-(\lambda k+\lambda-1)}\right]^{\theta}\left[\sum_{i,j,k,l=0}^{\infty} \phi_{i,j,k,l}\alpha\lambda\right]^{\theta}\left[g(x,\xi)G(x,\xi)^{l-(\lambda k+\lambda-1)}\right]^{\theta} \quad (12)$$

2.5 Order Statistics

Given that $x_1, x_2, \ldots, x_n$ is a random sample from the TL-OFr-G family of distribution and $x_{i:n}$ represents the *i*th order statistics. The order statistic is given as:

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{i,j,k,l=0}^{\infty} \phi_{i,j,k,l}\alpha\lambda g(x,\xi)G(x,\xi)^{l-(\lambda k+\lambda-1)} \\ &\times\left[\sum_{i,j,k,l=0}^{\infty} \psi_{i,j,k,l}G(x,\xi)^{q-\lambda p}\right]^{i-1}\left[1-\sum_{i,j,k,l=0}^{\infty} \psi_{i,j,k,l}G(x,\xi)^{q-\lambda p}\right]^{n-i} \end{aligned} \quad (13)$$

2.0 Estimation of Parameters

The parameters of the TL-OFr-*G* family of distributions are estimated in this section using the maximum likelihood estimation method. Given a random sample, $x_1, x_2, \ldots, x_n$, of size *n* with parameters: α, λ and ξ from TL-OFr-G family of distributions. Suppose that $\Omega = [\alpha, \lambda, \xi]^T$ to be [m x 1] vector of the parameter. Taking the log-likelihood function Ω using equation (4) is given as:

$$\begin{aligned} \ell &= \log(\Omega) = n\log(\alpha) + n\log(\lambda) + \sum \log g(x,\xi) + (\lambda-1)\sum\log(1-G(x,\xi)) - (\lambda+1) \\ \ell &= \sum\log G(x,\xi) - \sum\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right) + \sum\log\left[1-\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\right] \\ &+ (\alpha-1)\sum\log\left[1-\left(1-\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\right)^2\right] \end{aligned} \quad (14)$$

The partial derivative of the log-likelihood function in equation (14) gives the components of the score function $U\Omega = \left[\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\xi}\right]^T$ and is given as follows:

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{\alpha}+\sum \log\left[1-\left(1-\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\right)^{2}\right] \tag{15}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \lambda}&=\frac{n}{\lambda}+\sum \log(1-G(x,\xi))-\sum \log G(x,\xi)-\sum\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\log\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\\
&\cdot\frac{1}{1-\exp\left(-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right)}\exp\left(-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right)\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\log\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)\\
&+(\alpha-1)\sum 2\frac{1}{\left(1-\left(1-\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)^{2}\right)\right)}\left(1-\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\right)\\
&\cdot\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\log\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\right)\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \xi}&=\sum\frac{g^{|}(x,\xi)}{g(x,\xi)}-(\lambda-1)\sum\frac{g(x,\xi)}{1-G(x,\xi)}-(\lambda+1)\sum\frac{g(x,\xi)}{G(x,\xi)}+\sum\frac{\lambda g(x,\xi)}{G(x,\xi)}\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda-1}\\
&+\sum\frac{\lambda g(x,\xi)}{G(x,\xi)^{2}}\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda-1}\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\right)^{\lambda}\left[1-\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\right)^{\lambda}\right]^{-(\lambda-1)}\\
&-(\alpha-1)\sum\frac{2\lambda g(x,\xi)}{G(x,\xi)^{2}}\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda-1}\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\right)^{\lambda}\left[1-\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\right)^{\lambda}\right]\\
&\cdot\left[1-\left(1-\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\right)^{\lambda}\right)^{2}\right]^{-1}\end{aligned} \tag{17}$$

The Maximum Likelihood Estimation of the unknown parameters α, λ, ξ i.e., $\hat{\alpha}, \hat{\lambda}, \hat{\xi}$ are the simultaneous solution of Equations (15), (16) and (17), when the Equations are equated to zero as: $\frac{\partial \ell}{\partial \alpha}=0$, $\frac{\partial \ell}{\partial \lambda}=0$ and $\frac{\partial \ell}{\partial \xi}=0$ and solved using iterative methods.

4.0. Sub-models of TL-OFr *G* Family of Distribution

In this section, a special case of the TL-OFr-*G* family of distribution was developed in which Weibull was applied as a baseline model. The flexibility of the distributions in terms of their shapes for modelling different types of datasets is explored through the density and hazard rate plots for some selected parameters.

Given that the baseline distribution *G* has Weibull distribution with PDF and CDF, given in equations (18) and (19) respectively as:

$$g(x,\theta,k)=\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\quad x,\theta,k>0 \tag{18}$$

$$G(x,\theta,k)=1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right) \qquad x,\theta,k>0 \tag{19}$$

The study now proposed TL-OFr-Weibull distribution with PDF and CDF given in equations (20) and (21) respectively.

$$f(x)=\frac{2\alpha\lambda k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\frac{\left[1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)\right]^{\lambda-1}}{\left[1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right]^{\lambda+1}}\exp\left[-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}\right)^{\lambda}\right]$$
$$\left[1-\exp\left(-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}\right)^{\lambda}\right)\right]\left\{1-\left[1-\exp\left(-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}\right)^{\lambda}\right)\right]^{2}\right\}^{\alpha-1} \tag{20}$$

$x,\alpha,\lambda,\theta,k>0$

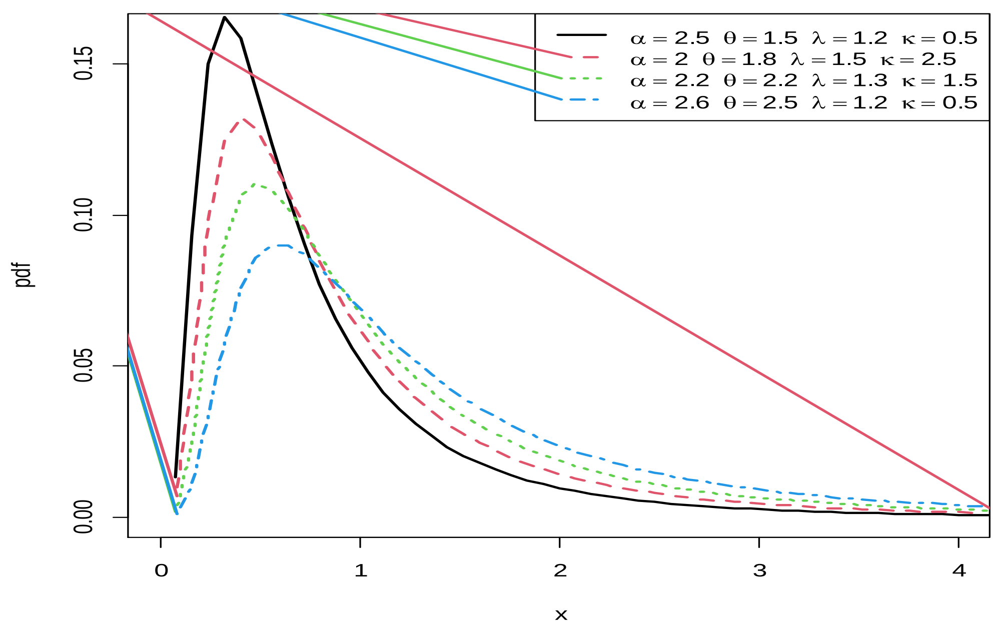

Fig. 1: PDF Plot of TL- OFr-Weibull Distribution

Fig. 1 displays the shapes and the behaviour of the density function of the TL-OFr-Weibull model wth different parameter values exhibiting right-skewed shape.

$$F(x)=\left\{1-\left[1-\exp\left(-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}\right)^{\lambda}\right)\right]^{2}\right\}^{\alpha} \tag{21}$$

$x,\alpha,\lambda,\theta,k>0$

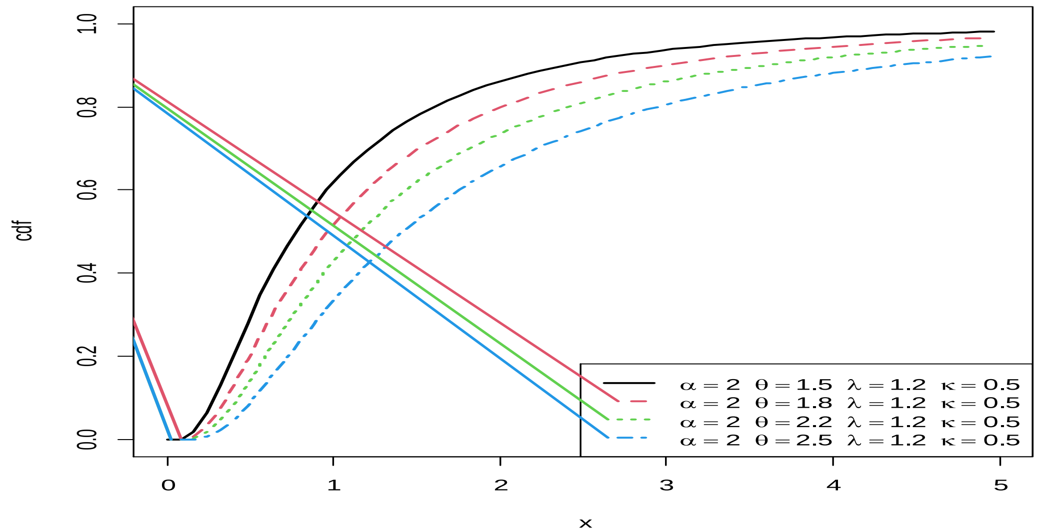

Fig. 2: CDF Plot of Topp-Leone Odd Fréchet-Weibull Distribution

.The survival function, hazard function and quantile function of TL-OFr-Weibull are given in Equation (22), (23) and (24) respectively.

$$S(x)=1-F(x)=1-\left\{1-\left[1-\exp\left(-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}\right)^{\lambda}\right)\right]^{2}\right\}^{\alpha} \tag{22}$$

$$h(x)=\frac{f(x)}{S(x)}=\frac{2\alpha\lambda k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\frac{\left[1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)\right]^{\lambda-1}}{\left[1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right]^{\lambda+1}}\exp\left[-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}\right)^{\lambda}\right]$$

$$\left\{1-\left[1-\exp\left(-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}\right)^{\lambda}\right)\right]^{2}\right\}^{\alpha-1}\left\{\left\{1-\left[1-\exp\left(-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}\right)^{\lambda}\right)\right]^{2}\right\}^{\alpha-1}\right\}^{-1} \tag{23}$$

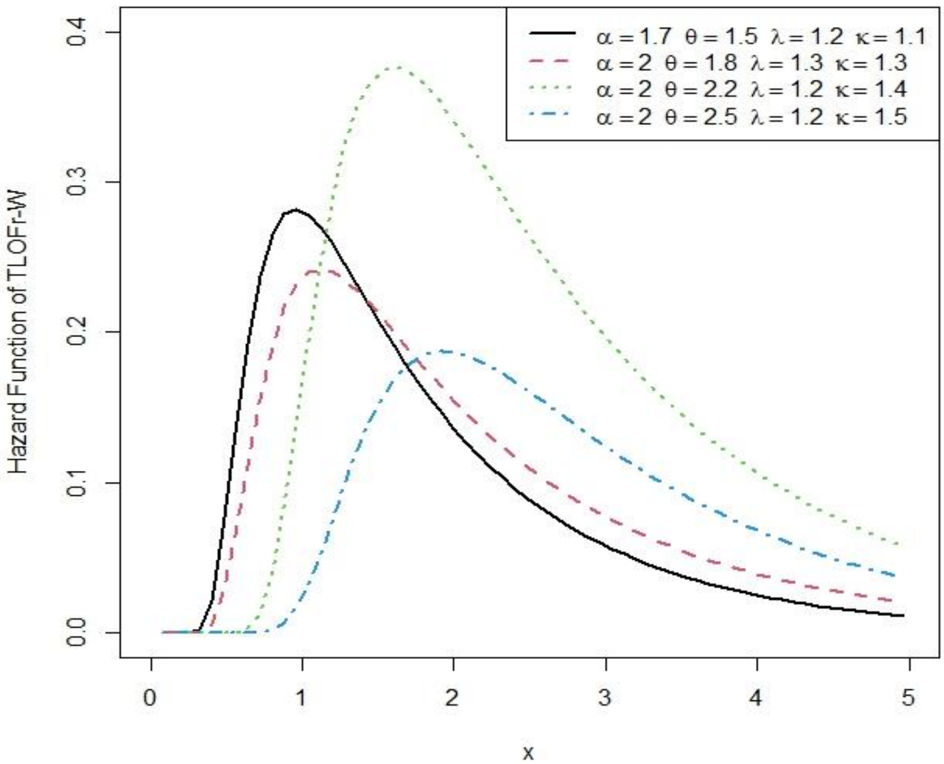

Fig. 3: Hazard Function Plot of TL-OFr-Weibull Distribution

Fig. 3 displays the shapes and the behaviour of the hazard function of the TL-OFr-Weibull distribution with different values of the parameters. The graph shows the parameter values exhibiting upside-down bathtub failure rates.

The quantile function in Equation (24), also known as the inverse CDF, is essential for simulation as it allows random sampling from the distribution and supports Monte Carlo techniquesfacilitates Monte Carlo techniques, and ensures realistic data generation in various fields. Many simulation methods rely on generating random numbers from a uniform distribution U(0,1).

$$x=\theta\left\{-\log\left(1+\left[\left\{\log\left[1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2}}\right]\right\}\right]^{-\frac{1}{\lambda}}\right)\right\}^{\frac{1}{k}} \tag{24}$$

5.0 Simulation Study

A simulation study was conducted using the Monte Carlo Simulation method to compute the mean, bias and mean square error (MSE) of the estimated parameters using the maximum likelihood estimation method. The stimulated data was generated using the quantile function defined in equation (24) for different sample sizes, n = 20, 50, 75, 100, 150 and 250, replicated 1000 times. For each sample size, the parameter values are $\alpha=2.0,\ \theta=1.6,\ \lambda=0.5,\ k=0.3$. Table 1 presents the results of the Simulation with estimates, bias and RME from the proposed distribution.

Table 1, presents the results obtained from the Monte Carlo Siimulation study from TL-OFr-Weibull Distribution. The results indicated that the bias and RMSE decrease toward zero with an increase in sample size. However, the actual value of the parameters and the estimated values are almost the same at different sample sizes and iterative levels for the MLE technique. —These results support the consistency of the parameter estimates

Table 1: Results of the Simulated Data from TL-OFr-Weibull Distribution

N	Properties	α=2.0	θ=1.6	λ=0.5	κ=0.3
20	Estimate	2.0088	1.6762	0.8259	0.3597
	Bias	0.0088	0.0762	0.0259	0.0597
	MSE	0.2874	0.2143	0.1075	0.0259
50	Estimate	2.0029	1.668	0.8232	0.3223
	Bias	0.0029	0.068	0.0232	0.0223
	MSE	0.1521	0.1553	0.0569	0.0082
75	Estimate	2.0098	1.6458	0.819	0.3145
	Bias	0.0098	0.0458	0.019	0.0145
	MSE	0.1021	0.1024	0.0406	0.0055
100	Estimate	2.0156	1.6448	0.8173	0.3105
	Bias	0.0156	0.0448	0.0173	0.0105
	MSE	0.0861	0.0924	0.0317	0.0041
150	Estimate	2.0061	1.6429	0.8118	0.3081
	Bias	0.0061	0.0429	0.0118	0.0081
	MSE	0.0589	0.069	0.0258	0.0031
250	Estimate	1.9939	1.6427	0.8049	0.3067
	Bias	-0.0061	0.0427	0.0049	0.0067
	MSE	0.0413	0.0531	0.0165	0.0021

5.1 Application with Real Datasets

In this section, two real-life datasets are applied to demonstrate the performance of the Topp Leone Odd Fréchet-Weibull distribution. Comparisons were made with Odd Fréchet Inverse (OFIE), Exponential Generalized Fréchet (EGF), Topp Leone Kumaraswamy Exponential (TLKE) and Weighted Weibull Exponential (WWE) distributions for their log-likelihood (LL), Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Kolmogorov Smirnov (KS) with p-value values.

The first dataset used, reports the 61 strength of carbon fibres tested under tension at gauge lengths of 10 mm. The data was used and analyzed by Bi and Gui (2017). The observations are:
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 4.024, 4.027, 4.225, 4.395, 5.020
Table 2, presents the results of the analysis as follows:

Table 2: MLEs, LL, AIC, BIC and KS(*p*-value)

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\kappa}$	$\hat{\beta}$	$\hat{\sigma}$	LL	AIC	BIC	KS (*p*-value)
TLOFW	0.309	1.177	4.099	3.309	-	-	**52.731**	**113.907**	**120.907**	**0.077(0.867)**
OFIE	1.864	-	3.977	-	-	-	56.387	116.774	120.996	0.100(0.570)
EGF	10.100	1.502	-	-	4.102	4.311	53.050	114.100	122.543	0.089(0.715)
TLKE	8.341	6.532	0.916	-	0.916	-	52.993	113.986	122.429	0.084(0.778)
WWE	0.085	3.124	4.250	-	-	-	60.265	126.529	132.862	0.107(0.492)

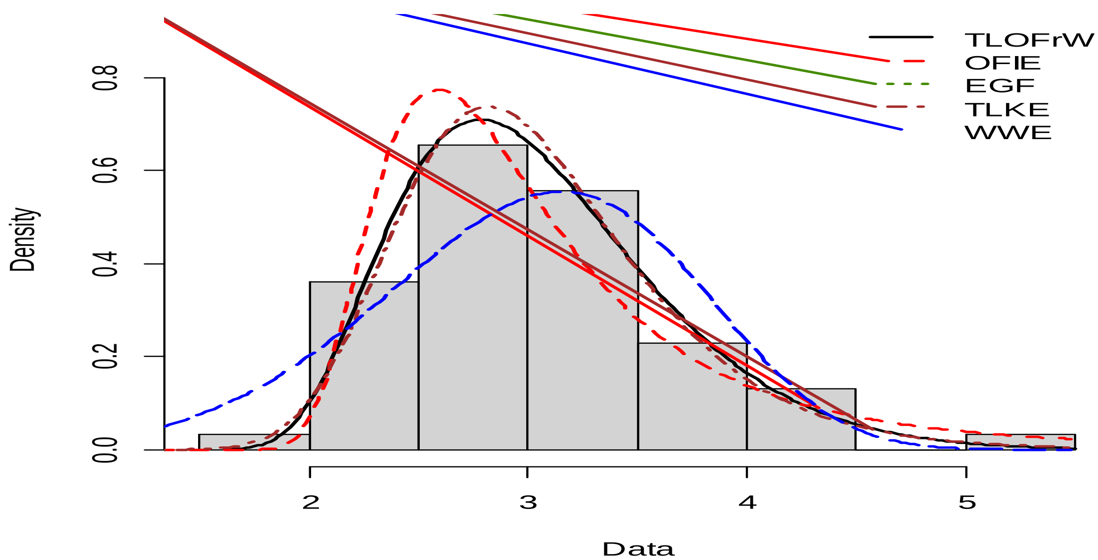

Fig. 4: Density plot of Topp-Leone-Odd Frechet Weibull distribution

The second dataset is on 63 observations for strengths of 1.5 cm glass fibres: This Data has previously been used by Sanusi *et al.*, (2020). The data is by workers at the UK National Physical Laboratory Study.

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24

Table 3, presents the results of the analysis as follows

Table 3: MLEs, LL, AIC, BIC and KS(*p*-value)

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\kappa}$	$\hat{\beta}$	$\hat{\sigma}$	LL	AIC	BIC	KS(*p*-value)
TLOFW	5.135	0.114	1.054	4.889	-	-	**11.880**	**31.759**	**40.332**	**0.1040(0.5030)**
OFIE	0.845	-	1.930	-	-	-	53.899	111.799	116.085	0.2612(0.0004)
EGF	19.735	1.640	-	-	0.548	3.419	25.588	59.175	67.748	0.2490(0.001)
TLKE	15.189	12.627	0.618	-	0.957	-	17.673	43.346	51.919	0.2008(0.012)
WWE	0.425	0.162	4.229	-	-	-	14.612	32.225	41.654	0.1488(0.123)

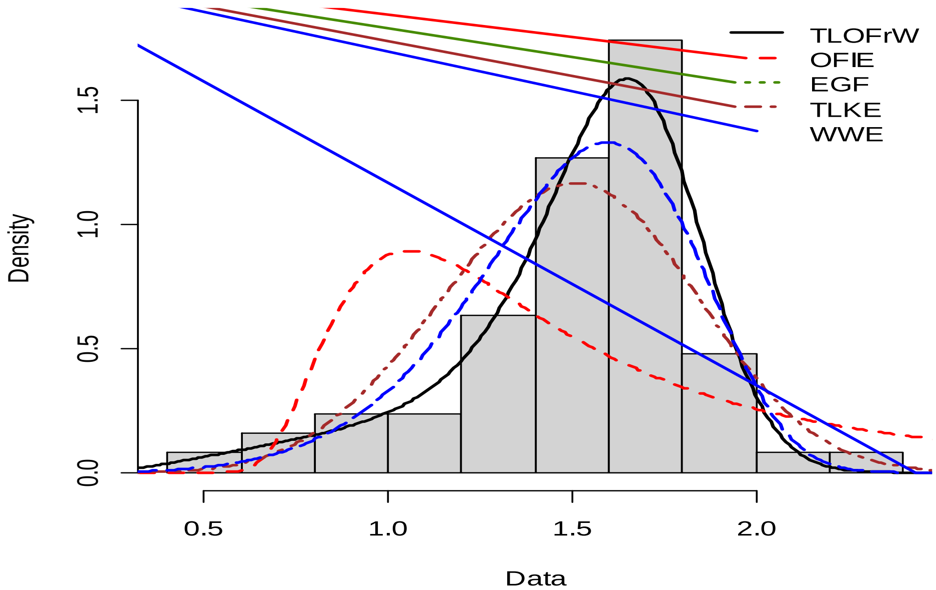

Fig. 5: Density plot of Topp-Leone-Odd Fréchet Weibull distribution

Tables 2 and 3 present the results of the analysis of the two datasets. The results of the analysis of Topp-Leone-Odd Fréchet Weibull were compared with Odd Fréchet Inverse Exponential Weibull (OFIE), Exponential Generalized Fréchet (EGF), Topp Leone Kumaraswamy Exponential (TLKE), and Weighted Weibull Exponential (WWE) distributions. The proposed Topp-Leone-Odd Fréchet-Weibull distribution has displayed good potential and has proven to be the better distribution because it has the least AIC, BIC and LL. Also, the proposed model displays the least Kolmogorov Smirnova value with the highest corresponding p-value. To further validate the results obtained, the visual examination of the fit presented in Figures 4 and 5 also, confirms the superiority of the proposed distribution amongst its comparators. Thus, the proposed distribution fits the two datasets better.

7.0 Conclusion

This research underscores the critical role of flexible lifetime models in effectively capturing the dynamic characteristics inherent in real-world datasets. To address this need, the study introduced a novel distributional framework known as the Topp-Leone-Odd Fréchet family of distributions, developed by embedding the Odd Fréchet-G family into the Topp-Leone-G distribution. The resulting model is distinguished by its exceptional flexibility, enabling it to model a wide range of data behaviors with high precision. The empirical application of this model to two distinct real-life datasets provided compelling evidence of its superior performance, as it consistently delivered better fits compared to existing competing models. This finding affirms the model's practical relevance and adaptability to complex data structures. The study concludes that the Topp-Leone-Odd Fréchet distribution is not only a valuable addition to the growing body of generalized distributions but also a powerful tool for statistical modeling and inference. Its high adaptability and robust fitting capability make it a promising candidate for applications in various scientific, engineering, and industrial fields where modeling lifetime or extreme value data is essential.

In light of these findings, it is recommended that future researchers consider the Topp-Leone-Odd Fréchet distribution as a baseline or alternative model when dealing with lifetime and reliability data. Additionally, the model should be further examined and extended to multivariate contexts or censored data frameworks, which would broaden its utility and enhance its theoretical underpinnings. This study serves not only as a demonstration of the model's applicability but also as a foundation for future investigations into more sophisticated distributional structures.

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