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Closed Form Solutions of a Re-Insurer's Surplus, Stochastic and Time-Dependent Investment Returns with Random Parameters

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Abstract: *One of the challenges faced by insurance companies is the ability to manage the risk of their client (policyholders) and pay their claims whenever the need arises; hence there is a need to study the portfolio of a reinsurer which is a combination of his surplus and invested funds in the financial market. To achieve this, we consider cases where the expected rates of return of the risky asset are linear and quadratic. Ito's lemma, maximum principle and variable separation technique were used in solving for the closed-form solutions of the prices of the risky assets for all considered cases. Similarly, the closed-form solutions of the reinsurer's surplus were obtained for both time-dependent and stochastic cases. Finally, some numerical simulations were presented to study the effect of some sensitive parameters on the price process of the risky asset and also the behaviour of the reinsurer's surpluses with time. It was observed that the price process of the risky asset is an increasing function of the expected rate of return and a decreasing function of the instantaneous volatilities while the surplus process does not necessarily depend on the expected rate of returns but on contribution rate and the number claims to be serviced at any given time.*

Keywords: Re-insurer's surplus, Risky assets, Ito's lemma, Geometric Brownian motion, financial market

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1.0 Introduction

Most financial institutions such as banks, pensions and insurance companies are majorly in operation with the sole aim of making a profit. Insurance companies are generally seen as risk transfer institutions which offer insurance to policyholders and the insurance offered is seen as a risk transfer tool and can also be seen as a risk preventive tool (Akpanibah & Osu, 2017; Malik *et al*, 2020). Since risk is part of our daily life, the importance of insurance cannot be underestimated in our day-to-day activities. However, some insurance companies can also buy reinsurance policies from other insurance companies to effectively manage the risk of its members by transferring part of the burden of payment of claims to the reinsurer.(Osu et al, 2022; Ini*et al*, 2021;Zhu and Li, 2020; Xiao *et al*, 2019; Li *et al*, 2014; Mwanakatwe, 2019). Reinsurance is a powerful tool for insurers to transfer insurance risk (Zhu & Li, 2020).

Because most insurers and reinsurers are authorized to invest in the financial market, they can make profits not only by the premiums paid by clients but also by investing in different assets obtainable in the financial market. Different from the other institutional investors, both the insurer and the reinsurer are faced with double risks that exist both in the financial and insurance markets. To reduce the risk of claims, the insurer can buy reinsurance contracts from the reinsurer and transfer part of the risk of claims to the reinsurer, since the reinsurer is more risk-seeking than the insurer (Xiao *et al*, 2019). Hence a good number of authors have studied some investment strategies and reinsurance policies for an insurer and reinsurer respectively.

In (Li *et al*, 2014) the optimal reinsurance investment model for maximizing the product of the insurer's and the reinsurer's utilities was studied under a CEV model by maximizing the expected two exponential utility functions where the claim process was using Brownian motion with drift.In (Akpanibah & Osu, 2017), an investment strategy of an insurer with stochastic premium was obtained with the Legendre transform technique under exponential utility. In their work, the premium received by the insurance company was assumed to be stochastic. Lin and Li, (2011), and Wang *et al*. (2018) modelled their investment problem under the Jump diffusion process and proceed to obtain an insurer's investment strategy under exponential utility maximation. Osu & Ihedioha(2013), obtained an insurer's investment strategy and also studied the probability of ruin of the insurance company. Sultan and Aqsa(2017) studied the wealth investment strategy for insurance companies and also the probability of ruin of such a company, they observed that the probability of ruin helps the insurance company to know the true state of the company.In (Ini *et al*, 2021) a mathematical model for an insurer's portfolio and reinsurance strategy was studied under the constant elasticity of variance model using a utility function which exhibits constant relative risk aversion. In their work, they obtained an insurer's investment strategy and that of a reinsurer under exponential utility.

In (Malik *et al* , 2020) the investment strategy and the reinsurance policy wereobtained under the CEV model using the fractional power utility function; here, they obtained the optimal reinsurance policy and investment strategy by solving the resultant non-linear partial differential equation by power transformation technique and find out thathigher value of the reinsurer's safety loading leads to a decrease in the optimal reinsurance policy and to maintain a stable income, the insurer should buy less reinsurance policy. (Mwanakatwe *et al*, 2019), studied optimal investment and risk control for an insurance company where the investment model followed a stochastic framework and obtained an investment strategy and reinsurance policy which maximisedthe expected utility function.

In (Gu *et al*, 2012, Li & Gu, 2013), the optimal excess of loss reinsurance and investment strategy was obtained by maximizing the value function under exponential utility.

(Chunxiang *et al*., 2018) studied optimal excess of loss reinsurance and investment problems with delay and jump-diffusion risk. In (Ihedioha, 2015 & Sheng, 2016) the optimal reinsurance and investment problem of maximizing the expected power utility function was investigated using a promotional budget. In general, the study of investment strategy has been done by other researchers such as (Akpanibah *et al*, 2019; Amadi et al, 2022; Osu *et al*, 2017;Akpanibah & Samaila, 2017;Osu *et al*, 2017; Egbe *et al*, 2013; Deng *et al* 2019)

However, the need to analyze the stochastic model prices of the risky asset which plays a vital role in the determination of the investment strategy and reinsurance policy has become very important and necessary. In (Davies *et al*, 2019), stability analysis of a stochastic model of price change at the floor of a stock market was investigated; In their analysis, real steps were derived and were useful in the determination of equilibrium price and growth rate of stock prices.

In (Osu *et al*, 2009), the unstable nature of stock market prices was investigated using the proposed differential equation model.(Adeosun *et al*, 2015) studiedthe stochastic analysis of the behaviour of stock prices. Results reveal that the proposed model is efficient for the prediction of stock prices. Also, (Ofomata *et al*, 2019) investigated some selected stock prices in the Nigerian Stock Exchange (NSE) by stochastic formulations. They discovered that the drift and volatility coefficients for the stochastic differential equations were determined and they used the Euler-Maruyama method for the system of SDE'S to stimulate the stock prices. Similarly, (Akpanibah & Catherine, 2023), studied stock market price prediction under geometric Brownian motion (GBM).

More so (Osu, 2010) studied the fluctuation of stock market prices by determining the conditions for finding the equilibrium price, necessary and sufficient conditions for dynamic stability and convergence to equilibrium of the growth rate of the valued function of shares.

From the work of (Ihedioha and Osu, 2015 and Osu *et al*, 2022), It is obvious that the reserve of risk of the two insurance variables revolves according to a Brownian motion under which their optimal probability of existence was tackled. In their work, the point of interest was the risky asset price, risk-free asset price and over-plus of company's claims which changes over time between the one who insures and reinsures. Due to these changes affecting stock variables, they added sources of randomness in their formulated model and obtained an analytical solution for time-varying investment returns. Hence, in this paper, we study a time-varying investment model for the risky asset prices different from the ones in (Ihedioha & Osu, 2015 and Osu *et al*, 2022) by introducing some random parameters, and obtain deterministic and stochastic time-varying investment returns for our assets. Also obtained, was the surplus of the reinsurer for both linear and quadratic appreciation rates of the risky asset under power utility.

2.0 Methodology and Model Formulations

2.1 Methodology

Definition 2.1 A probability space $(\Omega, \mathcal{F}, \mathbb{p})$, consist of a set Ω, a collection of subsets $\mathcal{F}$ of Ω and a probability measure $\mathbb{p}$, which specifies the probability of each event $V \in \mathcal{F}$. The collection $\mathcal{F}$ is assumed closed under the operation of countable union and taking complement (σ- field).

Definition 2.2 Let $\mathfrak{T}_0$ be a set of all open subsets in a topological space Ω and $\mathfrak{U}_{\mathfrak{T}_0}$ a σ-algebra generated by set $\mathfrak{T}_0$. Then $\mathfrak{U}_{\mathfrak{T}_0}$ is known as the Borel σ-algebra on Ω and the elements $\mathfrak{B} \in \mathfrak{U}_{\mathfrak{T}_0}$ are known as the Borel sets.

Definition 2.3 $\mathcal{W}_{\mathfrak{U}}(S,T)$ denotes the class of processes $f(t, w) \in \mathfrak{R}$ satisfying:

a. $(t, w) \to f(t, w)$ *is* $\mathfrak{B} \times \mathcal{F}$-measurable, where $\mathfrak{B}$ denotes the Borel σ-algebra on $[0, \infty)$;

b. There exists an increasing family of σ-algebras $\mathfrak{U}(t)$ with $t \geq 0$, such that $\mathfrak{B}_0$ is a martingale with respect to $\mathfrak{U}(t)$ and that $f(t)$ is $\mathfrak{U}(t)$-adapted*;*

c. $\mathbb{p}\left[\int_S^T f(t, w)^2\, ds < \infty\right] = 1.$

Definition 2.4 Let $\mathfrak{B}_t$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{p})$. A (one-dimensional) Itô process (or stochastic integral) is a stochastic process $\mathcal{Y}(t)$ on $(\Omega, \mathcal{F}, \mathbb{p})$ of the form

$$\mathcal{Y}(t) = \mathcal{Y}\begin{pmatrix} (0) + \int_S^T n(s,\omega)\, ds \\ + \int_S^T \sigma(s,\omega)\, d\mathfrak{B}_t \end{pmatrix} \quad (1)$$

where $\sigma, n \in W_H$ so that

$$\mathbb{p}\left[\int_S^T n(s,\omega)^2\, ds < \infty, \forall\, t \geq 0\right] = 1.$$

We also assume that $\mathfrak{T}$ is $\mathfrak{U}(t)$ −adapted, where $\mathfrak{U}(t)$ is an increasing family of σ-algebras, $\{\mathfrak{U}(t)\}\ t \geq 0$, such that $\mathfrak{B}_t$ is a martingale concerning $\mathfrak{U}(t)$, and

$$\mathbb{p}\left[\int_S^T |\sigma(s,\omega)|\, ds < \infty, \forall\, t \geq 0\right] = 1.$$

Assuming $\mathcal{Y}(t)$ is an Itô process in the of form (3.1), the differential form of (3.1) is given as

$$d\mathcal{Y}(t) = n dt + \sigma d\mathfrak{B}_t \quad (2)$$

Where n is the drift and σ, isthe standard deviation representing the instantaneous volatility.

Remark 1. Let $\mathcal{Y}(t)$ be an Itô process given by (2) Let $h(t, y) \in C^2([0, \infty) \times \mathfrak{R})$. Then $h(t, \mathcal{Y}(t))$ is again an Itô process, and

$$dh(t, \mathcal{Y}(t)) = \begin{pmatrix} \frac{\partial h(t,\mathcal{Y}(t))}{\partial t} dt + \frac{\partial h(t,\mathcal{Y}(t))}{\partial y} d\mathcal{Y}(t) \\ + \frac{1}{2}\frac{\partial^2 h(t,\mathcal{Y}(t))}{\partial y^2} (d\mathcal{Y}(t))^2 \end{pmatrix}, \quad (3)$$

where $(d\mathcal{Y}(t))^2 = d\mathcal{Y}(t) d\mathcal{Y}(t)$ is computed according to the rules

$dtdt = dtd\mathfrak{B}_t = d\mathfrak{B}_t dt = 0;\ d\mathfrak{B}_t d\mathfrak{B}_t = dt.$

2.2 Model Formulation

In this section, we derived the insurance risk model with reinsurance and investment. The model surplus process via the Classical Cramer-Lundberg model, and assumed that an insurer can invest in a risky asset modelled by the GBM model. We finish this section by presenting the wealth process model and surplus process.

2.2.1 Surplus Process

In this subsection, we formulate the surplus process of the insurer. In insurance, the surplus process is the process of accumulation of wealth. To derive the surplus process, we need the claim process. From the work of Cao & Wan (2009) and Li *et al* (2015), the claim process $\mathcal{A}(t)$ which follows the Brownian motion with drift is given thus

$$d\mathcal{A}(t) = e_1 dt - e_2 d\mathfrak{B}_t(t) \quad (4)$$

Where e_1 and e_2 are positive constants, and $\mathfrak{B}_t(t)$ is a standard Brownian motion defined on the complete probability space $(\Omega;\ \mathcal{F}_t;\ \mathcal{P})$.

From the expected value principle in Nozadi (2014), an insurer's premium rate is given as

$$\mathscr{k} = (1+\theta)e_1 \quad (5)$$

where $\theta > 0$ is the safety loading of the insurer.

Also, from Nozadi (2014), a classical Cramer-Lundberg model for surplus process is given by

$$\mathcal{N}(t) = \mathscr{n}_0 + \mathscr{k}t - \mathcal{A}(t)t \ \geq\ 0 \quad (6)$$

where $\mathcal{N}(t)$ and $\mathscr{n}_0$ are the insurers capital at time t and initial capital $\mathcal{N}(0) = \mathscr{n}_0$, respectively. The differential form of (6) is given thus

$$\frac{d\mathcal{N}(t)}{\mathcal{N}(t)} = \mathscr{k}dt - d\mathcal{A}(t) = e_1\theta dt + e_2 d\mathfrak{B}_t(t), \quad (7)$$

Furthermore, the insurer can buy reinsurance contracts to reduce risk. Suppose the insurer pays reinsurance premiums continuously at rate $\mathscr{k}_1 = (1+\eta)e_1$ where $\eta > \theta > 0$ is thesafety loading of the reinsurer. So, the surplus process $\mathcal{N}_1(t)$ associated with the reinsurance of the insurer follows

$$\frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)} = \begin{pmatrix} \mathscr{k}dt - \big(1-m(t)\big)d\mathcal{A}(t) \\ -\mathscr{k}_1 m(t)dt \end{pmatrix} \quad (8)$$

$$\frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)} = \begin{pmatrix} \big(\theta - \eta m(t)\big)e_1 dt \\ +e_2\big(1-m(t)\big)d\mathfrak{B}_t(t) \end{pmatrix} \quad (9)$$

where $m(t)$ is proportion reinsurance at time t

2.2.2 The Financial Market Model

Consider a portfolio comprising of one risk-free asset and two risky assets in a financial market which is open continuously for an interval $t \in [0,T]$ where T is the expiration date of the investment. Let $\{\mathfrak{B}_0(t), \mathfrak{B}_1(t), \mathfrak{B}_2(t) : t \geq 0\}$ be standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{p})$where Ω is a real space and $\mathbb{p}$ is a probability measure and $\mathcal{F}$ is the filtration which represents the information generated by the three Brownian motions.

Let $\mathcal{S}_0(t)$ denote the price of the risk-free asset at time t and the model is given as follows

$$\frac{d\mathcal{S}_0(t)}{\mathcal{S}_0(t)} = \mathscr{r}(t)dt \qquad \mathcal{S}_0(0) = \mathscr{s}_0 > 0 \quad (10)$$

where $\mathscr{r}$ is the predetermined interest rate process Let $\mathcal{K}_1(t)$ denote the price of the riskless asset described by the GBM model when the expected rate of return is linear and the dynamics of the riskless asset is given at $t \geq 0$ as follows for the deterministic case.

$$\frac{d\mathcal{K}_1(t)}{\mathcal{K}_1(t)} = \rho\mu_1 dt \quad (11)$$

Similarly, let $\mathcal{K}_2(t)$ denote the price of the riskless asset described by the GBM model when the expected rate of return is linear and the dynamics for the stochastic form is given thus;.

$$\frac{d\mathcal{K}_2(t)}{\mathcal{K}_2(t)} = \begin{pmatrix} \rho u_1 dt + \ \sigma_1(t)d\mathfrak{B}_1(t) \\ +\sigma_2(t)d\mathfrak{B}_2(t) \end{pmatrix} \quad (12)$$

Also, Let $\mathcal{M}_1(t)$ denote the price of the risky asset described by the GBM model when the expected rate of return is quadratic and the dynamics for both deterministic and stochastic forms are given as follows

$$\frac{d\mathcal{M}_1(t)}{\mathcal{M}_1(t)} = \rho u_1^2 dt \quad (13)$$

$$\frac{d\mathcal{M}_2(t)}{\mathcal{M}_2(t)} = \begin{pmatrix} \rho u_1^2 dt + \sigma_2(t)d\mathfrak{B}_1(t) \\ +\sigma_3(t)d\mathfrak{B}_2(t) \end{pmatrix} \quad (14)$$

where u_1 and u_2 are the appreciation rates of the stock,σ_1, σ_2 and σ_3 is theinstantaneous volatilities of the risky asset and $\rho$is random noise due to environmental effects.

3.0 Optimization Problem

Let ℓ be the optimal portfolio strategy and we define the utility attained by the investor from a given state x at time t as

$$\mathcal{H}_\ell(t,z) = E_\ell\left[\mathcal{V}(\mathcal{Z}(T)) \mid \mathcal{Z}(t) = z\right], \quad (15)$$

where$t$ is the time, r is the risk free interest rate and z is the wealth. The objective here is to determine the optimal portfolio strategy and the optimal value function of the investor given as

$$\ell^* \text{and} \mathcal{H}(t,z) = \sup_{\ell} \mathcal{H}_\ell(t,z) \quad (16)$$

Respectively such that

$$\mathcal{H}_{\ell^*}(t,z) = \mathcal{H}(t,z). \quad (17)$$

Let $\mathcal{Z}(t)$ be the insurer's wealth at time t and the amount of wealth of the insurer invested on risk asset at time t denoted by $\ell(t)$ and $\mathcal{Z}(t)$represents the wealth of theinsurer. So, the remainder $\mathcal{Z}(t) - \ell(t)$invested in risk-freeasset. Then the differential form associated with the wealth size is given as:

$$d\mathcal{Z}(t) = \begin{pmatrix} (\mathcal{Z}(t) - \ell)\frac{d\mathcal{S}_0(t)}{\mathcal{S}_0(t)} + \ell\frac{dM_2(t)}{M_2(t)} \\ + \frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)} \end{pmatrix} \quad (18)$$

substituting (9), (10) and (3.12) into (18), we have

$$d\mathcal{Z}(t) = \begin{pmatrix} (r\mathcal{Z}(t) + \ell(\rho\mu_1 - r) + (\theta - \eta m(t))e_1)dt \\ +e_2(1 - m(t))d\mathcal{B}_0(t) \\ +\ell(\sigma_1(t)d\mathcal{B}_1(t) + \sigma_2(t)d\mathcal{B}_2(t)) \end{pmatrix} \quad (19)$$

Applying Ito's lemma and maximum principle, the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE is obtained by maximizing (15) subject to (19) as follows

$$\begin{Bmatrix} \mathcal{H}_t + \begin{bmatrix} r\mathcal{Z}(t) + \ell(\rho\mu_1 - r) \\ +(\theta - \eta m(t))e_1 \end{bmatrix}\mathcal{H}_z \\ +\frac{1}{2}\begin{pmatrix} e_2^2(1 - m(t))^2 \\ +\ell^2(\sigma_1^2 + \sigma_1^2) \end{pmatrix}\mathcal{H}_{zz} \end{Bmatrix} = 0 \quad (20)$$

Differentiating (3.6) concerning ℓ *and* m, we obtain the first-order maximizing condition for equation (20) as

$$\ell^* = -\frac{(\rho\mu_2 - r)\mathcal{H}_z}{(\sigma_1^2 + \sigma_2^2)\mathcal{H}_{zz}} \quad (21)$$

$$m^* = \frac{\eta e_1}{e_2^2}\frac{\mathcal{H}_z}{\mathcal{H}_{zz}} + 1 \quad (22)$$

Substituting (21) and (22) into (20), we have

$$\begin{Bmatrix} \mathcal{H}_t + +[rz + (\theta - \eta)e_1]\mathcal{H}_z \\ -\frac{1}{2}\left[\frac{(\rho\mu_2 - r)^2}{\sigma_1^2 + \sigma_1^2} - \frac{\eta^2 e_1^2}{e_2^2}\right]\frac{\mathcal{H}_z^2}{\mathcal{H}_{zz}} \end{Bmatrix} = 0 \quad (23)$$

Since we are interested in obtaining the overplus of the reinsurer, we will need to find a solution for m^* by solving (23).

From the work of Malik *et al* (2020), the optimal reinsurance policy (ORP) under the power utility function is given as

$$m^* = 1 + \frac{\eta e_1 z}{e_2^2(\gamma - 1)}, \quad (24)$$

Where z is the insurance wealth level of an insurer and γ is the constant relative risk aversion coefficient.

Substituting (24) into (9), we have

$$\frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)} = \begin{pmatrix} \left(\theta - \eta\left(1 + \frac{\eta e_1^2 z}{e_2^2(\gamma - 1)}\right)\right)dt \\ +\frac{\eta e_1 z}{e_2(1 - \gamma)}d\mathcal{B}_0(t) \end{pmatrix} \quad (25)$$

Hence the deterministic surplus of the reinsurer is given as

$$\frac{d\mathcal{N}_2(t)}{\mathcal{N}_2(t)} = \left(\theta - \eta(1 + \frac{\eta e_1^2 z}{e_2^2(\gamma-1)})\right) dt \quad (26)$$

4.0 Main Results

In this section, we are interested in solving for the prices of risk-free assets and risky assets under different assumptions. We will give solutions to the problems modeled in (3.9)-(3.12) and (3.23)-(3.24).

4.1 Solution of Price Process of Risky Asset for Deterministic Case with Linear Rate of Expected Returns

$$\begin{cases} \frac{dK_1(t)}{K_1(t)} = \rho\mu_1 dt \\ K_1(0) = e^{-1} \end{cases} \quad (27)$$

Solving (27), we have

$$K_1(t) = G_1 e^{\rho\mu_1 t} \quad (28)$$

Applying the initial condition to (28), we have

$$K_1(t) = e^{\rho\mu_1 t - 1} \quad (29)$$

Hence the price process of the riskless asset in (27) is given in (29)

4.2 Solution of Price Process of Risky Asset for Stochastic Case with Linear Rate of Expected Returns

$$\begin{cases} \frac{dK_2(t)}{K_2(t)} = \begin{pmatrix} \rho\mu_1 dt + \sigma_1(t)d\mathcal{B}_1(t) \\ +\sigma_2(t)d\mathcal{B}_2(t) \end{pmatrix} \\ K_2(0) = e^{-t} \end{cases} \quad (30)$$

To solve (30), we apply the Itô process in (3) as follows

Let

$$\hbar\left(t, K_2(t)\right) = lnK_2(t)\ , \quad (31)$$

Then from Itô process,

$$d\hbar\left(t, K_2(t)\right) = \begin{pmatrix} \frac{\partial\hbar(t,K_2(t))}{\partial t} dt \\ + \frac{\partial\hbar(t,K_2(t))}{\partial K_2(t)} dK_2(t) \\ + \frac{1}{2}\frac{\partial^2\hbar(t,K_2(t))}{\partial K_2(t)^2}(dK_2(t))^2 \end{pmatrix}, \quad (32)$$

where $(dK_2(t))^2 = dK_2(t)dK_2(t)$ is computed according to the rules

$dtdt = dtd\mathcal{B}_t = d\mathcal{B}_t dt = 0;\ d\mathcal{B}_t d\mathcal{B}_t = dt.$ From (32),

$$dK_2(t) = K_2(t)\begin{pmatrix} \rho\mu_1 dt + \sigma_1(t)d\mathcal{B}_1(t) \\ +\sigma_2(t)d\mathcal{B}_2(t) \end{pmatrix} \quad (33)$$

Substituting (33) into (32), we have

$$d\hbar\left(t, K_2(t)\right) = \begin{pmatrix} \hbar_t dt \\ +\hbar_{K_2}K_2(t)\begin{pmatrix} \rho\mu_1 dt \\ +\sigma_1(t)d\mathcal{B}_1(t) \\ +\sigma_2(t)d\mathcal{B}_2(t) \end{pmatrix} \\ +\frac{1}{2}\hbar_{K_2K_2}K_2^2(\sigma_1^2 + \sigma_2^2)dt \end{pmatrix}, \quad (34)$$

Differentiating (31), we have

$$\begin{cases} \hbar_t = 0 \\ \hbar_{K_2} = \frac{1}{K_2} \\ \hbar_{K_2K_2} = -\frac{1}{K_2^2} \end{cases} \quad (35)$$

Substituting (35) into (34), we have

$$d\hbar\left(t, K_2(t)\right) = \begin{pmatrix} (\rho\mu_1 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))dt \\ +\sigma_1(t)d\mathcal{B}_1 + \sigma_2(t)d\mathcal{B}_2 \end{pmatrix}, \quad (36)$$

Integrating (36), we have

$$\hbar\left(t, K_2(t)\right) = \begin{pmatrix} (\rho\mu_1 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))t \\ +\sigma_1(t)\mathcal{B}_1 + \sigma_2(t)\mathcal{B}_2 \end{pmatrix} \quad (37)$$

From (30),

$$lnK_2(t) - lnK_2(0) = \begin{pmatrix} (\rho\mu_1 \\ -\frac{1}{2}(\sigma_1^2 + \sigma_2^2))t \\ +\sigma_1(t)\mathcal{B}_1 \\ +\sigma_2(t)\mathcal{B}_2) \end{pmatrix} \quad (38)$$

$$ln(\frac{K_2(t)}{K_2(0)}) = \begin{pmatrix} (\rho\mu_1 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))t \\ +\sigma_1(t)\mathcal{B}_1 + \sigma_2(t)\mathcal{B}_2 \end{pmatrix} \quad (39)$$

$$K_2(t) = K_2(0)Exp\begin{bmatrix} (\rho\mu_1 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))t \\ +\sigma_1(t)\mathcal{B}_1 + \sigma_2(t)\mathcal{B}_2 \end{bmatrix} (40)$$

But $K_2(0) = e^{-t}$, hence

$$K_2(t) = Exp\begin{bmatrix} (\rho\mu_1 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))t \\ +\sigma_1(t)\mathcal{B}_1 + \sigma_2(t)\mathcal{B}_2) \end{bmatrix} \quad (41)$$

Hence the price process of the risky asset in (30) is given in (41)

4.3 Solution of Price Process of Risky Asset for Deterministic Case with Quadratic Rate of Expected Returns

$$\begin{cases} \frac{dM_1(t)}{M_1(t)} = \rho\mu_1^2 dt \\ M_1(0) = e^{-1} \end{cases} \quad (42)$$

Solving (4.16), we have

$$M_1(t) = H_1 e^{\rho\mu_1^2 t} \quad (43)$$

Applying the initial condition to (42), we have

$$M_1(t) = e^{\rho\mu_1^2 t - 1} \quad (44)$$

4.4 Solution of Price Process of Risky Asset for Stochastic Case with Quadratic Rate of Expected Returns

$$\begin{cases} \frac{dM_2(t)}{M_2(t)} = \begin{pmatrix} \rho\mu_1^2 dt + \sigma_1(t)d\mathcal{B}_1(t) \\ +\sigma_2(t)d\mathcal{B}_2(t) \end{pmatrix} \\ M_2(0) = e^{-t} \end{cases} \quad (45)$$

To solve (45), we follow the same approach in solving (30) and obtain

$$M_2(t) = M_2(0)Exp\begin{bmatrix} \begin{pmatrix} \rho\mu_1^2 \\ -\frac{1}{2}(\sigma_1^2 + \sigma_2^2) \end{pmatrix} t \\ +\sigma_1(t)\mathcal{B}_1(t) + \sigma_2(t)\mathcal{B}_2(t) \end{bmatrix} \quad (46)$$

Applying the initial condition in (45), we have

$$M_2(t) = Exp\begin{bmatrix} \left(\rho\mu_1^2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) - 1\right) t \\ +\sigma_1(t)\mathcal{B}_1(t) + \sigma_2(t)\mathcal{B}_2(t) \end{bmatrix} \quad (47)$$

Hence the surplus process in (45) is given in (47)

4.5 Solution of Insurer's Surplus for Deterministic Case

$$\begin{cases} \frac{dN_1(t)}{N_1(t)} = \left(\theta - \eta(1 + \frac{\eta e_1^2 z}{e_2^2(\gamma-1)})\right) dt \\ N_1(0) = 1 \end{cases} \quad (48)$$

Solving (48), we have

$$N_1(t) = I_1 e^{\left(\theta - \eta(1 + \frac{\eta e_1^2 z}{e_2^2(\gamma-1)})\right)t} \quad (49)$$

Applying the initial condition in (48), we have

$$N_1(t) = e^{\left(\theta - \eta(1 + \frac{\eta e_1^2 z}{e_2^2(\gamma-1)})\right)t} \quad (50)$$

4.6 Solution of Surplus Process for Stochastic Case

$$\begin{cases} \frac{dN_2(t)}{N_2(t)} = \begin{pmatrix} \left(\theta - \eta\left(1 + \frac{\eta e_1^2 z}{e_2^2(\gamma-1)}\right)\right) dt \\ + \frac{\eta e_1 z}{e_2(1-\gamma)} d\mathcal{B}_0(t) \end{pmatrix} \\ N_2(0) = 1 \end{cases} \quad (51)$$

To solve (51), we follow the same approach in solving (30) and obtain

$$N_2(t) = N_2(0)Exp\left[\begin{pmatrix}\left(\theta - \eta\left(1 + \frac{\eta e_1 z}{e_2^2(\gamma-1)}\right)\right)e_1 \\ -\frac{1}{2}\left(\frac{\eta e_1 z}{(1-\gamma)}\right)^2\end{pmatrix}t \\ + \frac{\eta e_1 z}{e_2(1-\gamma)}\mathcal{B}_0(t)\right] \quad (52)$$

Applying the initial condition in (51), we have

$$N_2(t) = Exp\left[\begin{pmatrix}\begin{pmatrix}\theta \\ -\eta\left(1 + \frac{\eta e_1 z}{e_2^2(\gamma-1)}\right)\end{pmatrix}e_1 \\ -\frac{1}{2}\left(\frac{\eta e_1 z}{(1-\gamma)}\right)^2\end{pmatrix}t \\ + \frac{\eta e_1 z}{e_2(1-\gamma)}\mathcal{B}_0(t)\right] \quad (53)$$

5.0 Numerical Analysis

In this section, some numerical simulations are presented to analyze the impact of some sensitive parameters on the optimal control plans under exponential utility. To achieve this, the following data will be used unless otherwise stated; , $r(0) = 0.07, T = 20, \hbar_0 = 1, \vartheta = 0.02, m_{11} = 1,$ $m_{12} = 0.85, m_{21} = 0.95, m_{22} = 0.8,, \ell_1 = 0.12, \ell_1 = 1.0, \gamma = 0.05, e = 100, e_0 = 20.$

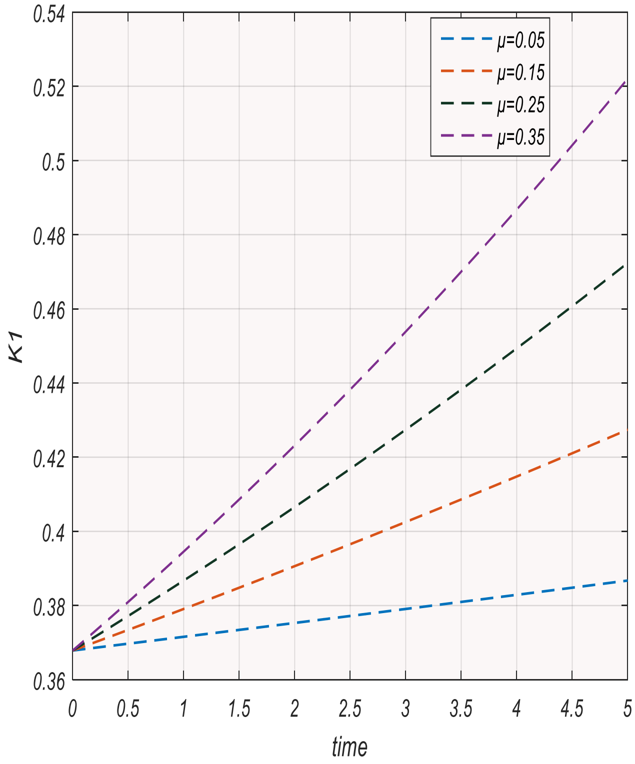

Fig. 1: The evolution of K_1 with different expected rate of returns

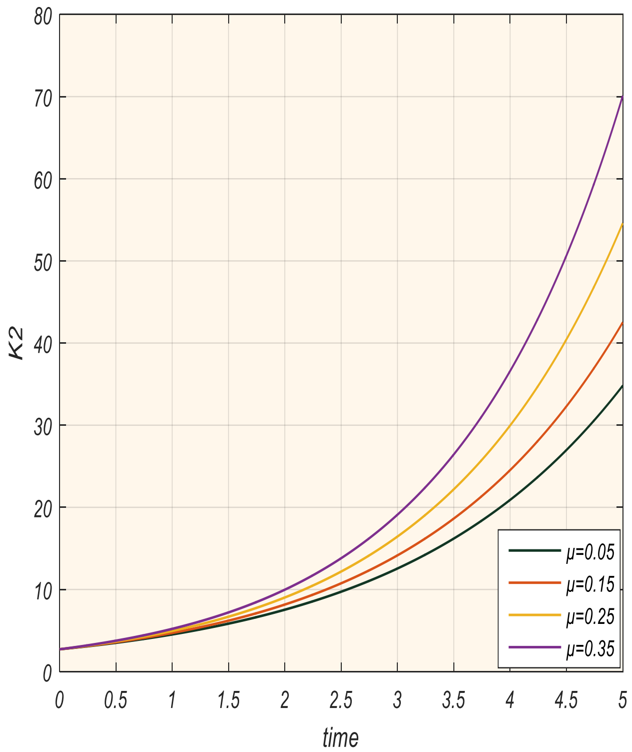

Fig. 2: The evolution of K_2 with different expected rate of returns

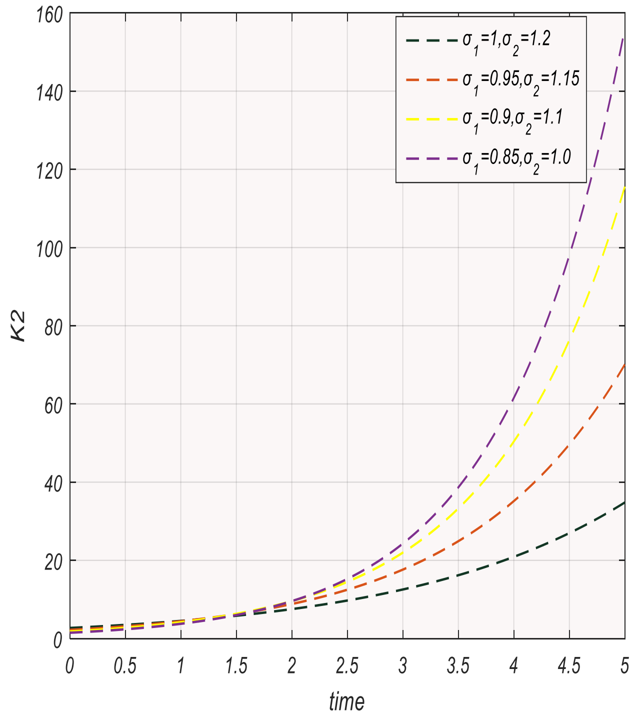

Fig. 3: The evolution of K_2 with different instantaneous volatility

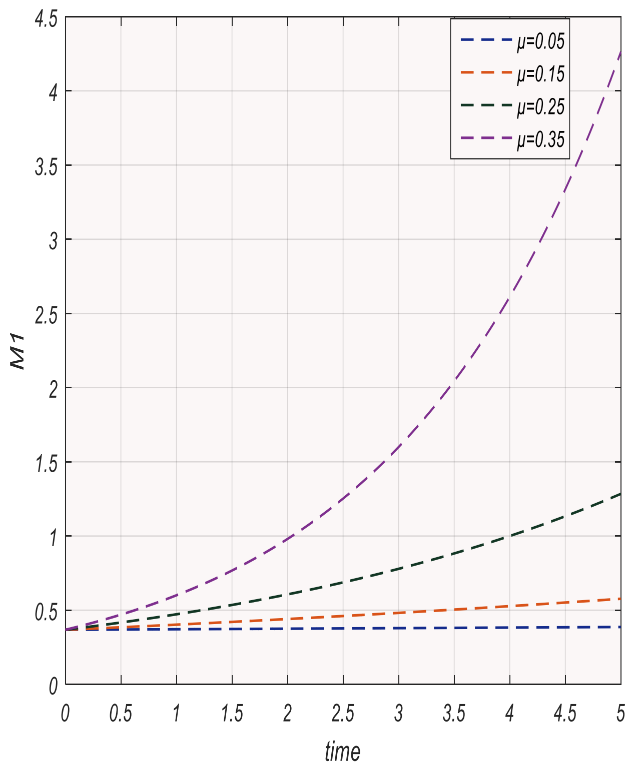

Fig. 4: The evolution of M_1 with different expected rate of returns

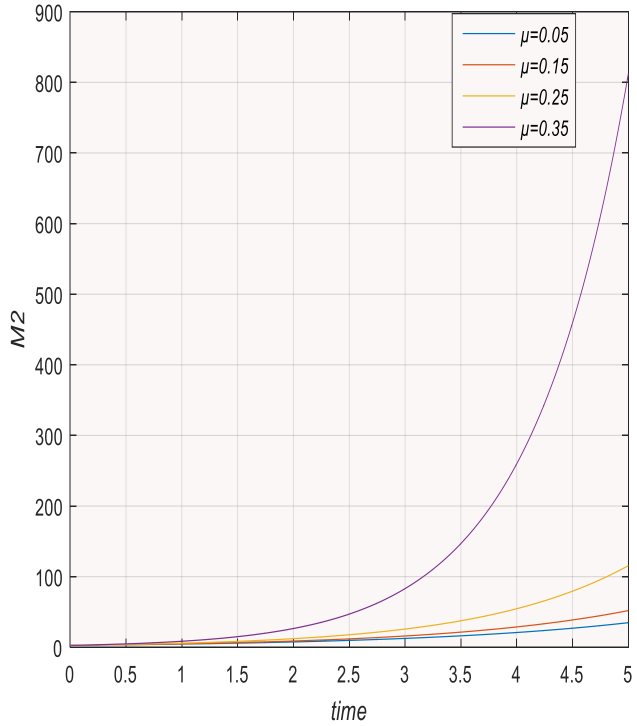

Fig. 5: The evolution of M_2 with different expected rate of returns

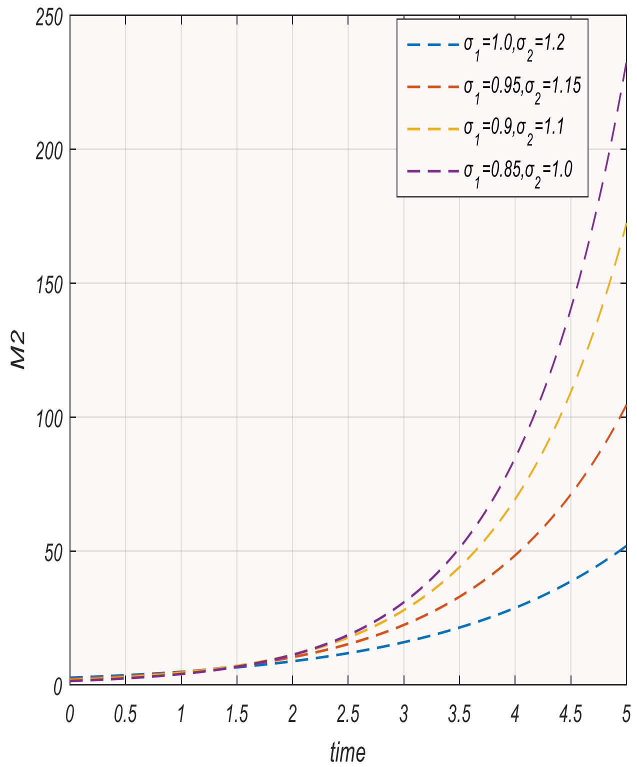

Fig. 6: The evolution of K_1 with different instantaneous volatility

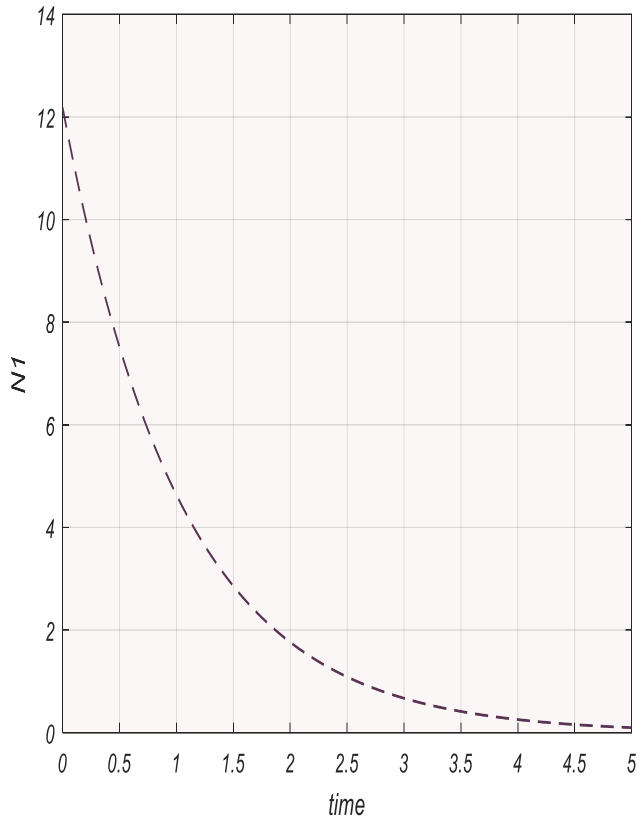

Fig. 7: The evolution of N_1 with time

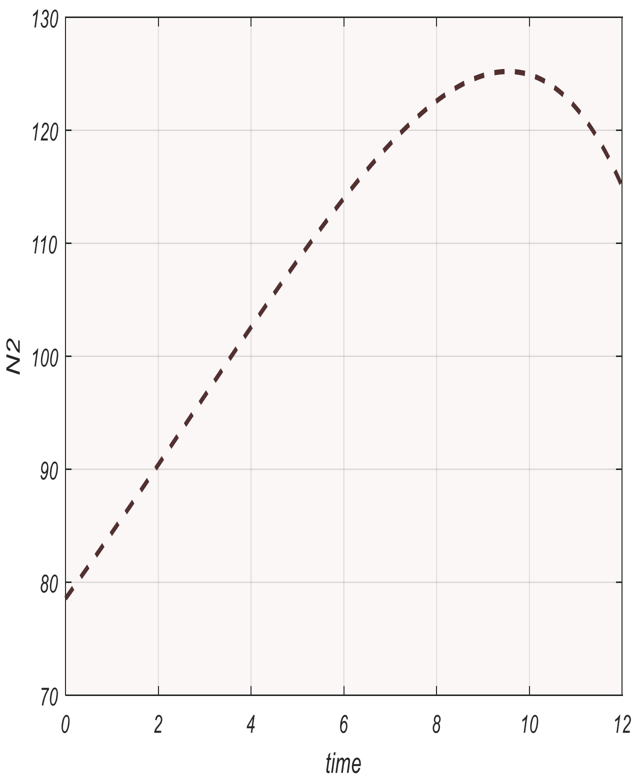

Fig. 8: The evolution of N_2 with time

6.0 Discussion of Results

In this paper, we investigated the explicit solutions of an insurer/reinsurer surplus and time-varying investment returns with random parameters. We also consider cases where the risky asset and the insurer/reinsurer's surplus are deterministic or stochastic and when the expected rate of returns from investment is linear or quadratic. Furthermore, some numerical solutions were presented to examine the behaviour of the risky asset and that of the reinsurer's surplus subject to some sensitive parameters. Figure 1 shows the evolution of the deterministic risky asset K_1 with time under the different expected rate of returns μ. It is obvious that an increase in the rate of return triggers an increase in deterministic risky asset K_1. This is very much consistent because any asset with a high expected rate of returns makes it attractive and appealing to an insurer or reinsurer thereby encouraging the insurer and reinsurer to continue enlarging its portfolio of investment for higher returns.. Risky assets have a higher expected return rate for higher risk. This result is in line with that of (Okoroafor & Osu, 2009; Giet *et al*, 2015). Figure 2 shows the evolution of the stochastic risky asset K_2 with time under the different expected rate of returns μ. It is obvious that an increase in the expected rate of return triggers an increase in stochastic risky asset K_2. This is very much similar to that of the deterministic case only that, for the stochastic case, the insurer expects more from any investment with higher risk involvement and also, we know from mean-variance utility and efficient frontier, there is a linear relationship between the expectation and the risk involved in any business (He & Liang, 2013; Akpanibah *et al*. 2020). Hence we observed that K_2 is higher than K_1.

Figure 3 shows the evolution of the stochastic risky asset K_2 with time under different instantaneous volatilities σ_1 andσ_2 It is obvious that an increase in the instantaneous volatilities σ_1 and σ_2 triggers a decrease in the stochastic risky asset K_2. This is very much so because higher instantaneous volatilities indicate the presence of more risk and this is sometimes scary to investors especially those that are not risk lovers. Hence, the insurer/reinsurer will prefer to increase his portfolio with K_2 when the risk involved is not too high and very considerate. This is very much consistent with existing literature such as (He & Liang, 2013; Akpanibah *et al*, 2020; Gao, 2009). Figure 4 shows the evolution of the deterministic risky asset M_1 with time under the different expected rate of returns μ. It is obvious that an increase in the rate of return triggers an increase in deterministic risky asset M_1. This is very similar to K_1 and is consistent because any asset with high expected rate of returns makes it attractive and appealing to an insurer or reinsurer thereby encouraging the insurer and reinsurer to continue enlarging its portfolio of investment for higher returns. The difference between Fig. 1 and Fig. 4 is that the square of the expected rate of return triggers a higher investment portion in M_1than K_1. Figure 5 shows the evolution of the stochastic risky asset M_2 with time under a different expected rate of returns μ. It is obvious that an increase in the expected rate of return triggers an increase in stochastic risky asset M_2. This is very similar to that of the deterministic case only that, for the stochastic case, the insurer expects more from any investment with higher risk involvement. Also, the difference between Figs 2 and 5 is that the square of the expected rate of return triggers a higher investment portion in M_2than K_2.

Figure 6 shows the evolution of the stochastic risky asset M_2 with time under different instantaneous volatilities σ_1 andσ_2 It is obvious to that an increase in the instantaneous volatilities σ_1 and$\sigma_2$triggers a decrease in the stochastic risky asset M_2. Furthermore, this will surely create a negative effect on the investment pattern of the insurer/reinsurer. Fig. 7, shows the evolution of the deterministic surplus process N_1 of an insurer and reinsurer. It was observed that the surplus process of the insurance company is inversely proportional to the time of investment, it is quite obvious in the sense that funds are allocated towards public debt such as payment of claims etc; which reduces the interest rate and helps the economy. It was also observed that for the deterministic case, the surplus process decreases with risk-averse coefficient and is not dependent on the expected rate of returns. Fig. 8, shows the evolution of the deterministic surplus process N_2 of an insurer and reinsurer. It was observed that the surplus process of the insurance company increases at the early stage of the investment and as the need for payment of claim begins to set in, the surplus of the insurance company begins to go down. It is clear to see in the sense that funds are allocated towards payment of claims etc; which reduces interest rates and helps economy it was also observed that for the deterministic case, the surplus process decreases with risk-averse coefficient and is not dependent on the expected rate of returns.

7.0 Conclusion

In this research work, we present a reinsurer's portfolio which is a combination of his surplus and risky assets for both stochastic and deterministic cases. Also, we consider linear and quadratic cases for the expected rate of returns from the risky asset. We used Ito's lemma and maximum principle to solve for the explicit solutions of the risky assets for all considered cases. Furthermore, the explicit solutions of the reinsurer's surplus were obtained for both deterministic and stochastic cases. Finally, some numerical simulations were presented to study the impact of the expected rate of return and instantaneous volatilities of the stock market prices on the risky asset and also the behaviour of the reinsurer's surpluses with time. It was observed that (i) the price process of the risky asset is directly proportional to the expected rate of return

(ii) the price process of the risky asset is inversely proportional to the instantaneous volatilities (iii) the surplus process does not necessarily depend on the expected returns but mostly on the amount of claims to be serviced at any given time.

Conflict of Interests

The authors declare that there is no conflict of interests regarding this paper.

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The first author developed the manuscript after discussion with the second and third authors. The second author solved the formulation and solved

Compliance with Ethical Standards Declarations

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Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data public.

Competing interests

the problem while the third author carried out numerical simulations for the results. The three authors also proofread the work.