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A Mathematical Investigation of Fuel Subsidy Removal and its Effects on Nigerian Economy

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Abstract: *The objective of this paper is to investigate and provide a solution to the adverse effects of fuel subsidy removal on the Nigerian Economy through a mathematical investigation. Recently, the removal of fuel subsidies by the Nigerian Government is of good interest to strengthen the economy of the country but it resulted in information and macroeconomic adverse effects of uncertainties directly to petroleum marketers, fuel dealers, transport operators, production companies and marketers of produced products and the general public thereby inflicting sufferings on the masses through inflation. The adverse effects of these uncertainties are capable of resulting in future delay and uncertainty noise in the financial market during business transactions which are therefore modeled mathematically as an Advanced Stochastic Delay Differential Equation (ASDDE). The modelled equation is solved using the Hybrid Extended Second Derivative Block Backward Differentiation Formulae Method (HESDBBDFM) with three new theorems of mathematical expressions developed for the evaluations of the delay term and noise term. To reduce the adverse effects of uncertainties of fuel subsidy removal, the government of Nigeria should diversify and develop other economies and conduct a well-designed communication campaign to highlight and tackle the negative impact of fuel subsidy and the benefits from its removal and compensating measures for the poor to cushion the adverse effects of fuel subsidy removal. This can be expressed mathematically by solving some examples of ASDDE numerically using the proposed method. The analysis of the numerical solutions of the modelled equation with its comparison and graphical presentations proved that the proposed method performs better by producing the Least Minimum Absolute Random Errors (LMAREs) at Lesser Computational Processing Unit Time (LCPUT) which indicates a reduction in the adverse effects of uncertainties of fuel subsidy removal for better economy than other existing methods in terms of accuracy and efficiency.*

Keywords: Absolute Random Error, ASDDE, Fuel Subsidy Removal**,** HESDBBDFM, Nigerian Government, Uncertainty

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1.0 Introduction

Following the hasty declaration of the

removal of fuel subsidy by the present government of Nigeria which did not follow through the introduction of a market-based pricing mechanism leaves the determination of fuel price in the hands of the market forces, especially the oil marketers. As a result of this fire-brigade declaration, the price of fuel per litre skyrocketed instantly resulting in devastating uncertainties across the filling stations in the country which caused inflation in every sector of the Nigerian economy. Arze *et al*. (2012) analyzed the adverse impact of fuel subsidy removal on the poor and developing countries which can be mitigated with a fraction of fiscal resources using the data collected by the International Monetary Fund in 2019. Bashir (2013) studied that removing fuel subsidies would reduce income inequality. A fuel price increase to cost-recovery level reduces customers' purchasing ability which calls for a distributional analysis of the effect on the customers' standard of living. The 2023 fuel subsidy removal sent future uncertainty signals to the petroleum marketers and fuel dealers which prompted a rise in pump prices of petroleum products. Recently, most companies and organizations have laid off some of their staff to curtail expenditures as a result of the fuel subsidy removal and this resulted in one worker per more duties which amounts to inefficiencies in service delivery. For manufacturing companies to survive the adverse effects of the fuel subsidy removal, they introduce the additional expenditure into their cost of production and geometrically reflect it as a profit in the marketing of their product which triggers a wave of inflation as future uncertainties (Okigbo & Enekebe, 2011). According to Soile & Mu (2015), there is a macroeconomic effect between the price of fuel, cost of transportation and cost of goods in the market. Since fuel is a major input in transportation and electricity, any increase in its price triggers inflation in the cost of electricity tariff, and transportation costs and eventually leads to a rise in the cost of goods and services. There are several complaints and sufferings experienced by the general public as a result of the fuel subsidy removal. The existing literature on this concept revealed that no research has been carried out mathematically in addressing the adverse effects of fuel subsidy removal on the Nigerian economy which left a huge research gap. There is a need for urgent evaluations and analysis to address these challenges to reduce the adverse effects of future uncertainties caused by the removal of fuel subsidies. This can be done by applying the proposed method in solving some examples of ASDDE developed by Zhang, Gan & Hu (2009) which takes into consideration the future delay term and adverse effects of future uncertainties as a noise term and was expressed as;

$$dy(t) = \lambda(y(t), y(t+\tau), t)dt + \Phi(y(t), y(t+\tau), t)d\alpha(t) \text{ for } t > 0, \tau > 0$$

$$y(t) = \beta(t), \qquad \text{for } t > 0 \tag{1}$$

where $\beta(t)$ is the initial function, λ, Φ are drift and volatility coefficients, $y(t)$ represents the information and macroeconomic adverse effects of uncertainties on the Nigerian economy, t is the time of delay in days, τ delay, $(t+\tau)$ is called the future delay term and $y(t+\tau)$ is the solution of the future delay term on the drift part of (1). $\alpha(t)$ is the Standard Brownian motion with its differential equivalence $d\alpha(t)$ as the noise term together with the solution of the future delay term as $y(t+\tau)d\alpha(t)$ on the stochastic or diffusion part of (1). The drift part of the equation (1) $dy(t) = \lambda(y(t), y(t+\tau), t)dt$ is deterministic and takes care of the average time rate of the system without any risk. The stochastic or diffusion part $dy(t) = \Phi(y(t), y(t+\tau), t)d\alpha(t)$ is stochastic, which takes care of the future uncertainties or random changes in the modeled system (1). In Ugbebor (1991), a noise term is a stochastic process of uncertainties governed by probabilistic laws. The Nigerian government has attempted to remove fuel subsidies several times, but it has not succeeded, mainly due to a strong popular opposition to its removal causing hardship and suffering on the poor McCulloch, Moerenhout & Yang (2021). Moreover, the fuel subsidy removal declaration by President Bola Ahmed Tinubu's administration was done by the mere introduction of newly increased unregulated prices instead of introducing a market-based pricing mechanism. As a result, fuel subsidies always appear again due to persistent currency depreciation and related increases in inflation (Ogbu, 2012). Empirical studies have also supported that fuel subsidy removal is a costly approach to improving the economy which requires wide economic consultations and analysis (Umunna, Chike, Harold & Uloma, 2013). According to Okigbo & Enekebe (2011) estimated the distributional impact of fuel price increase in Nigeria, a 40 percent increase in PMS price (recovery of current costs) reduces the disposable income of rich households and decreases income inequality (measured by the Gini coefficient) by ¼ point. It calculated both the direct effects (for consumers of fuel) and the indirect effects (for consumers of goods and services that use fuel as an input). Recently, this removal of fuel subsidy led to widespread protests across the states of the federation including the FCT-Abuja in expression of its bad effects on Nigerians. Therefore, these adverse effects of fuel subsidy removal as stated above result in future uncertainties as a stochastic process which affects the economy of Nigeria. The adverse effects of fuel subsidy removal can be reduced by solving some ASDDE numerically using the proposed method. Most scholars such as Akhtari, Babolian, Neuenkirch (2015), Wang, Gan, (2011) and Bahar (2019)in the quest to solve ASDDE numerically, used interpolation techniques for evaluations of the delay term and noise term and experienced difficulty in obtaining the Least Minimum Absolute Random Error (LMARE) at the Lowest Computer Processing Unit Time (LCPUT). Osu, Amadi & Azor (2023) obtained the analytic solution of a deterministic and stochastic model of time-varying investment returns with random parameters and revealed that the lower the absolute random errors of a stochastic process, the lower the risk effect of economic fluctuations in any financial model. So therefore, the lower the absolute random error of a stochastic process, the lower the uncertainties on the financial model of fuel subsidy removal and its effects on the Nigerian economy. These difficulties experienced by these researchers in literature and the adverse effects of future uncertainties caused by the removal of fuel subsidies left a huge research gap that needs to be filled which is the motivation behind this study.

The purpose of this study is to evaluate, analyze, unveil strategic measures and recommend the best ways or approaches which will be of great importance to the government, companies or organizations to drastically reduce the adverse effects of uncertainties caused by the removal of fuel subsidies to improve the economy of the country and peoples' standard of living through numerical solution of some ASDDE.

2.0 Derivation of the Method

The discrete schemes of the HESDBBDFM for step numbers $k = 2$, 3 and 4 were derived through matrix inversion techniques of the *k*-step multistep collocation method and expressed as;

For $k=2$ of (HESDBBDFM)

$$y_{n+1}=\frac{3195957}{40880}h^2r_{n+2}+\frac{1204859}{6132}h^2r_{n+\frac{5}{2}}-\frac{4365586}{22995}h^2r_{n+\frac{9}{4}}-\frac{105674}{1095}h^2r_{n+\frac{11}{4}}+\frac{6833377}{367920}h^2s_{n+3}+\frac{34273}{4088}hf_{n+1}-\frac{30185}{4088}hf_{n+2}+y_n$$

$$y_{n+2}=-\frac{799}{34273}y_n+\frac{35072}{34273}y_{n+1}+\frac{33474}{34273}hf_{n+2}-\frac{1591042}{514095}h^2r_{n+2}+\frac{3420416}{514095}h^2r_{n+\frac{9}{4}}-\frac{3376096}{514095}h^2r_{n+\frac{5}{2}}+\frac{537344}{171365}h^2r_{n+\frac{11}{4}}-\frac{20292}{34273}h^2s_{n+3}$$

$$y_{n+\frac{9}{4}}=-\frac{409195}{17547776}y_n+\frac{17956971}{17547776}y_{n+1}+\frac{21525525}{17547776}hf_{n+2}-\frac{108033821}{35095552}h^2r_{n+2}+\frac{14640587}{2193472}h^2r_{n+\frac{9}{4}}-\frac{115404915}{17547776}h^2r_{n+\frac{5}{2}}+\frac{6885593}{2193472}h^2r_{n+\frac{11}{4}}-\frac{5199671}{8773888}h^2s_{n+3}$$

$$y_{n+\frac{5}{2}}=-\frac{1599}{68546}y_n+\frac{70145}{68546}y_{n+1}+\frac{50610}{34273}hf_{n+2}-\frac{3352191}{1096736}h^2r_{n+2}+\frac{1387699}{205638}h^2r_{n+\frac{9}{4}}-\frac{3609059}{548368}h^2r_{n+\frac{5}{2}}+\frac{215441}{68546}h^2r_{n+\frac{11}{4}}-\frac{1952125}{3290208}h^2s_{n+3}$$

$$y_{n+\frac{11}{4}}=-\frac{409493}{17547776}y_n+\frac{17957269}{17547776}y_{n+1}+\frac{30299115}{17547776}hf_{n+2}-\frac{1597733137}{526433280}h^2r_{n+2}+\frac{224660513}{32902080}h^2r_{n+\frac{9}{4}}-\frac{1720324991}{263216640}h^2r_{n+\frac{5}{2}}+\frac{34581701}{10967360}h^2r_{n+\frac{11}{4}}-\frac{3258717}{5483680}h^2s_{n+3}$$

$$y_{n+3}=-\frac{800}{34273}y_n+\frac{35073}{34273}y_{n+1}+\frac{67746}{34273}hf_{n+2}-\frac{309899}{102819}h^2r_{n+2}+\frac{3552352}{514095}h^2r_{n+\frac{9}{4}}-\frac{1111732}{171365}h^2r_{n+\frac{5}{2}}+\frac{1653856}{514095}h^2r_{n+\frac{11}{4}}-\frac{303517}{514095}h^2s_{n+3}$$

(2)

For $k=3$ of (HESDBBDFM)

$$y_{n+1}=\frac{626087947}{336161310}h^2r_{n+3}-\frac{680587916}{168080655}h^2r_{n+\frac{7}{2}}+\frac{630800768}{168080655}h^2r_{n+\frac{15}{4}}-\frac{688630997}{672322620}h^2s_{n+4}-\frac{56631783}{89643016}hf_{n+1}-\frac{46703841}{89643016}hf_{n+3}-\frac{855788}{11205377}y_n+\frac{12061165}{11205377}y_{n+2}$$

$$y_{n+2}=\frac{2573430208}{541733385}h^2r_{n+3}-\frac{4603388288}{541733385}h^2r_{n+\frac{7}{2}}+\frac{1356812288}{180577795}h^2r_{n+\frac{15}{4}}-\frac{1072270634}{541733385}h^2s_{n+4}+\frac{113263566}{36115559}hf_{n+2}-\frac{79330800}{36115559}hf_{n+3}-\frac{2182793}{36115559}y_n+\frac{38298352}{36115559}y_{n+1}$$

$$y_{n+3}=\frac{106888}{18877261}y_n-\frac{1653939}{18877261}y_{n+1}+\frac{20424312}{18877261}y_{n+2}+\frac{17437098}{18877261}hf_{n+3}-\frac{75518022}{94386305}h^2r_{n+3}+\frac{105233472}{94386305}h^2r_{n+\frac{7}{2}}-\frac{6899712}{7260485}h^2r_{n+\frac{15}{4}}-\frac{23054436}{94386305}h^2s_{n+4}$$

$$y_{n+\frac{7}{2}}=\frac{6938495}{1208144704}y_n-\frac{1671544}{18877261}y_{n+1}+\frac{1308185025}{1208144704}y_{n+2}+\frac{859557615}{604072352}hf_{n+3}-\frac{2642247503}{3624434112}h^2r_{n+3}+\frac{548289371}{453054264}h^2r_{n+\frac{7}{2}}-\frac{1453984}{1452097}h^2r_{n+\frac{15}{4}}+\frac{924056539}{3624434112}h^2s_{n+4}$$

$$y_{n+\frac{15}{4}}=-\frac{224660513}{38660630528}y_n-\frac{863375625}{9665157632}y_{n+1}+\frac{41889472515}{38660630528}y_{n+2}+\frac{32325960975}{19330315264}hf_{n+3}-\frac{26287430631}{38660630528}h^2r_{n+3}+\frac{6514954677}{4832578816}h^2r_{n+\frac{7}{2}}-\frac{48336519}{46467104}h^2r_{n+\frac{15}{4}}+\frac{10208375793}{38660630528}h^2s_{n+4}$$

$$y_{n+4}=\frac{111011}{18877261}y_n-\frac{1701312}{18877261}y_{n+1}+\frac{20467562}{18877261}y_{n+2}+\frac{36275232}{18877261}hf_{n+3}-\frac{178606424}{283158915}h^2r_{n+3}+\frac{422132864}{283158915}h^2r_{n+\frac{7}{2}}-\frac{22360064}{21781455}h^2r_{n+\frac{15}{4}}+\frac{78773252}{283158915}h^2s_{n+4}$$

(3)

For $k=4$ of (HESDBBDFM)

$$y_{n+1}=-\frac{6611194}{8119095}h^2r_{n+4}+\frac{3940736}{8119095}h^2r_{n+\frac{9}{2}}-\frac{4442006}{40595475}h^2s_{n+5}-\frac{26663494}{40595475}hf_{n+1}+\frac{19370614}{40595475}hf_{n+4}-\frac{1184599}{13531825}y_n+\frac{28186487}{13531825}y_{n+2}-\frac{13470063}{13531825}y_{n+3}$$

$$y_{n+2}=\frac{395872742}{590161785}h^2r_{n+4}-\frac{214095616}{590161785}h^2r_{n+\frac{9}{2}}+\frac{45955496}{590161785}h^2s_{n+5}-\frac{239971446}{196720595}hf_{n+2}+\frac{90831684}{196720595}hf_{n+4}+\frac{4928131}{196720595}y_n-\frac{74433464}{196720595}y_{n+1}+\frac{266225928}{196720595}y_{n+3}$$

$$y_{n+3}=\frac{43795628}{44681305}h^2r_{n+4}-\frac{19560448}{44681305}h^2r_{n+\frac{9}{2}}+\frac{3878126}{44681305}h^2s_{n+5}-\frac{79990482}{44681305}hf_{n+3}-\frac{41632092}{44681305}hf_{n+4}+\frac{662506}{44681305}y_n-\frac{7647927}{44681305}y_{n+1}+\frac{51666726}{44681305}y_{n+2}$$

$$y_{n+4}=-\frac{3369}{1025519}y_n-\frac{479168}{13331747}y_{n+1}-\frac{2795592}{13331747}y_{n+2}+\frac{15691968}{13331747}y_{n+3}+\frac{11363100}{13331747}hf_{n+4}-\frac{5552072}{13331747}h^2r_{n+4}+\frac{1828864}{13331747}h^2r_{n+\frac{9}{2}}-\frac{337472}{13331747}h^2s_{n+5}$$

$$y_{n+\frac{9}{2}}=-\frac{227185}{65633216}y_n+\frac{32121945}{853231808}y_{n+1}-\frac{185302467}{853231808}y_{n+2}+\frac{1009365735}{853231808}y_{n+3}-\frac{143741115}{106653976}hf_{n+4}-\frac{277116945}{853231808}h^2r_{n+4}+\frac{19198095}{106653976}h^2r_{n+\frac{9}{2}}-\frac{24813285}{853231808}h^2s_{n+5}$$

$$y_{n+5}=-\frac{10918}{3076557}y_n+\frac{1543415}{39995241}y_{n+1}-\frac{8878930}{39995241}y_{n+2}+\frac{15824230}{13331747}y_{n+3}+\frac{24590860}{13331747}hf_{n+4}-\frac{25957700}{119985723}h^2r_{n+4}+\frac{52145920}{119985723}h^2r_{n+\frac{9}{2}}-\frac{1560500}{119985723}h^2s_{n+5}$$

(4)

3.0 Basic Properties of the Method

The necessary and sufficient conditions to be satisfied by the proposed method are investigated below following the steps formulated by (Dahlquist, 1956 & Lambert,1973).

3.1. Order and Error Constant

Following that the method is one of the families of Linear Multistep Method (LMM), Dahlquist, (1956) developed and examined that the LMM is said to be of order m if $C_0 = C_1 = 0,\ldots,C_m = 0$ but $C_{m+1} \neq 0$ and C_{m+1} is the error constant.
The order and error constants for (2) are analyzed and expressed as follows;

$C_0 = C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$ but

$$C_2 = \begin{pmatrix} \frac{28141}{4088} & -\frac{15938}{34273} & -\frac{15223455}{35095552} & -\frac{93225}{274184} & -\frac{6448673}{35095552} & \frac{1200}{34273} \end{pmatrix}^T$$
.
Therefore, (2) has order $m=1$ and error constant,

$$\frac{28141}{4088}, -\frac{15938}{34273}, -\frac{15223455}{35095552}, -\frac{93225}{274184}, -\frac{6448673}{35095552}, \frac{1200}{34273}$$

Following the same approach to (3), we obtained

$C_0 = C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$ but

$$C_2 = \begin{pmatrix} \frac{24293087}{44821508} & \frac{64547210}{36115559} & -\frac{568098}{1452097} & -\frac{6160385}{23233552} & -\frac{160592355}{1486947328} & \frac{162148}{1452097} \end{pmatrix}^T$$

Therefore, (3) has order $m=1$ and error constant,

$$\frac{24293087}{44821508}, \frac{64547210}{36115559}, -\frac{568098}{1452097}, -\frac{6160385}{23233552}, -\frac{160592355}{1486947328}, \frac{162148}{1452097}$$

Applying the same approach to (4), we obtained

$C_0 = C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$ but

$$C_2 = \begin{pmatrix} -\frac{5931432}{13531825} & \frac{75910874}{196720595} & \frac{28113306}{44681305} & -\frac{312360}{1025519} & -\frac{5705595}{32816608} & \frac{631480}{3076557} \end{pmatrix}^T$$

Therefore, (4) has order $m=1$ and error constant,

$$-\frac{5931432}{13531825}, \frac{75910874}{196720595}, \frac{28113306}{44681305}, -\frac{312360}{1025519}, -\frac{5705595}{32816608}, \frac{631480}{3076557}$$

3.2 Consistency

Since the order of the proposed method as investigated in sub-section 3.2.1 above satisfies the condition developed by Dahlquist (1956) of the order $m \geq 1$, the method is consistent.

3.3 Zero Stability Analysis

According to Lambert (1973), the zero stability of an LMM is satisfied if no simple or distinct roots $u_i, i = 1, 2, 3, \ldots, n$ of the first characteristic polynomial $P(u)$ expressed as $P(u) = \det\left(u N_2^{(1)} - N_1^{(1)}\right)$ is greater than 1 which satisfies $|u_i| \leq 1$.

The zero stability for (2) is analyzed as;

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{35072}{34273} & 1 & 0 & 0 & 0 & 0 \\ -\frac{17956971}{17547776} & 0 & 1 & 0 & 0 & 0 \\ -\frac{70145}{68546} & 0 & 0 & 1 & 0 & 0 \\ -\frac{17957269}{17547776} & 0 & 0 & 0 & 1 & 0 \\ -\frac{35073}{34273} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{9}{4}} \\ y_{n+\frac{5}{2}} \\ y_{n+\frac{11}{4}} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{799}{34273} \\ 0 & 0 & 0 & 0 & 0 & \frac{409195}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{1599}{68546} \\ 0 & 0 & 0 & 0 & 0 & \frac{409493}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{800}{34273} \end{pmatrix} \begin{pmatrix} y_{n-\frac{11}{4}} \\ y_{n-\frac{5}{2}} \\ y_{n-\frac{9}{4}} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix}$$

$$\begin{pmatrix} \frac{34273}{4088} & -\frac{30185}{4088} & 0 & 0 & 0 & 0 \\ 0 & \frac{33474}{34273} & 0 & 0 & 0 & 0 \\ 0 & \frac{21525525}{17547776} & 0 & 0 & 0 & 0 \\ 0 & \frac{50610}{34273} & 0 & 0 & 0 & 0 \\ 0 & \frac{30299115}{17547776} & 0 & 0 & 0 & 0 \\ 0 & \frac{67746}{34273} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{9}{4}} \\ f_{n+\frac{5}{2}} \\ f_{n+\frac{11}{4}} \\ f_{n+3} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{11}{4}} \\ f_{n-\frac{5}{2}} \\ f_{n-\frac{9}{4}} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}$$

$$+h^2 \begin{pmatrix} 0 & \frac{3195957}{40880} & -\frac{4365586}{22995} & \frac{1204859}{6132} & -\frac{105674}{1095} & 0 \\ 0 & -\frac{1591042}{514095} & \frac{3420416}{514095} & -\frac{3376096}{514095} & \frac{537344}{171365} & 0 \\ 0 & -\frac{108033821}{35095552} & \frac{14640587}{2193472} & -\frac{115404915}{17547776} & \frac{6885593}{2193472} & 0 \\ 0 & -\frac{3352191}{1096736} & \frac{1387699}{205638} & -\frac{3609059}{548368} & \frac{215441}{68546} & 0 \\ 0 & -\frac{1597733137}{526433280} & \frac{224660513}{32902080} & -\frac{1720324991}{263216640} & \frac{34581701}{10967360} & 0 \\ 0 & -\frac{309899}{102819} & \frac{3552352}{514095} & -\frac{1111732}{171365} & \frac{1653856}{514095} & 0 \end{pmatrix} \begin{pmatrix} r_{n+1} \\ r_{n+2} \\ r_{n+\frac{9}{4}} \\ r_{n+\frac{5}{2}} \\ r_{n+\frac{11}{4}} \\ r_{n+3} \end{pmatrix} + h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_{n-\frac{11}{4}} \\ r_{n-\frac{5}{2}} \\ r_{n-\frac{9}{4}} \\ r_{n-2} \\ r_{n-1} \\ r_{n} \end{pmatrix}$$

$$+h^2\begin{pmatrix} 0&0&0&0&0&\frac{6833377}{367920}\\ 0&0&0&0&0&-\frac{20292}{34273}\\ 0&0&0&0&0&-\frac{5199671}{8773888}\\ 0&0&0&0&0&-\frac{1952125}{3290208}\\ 0&0&0&0&0&-\frac{3258717}{5483680}\\ 0&0&0&0&0&-\frac{303517}{514095} \end{pmatrix}\begin{pmatrix} s_{n+1}\\ s_{n+2}\\ s_{n+\frac{9}{4}}\\ s_{n+\frac{5}{2}}\\ s_{n+\frac{11}{4}}\\ s_{n+3} \end{pmatrix}+h^2\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} s_{n-\frac{11}{4}}\\ s_{n-\frac{5}{2}}\\ s_{n-\frac{9}{4}}\\ s_{n-2}\\ s_{n-1}\\ s_{n} \end{pmatrix}$$

where

$$N_2^{(1)}=\begin{pmatrix} 1&0&0&0&0&0\\ -\frac{35072}{34273}&1&0&0&0&0\\ -\frac{17956971}{17547776}&0&1&0&0&0\\ -\frac{70145}{68546}&0&0&1&0&0\\ -\frac{17957269}{17547776}&0&0&0&1&0\\ -\frac{35073}{34273}&0&0&0&0&1 \end{pmatrix}, N_1^{(1)}=\begin{pmatrix} 0&0&0&0&0&-1\\ 0&0&0&0&0&\frac{799}{34273}\\ 0&0&0&0&0&\frac{409195}{17547776}\\ 0&0&0&0&0&\frac{1599}{68546}\\ 0&0&0&0&0&\frac{409493}{17547776}\\ 0&0&0&0&0&\frac{800}{34273} \end{pmatrix}, G_2^{(1)}=\begin{pmatrix} \frac{34273}{4088}&-\frac{30185}{4088}&0&0&0&0\\ 0&\frac{33474}{34273}&0&0&0&0\\ 0&\frac{21525525}{17547776}&0&0&0&0\\ 0&\frac{50610}{34273}&0&0&0&0\\ 0&\frac{30299115}{17547776}&0&0&0&0\\ 0&\frac{67746}{34273}&0&0&0&0 \end{pmatrix},$$

$$E_2^{(1)}=\begin{pmatrix} 0&\frac{3195957}{40880}&-\frac{4365586}{22995}&\frac{1204859}{6132}&-\frac{105674}{1095}&0\\ 0&-\frac{1591042}{514095}&\frac{3420416}{514095}&-\frac{3376096}{514095}&\frac{537344}{171365}&0\\ 0&-\frac{108033821}{35095552}&\frac{14640587}{2193472}&-\frac{115404915}{17547776}&\frac{6885593}{2193472}&0\\ 0&-\frac{3352191}{1096736}&\frac{1387699}{205638}&-\frac{3609059}{548368}&\frac{215441}{68546}&0\\ 0&-\frac{1597733137}{526433280}&\frac{224660513}{32902080}&-\frac{1720324991}{263216640}&\frac{34581701}{10967360}&0\\ 0&-\frac{309899}{102819}&\frac{3552352}{514095}&-\frac{1111732}{171365}&\frac{1653856}{514095}&0 \end{pmatrix} \text{ and}$$

The first characteristic polynomial is stated as;

$$P(u)=\det\left(uN_2^{(1)}-N_1^{(1)}\right) = \left|uN_2^{(1)}-N_1^{(1)}\right|=0 \quad (5)$$

$$R_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{6833377}{367920} \\ 0 & 0 & 0 & 0 & 0 & -\frac{20292}{34273} \\ 0 & 0 & 0 & 0 & 0 & -\frac{5199671}{8773888} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1952125}{3290208} \\ 0 & 0 & 0 & 0 & 0 & -\frac{3258717}{5483680} \\ 0 & 0 & 0 & 0 & 0 & -\frac{303517}{514095} \end{pmatrix}$$

Now we have,

$$P(u) = \left| u \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{35072}{34273} & 1 & 0 & 0 & 0 & 0 \\ -\frac{17956971}{17547776} & 0 & 1 & 0 & 0 & 0 \\ -\frac{70145}{68546} & 0 & 0 & 1 & 0 & 0 \\ -\frac{17957269}{17547776} & 0 & 0 & 0 & 1 & 0 \\ -\frac{35073}{34273} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{799}{34273} \\ 0 & 0 & 0 & 0 & 0 & \frac{409195}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{1599}{68546} \\ 0 & 0 & 0 & 0 & 0 & \frac{409493}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{800}{34273} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} u & 0 & 0 & 0 & 0 & 0 \\ -\frac{35072}{34273}u & u & 0 & 0 & 0 & 0 \\ -\frac{17956971}{17547776}u & 0 & u & 0 & 0 & 0 \\ -\frac{70145}{68546}u & 0 & 0 & u & 0 & 0 \\ -\frac{17957269}{17547776}u & 0 & 0 & 0 & u & 0 \\ -\frac{35073}{34273}u & 0 & 0 & 0 & 0 & u \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{799}{34273} \\ 0 & 0 & 0 & 0 & 0 & \frac{409195}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{1599}{68546} \\ 0 & 0 & 0 & 0 & 0 & \frac{409493}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{800}{34273} \end{pmatrix} \right|$$

$$\Rightarrow P(u) = \begin{pmatrix} u & 0 & 0 & 0 & 0 & 1 \\ -\frac{35072}{34273}u & u & 0 & 0 & 0 & -\frac{799}{34273} \\ -\frac{17956971}{17547776}u & 0 & u & 0 & 0 & -\frac{409195}{17547776} \\ -\frac{70145}{68546}u & 0 & 0 & u & 0 & -\frac{1599}{68546} \\ -\frac{17957269}{17547776}u & 0 & 0 & 0 & u & -\frac{409493}{17547776} \\ -\frac{35073}{34273}u & 0 & 0 & 0 & 0 & u-\frac{800}{34273} \end{pmatrix}$$

Using Maple (18) software,

$P(u) = u^5(u+1)$

$\Rightarrow u^5(u+1) = 0$

$\Rightarrow u_1 = -1, u_2 = 0, u_3 = 0, u_4 = 0, u_5 = 0, u_6 = 0$ .Since $|u_i| < 1,\ i = 1,2,3,4,5,6$ (2) is zero stable.

Using the same approach, then **(**3**)** is analyzed as;

$$\begin{pmatrix} 1 & -\frac{12061165}{11205377} & 0 & 0 & 0 & 0 \\ -\frac{38298352}{36115559} & 1 & 0 & 0 & 0 & 0 \\ \frac{1653939}{18877261} & \frac{20424312}{18877261} & 1 & 0 & 0 & 0 \\ \frac{1671544}{18877261} & -\frac{1308185025}{1208144704} & 0 & 1 & 0 & 0 \\ \frac{863375625}{9665157632} & -\frac{41889472515}{38660630528} & 0 & 0 & 1 & 0 \\ \frac{1701312}{18877261} & -\frac{20467562}{18877261} & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+\frac{15}{4}} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{855788}{11205377} \\ 0 & 0 & 0 & 0 & 0 & \frac{2182793}{36115559} \\ 0 & 0 & 0 & 0 & 0 & -\frac{106888}{18877261} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6938495}{1208144704} \\ 0 & 0 & 0 & 0 & 0 & -\frac{224660513}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{111011}{18877261} \end{pmatrix} \begin{pmatrix} y_{n-\frac{15}{4}} \\ y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix}$$

$$+h \begin{pmatrix} -\frac{56631783}{89643016} & 0 & -\frac{46703841}{89643016} & 0 & 0 & 0 \\ 0 & \frac{113263566}{36115559} & -\frac{79330800}{36115559} & 0 & 0 & 0 \\ 0 & 0 & \frac{17437098}{18877261} & 0 & 0 & 0 \\ 0 & 0 & \frac{859557615}{604072352} & 0 & 0 & 0 \\ 0 & 0 & \frac{32325960975}{19330315264} & 0 & 0 & 0 \\ 0 & 0 & \frac{36275232}{18877261} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+\frac{15}{4}} \\ f_{n+4} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{15}{4}} \\ f_{n-\frac{7}{2}} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}$$

$$+h^2 \begin{pmatrix} 0 & 0 & \frac{626087947}{336161310} & -\frac{680587916}{168080655} & \frac{630800768}{168080655} & 0 \\ 0 & 0 & \frac{2573430208}{541733385} & -\frac{4603388288}{541733385} & \frac{1356812288}{180577795} & 0 \\ 0 & 0 & -\frac{75518022}{94386305} & \frac{105233472}{94386305} & -\frac{6899712}{7260485} & 0 \\ 0 & 0 & -\frac{2642247503}{3624434112} & \frac{548289371}{453054264} & -\frac{1453984}{1452097} & 0 \\ 0 & 0 & -\frac{26287430631}{38660630528} & \frac{6514954677}{4832578816} & -\frac{48336519}{46467104} & 0 \\ 0 & 0 & -\frac{178606424}{283158915} & \frac{422132864}{283158915} & -\frac{22360064}{21781455} & 0 \end{pmatrix} \begin{pmatrix} r_{n+1} \\ r_{n+2} \\ r_{n+3} \\ r_{n+\frac{7}{2}} \\ r_{n+\frac{15}{4}} \\ r_{n+4} \end{pmatrix} + h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_{n-\frac{15}{4}} \\ r_{n-\frac{7}{2}} \\ r_{n-3} \\ r_{n-2} \\ r_{n-1} \\ r_{n} \end{pmatrix}$$

$$+h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{688630997}{672322620} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1072270634}{541733385} \\ 0 & 0 & 0 & 0 & 0 & \frac{23054436}{94386305} \\ 0 & 0 & 0 & 0 & 0 & \frac{924056539}{3624434112} \\ 0 & 0 & 0 & 0 & 0 & \frac{10208375793}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & \frac{78773252}{283158915} \end{pmatrix}\begin{pmatrix} s_{n+1} \\ s_{n+2} \\ s_{n+3} \\ s_{n+\frac{7}{2}} \\ s_{n+\frac{15}{4}} \\ s_{n+4} \end{pmatrix}+h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s_{n-\frac{15}{4}} \\ s_{n-\frac{7}{2}} \\ s_{n-3} \\ s_{n-2} \\ s_{n-1} \\ s_n \end{pmatrix}$$

where $$+h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{688630997}{672322620} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1072270634}{541733385} \\ 0 & 0 & 0 & 0 & 0 & \frac{23054436}{94386305} \\ 0 & 0 & 0 & 0 & 0 & \frac{924056539}{3624434112} \\ 0 & 0 & 0 & 0 & 0 & \frac{10208375793}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & \frac{78773252}{283158915} \end{pmatrix}\begin{pmatrix} s_{n+1} \\ s_{n+2} \\ s_{n+3} \\ s_{n+\frac{7}{2}} \\ s_{n+\frac{15}{4}} \\ s_{n+4} \end{pmatrix}+h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s_{n-\frac{15}{4}} \\ s_{n-\frac{7}{2}} \\ s_{n-3} \\ s_{n-2} \\ s_{n-1} \\ s_n \end{pmatrix}$$

$$E_2^{(2)}=\begin{pmatrix} 0 & 0 & \frac{626087947}{336161310} & -\frac{680587916}{168080655} & \frac{630800768}{168080655} & 0 \\ 0 & 0 & \frac{2573430208}{541733385} & -\frac{4603388288}{541733385} & \frac{1356812288}{180577795} & 0 \\ 0 & 0 & -\frac{75518022}{94386305} & \frac{105233472}{94386305} & -\frac{6899712}{7260485} & 0 \\ 0 & 0 & -\frac{2642247503}{3624434112} & \frac{548289371}{453054264} & -\frac{1453984}{1452097} & 0 \\ 0 & 0 & -\frac{26287430631}{38660630528} & \frac{6514954677}{4832578816} & -\frac{48336519}{46467104} & 0 \\ 0 & 0 & -\frac{178606424}{283158915} & \frac{422132864}{283158915} & -\frac{22360064}{21781455} & 0 \end{pmatrix}$$

$$E_2^{(2)}=\begin{pmatrix} 0 & 0 & \frac{626087947}{336161310} & -\frac{680587916}{168080655} & \frac{630800768}{168080655} & 0 \\ 0 & 0 & \frac{2573430208}{541733385} & -\frac{4603388288}{541733385} & \frac{1356812288}{180577795} & 0 \\ 0 & 0 & -\frac{75518022}{94386305} & \frac{105233472}{94386305} & -\frac{6899712}{7260485} & 0 \\ 0 & 0 & -\frac{2642247503}{3624434112} & \frac{548289371}{453054264} & -\frac{1453984}{1452097} & 0 \\ 0 & 0 & -\frac{26287430631}{38660630528} & \frac{6514954677}{4832578816} & -\frac{48336519}{46467104} & 0 \\ 0 & 0 & -\frac{178606424}{283158915} & \frac{422132864}{283158915} & -\frac{22360064}{21781455} & 0 \end{pmatrix} \text{ and}$$

$$R_2^{(2)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{688630997}{672322620} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1072270634}{541733385} \\ 0 & 0 & 0 & 0 & 0 & \frac{23054436}{94386305} \\ 0 & 0 & 0 & 0 & 0 & \frac{924056539}{3624434112} \\ 0 & 0 & 0 & 0 & 0 & \frac{10208375793}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & \frac{78773252}{283158915} \end{pmatrix}$$

Using the first characteristic polynomial;

$$\begin{aligned} P(u) &= \det\left(uN_2^{(2)} - N_1^{(2)}\right) \\ &= \left|uN_2^{(2)} - N_1^{(2)}\right| = 0 \end{aligned} \tag{6}$$

Now we have,

$$P(u)=\left| u\begin{pmatrix} 1 & -\frac{12061165}{11205377} & 0 & 0 & 0 & 0 \\ -\frac{38298352}{36115559} & 1 & 0 & 0 & 0 & 0 \\ \frac{1653939}{18877261} & \frac{20424312}{18877261} & 1 & 0 & 0 & 0 \\ \frac{1671544}{18877261} & -\frac{1308185025}{1208144704} & 0 & 1 & 0 & 0 \\ \frac{863375625}{9665157632} & -\frac{41889472515}{38660630528} & 0 & 0 & 1 & 0 \\ \frac{1701312}{18877261} & -\frac{20467562}{18877261} & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{855788}{11205377} \\ 0 & 0 & 0 & 0 & 0 & \frac{2182793}{36115559} \\ 0 & 0 & 0 & 0 & 0 & -\frac{106888}{18877261} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6938495}{1208144704} \\ 0 & 0 & 0 & 0 & 0 & -\frac{224660513}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{111011}{18877261} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} u & -\frac{12061165}{11205377}u & 0 & 0 & 0 & 0 \\ -\frac{38298352}{36115559}u & u & 0 & 0 & 0 & 0 \\ \frac{1653939}{18877261}u & \frac{20424312}{18877261}u & u & 0 & 0 & 0 \\ \frac{1671544}{18877261}u & -\frac{1308185025}{1208144704}u & 0 & u & 0 & 0 \\ \frac{863375625}{9665157632}u & -\frac{41889472515}{38660630528}u & 0 & 0 & u & 0 \\ \frac{1701312}{18877261}u & -\frac{20467562}{18877261}u & 0 & 0 & 0 & u \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{855788}{11205377} \\ 0 & 0 & 0 & 0 & 0 & \frac{2182793}{36115559} \\ 0 & 0 & 0 & 0 & 0 & -\frac{106888}{18877261} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6938495}{1208144704} \\ 0 & 0 & 0 & 0 & 0 & -\frac{224660513}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{111011}{18877261} \end{pmatrix} \right|$$

$$\Rightarrow P(u) = \begin{pmatrix} u & -\frac{12061165}{11205377}u & 0 & 0 & 0 & -\frac{855788}{11205377} \\ -\frac{38298352}{36115559}u & u & 0 & 0 & 0 & -\frac{2182793}{36115559} \\ \frac{1653939}{18877261}u & \frac{20424312}{18877261}u & u & 0 & 0 & \frac{106888}{18877261} \\ \frac{1671544}{18877261}u & -\frac{1308185025}{1208144704}u & 0 & u & 0 & \frac{6938495}{1208144704} \\ \frac{863375625}{9665157632}u & -\frac{41889472515}{38660630528}u & 0 & 0 & u & \frac{224660513}{38660630528} \\ \frac{1701312}{18877261}u & -\frac{20467562}{18877261}u & 0 & 0 & 0 & u+\frac{111011}{18877261} \end{pmatrix}$$

Using Maple (18) software,

$$P(u) = -\frac{57234288539337}{404688454160743}u^5(u+1)$$

$$\Rightarrow -\frac{57234288539337}{404688454160743}u^5(u+1) = 0$$

$\Rightarrow u_1 = -1, u_2 = 0, u_3 = 0, u_4 = 0, u_5 = 0, u_6 = 0$ .Since $|u_i| < 1,\ i = 1,2,3,4,5,6$ (3) is zero stable.

Following the same procedure, (4) can be analyzed as follows;

$$\begin{pmatrix} 1 & -\frac{28186487}{13531825} & \frac{13470063}{13531825} & 0 & 0 & 0 \\ \frac{744433464}{196720595} & 1 & -\frac{266225928}{196720595} & 0 & 0 & 0 \\ \frac{7647927}{44681305} & -\frac{51666726}{44681305} & 1 & 0 & 0 & 0 \\ -\frac{479168}{13331747} & \frac{2795592}{13331747} & -\frac{15691968}{13331747} & 1 & 0 & 0 \\ -\frac{32121945}{853231808} & \frac{185302467}{853231808} & -\frac{1009365735}{853231808} & 0 & 1 & 0 \\ -\frac{1543415}{39995241} & \frac{8878930}{39995241} & -\frac{15824230}{13331747} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1184599}{13531825} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4928131}{196720595} \\ 0 & 0 & 0 & 0 & 0 & -\frac{662506}{44681305} \\ 0 & 0 & 0 & 0 & 0 & \frac{3369}{1025519} \\ 0 & 0 & 0 & 0 & 0 & \frac{227185}{65633216} \\ 0 & 0 & 0 & 0 & 0 & \frac{10918}{3076557} \end{pmatrix} \begin{pmatrix} y_{n-\frac{9}{2}} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

$$
+h\begin{pmatrix} -\frac{26663494}{40595475} & 0 & 0 & \frac{19370614}{40595475} & 0 & 0 \\ 0 & -\frac{239971446}{196720595} & 0 & -\frac{90831684}{196720595} & 0 & 0 \\ 0 & 0 & \frac{79990482}{44681305} & -\frac{41632092}{44681305} & 0 & 0 \\ 0 & 0 & 0 & \frac{11363100}{13331747} & 0 & 0 \\ 0 & 0 & 0 & \frac{143741115}{106653976} & 0 & 0 \\ 0 & 0 & 0 & \frac{24590860}{13331747} & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+\frac{9}{2}} \\ f_{n+5} \end{pmatrix}+h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-\frac{9}{2}} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}
$$

$$
+h^2\begin{pmatrix} 0 & 0 & 0 & -\frac{6611194}{8119095} & \frac{3940736}{8119095} & 0 \\ 0 & 0 & 0 & \frac{395872742}{590161785} & -\frac{214095616}{590161785} & 0 \\ 0 & 0 & 0 & \frac{43795628}{44681305} & -\frac{19560448}{44681305} & 0 \\ 0 & 0 & 0 & -\frac{5552072}{13331747} & \frac{1828864}{13331747} & 0 \\ 0 & 0 & 0 & -\frac{277116945}{853231808} & \frac{19198095}{106653976} & 0 \\ 0 & 0 & 0 & \frac{25957700}{119985723} & \frac{52145920}{119985723} & 0 \end{pmatrix}\begin{pmatrix} r_{n+1} \\ r_{n+2} \\ r_{n+3} \\ r_{n+4} \\ r_{n+\frac{9}{2}} \\ r_{n+5} \end{pmatrix}+h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r_{n-\frac{9}{2}} \\ r_{n-4} \\ r_{n-3} \\ r_{n-2} \\ r_{n-1} \\ r_{n} \end{pmatrix}
$$

$$
+h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{4442006}{40595475} \\ 0 & 0 & 0 & 0 & 0 & \frac{45955496}{590161785} \\ 0 & 0 & 0 & 0 & 0 & \frac{3878126}{44681305} \\ 0 & 0 & 0 & 0 & 0 & -\frac{337472}{13331747} \\ 0 & 0 & 0 & 0 & 0 & -\frac{24813285}{853231808} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1560500}{119985723} \end{pmatrix}\begin{pmatrix} s_{n+1} \\ s_{n+2} \\ s_{n+3} \\ s_{n+4} \\ s_{n+\frac{9}{2}} \\ s_{n+5} \end{pmatrix}+h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s_{n-\frac{9}{2}} \\ s_{n-4} \\ s_{n-3} \\ s_{n-2} \\ s_{n-1} \\ s_{n} \end{pmatrix}
$$

where

$$N_2^{(3)}=\begin{pmatrix} 1 & -\frac{28186487}{13531825} & \frac{13470063}{13531825} & 0 & 0 & 0 \\ \frac{74433464}{196720595} & 1 & -\frac{266225928}{196720595} & 0 & 0 & 0 \\ \frac{7647927}{44681305} & -\frac{51666726}{44681305} & 1 & 0 & 0 & 0 \\ -\frac{479168}{13331747} & \frac{2795592}{13331747} & -\frac{15691968}{13331747} & 1 & 0 & 0 \\ -\frac{32121945}{853231808} & \frac{185302467}{853231808} & -\frac{1009365735}{853231808} & 0 & 1 & 0 \\ -\frac{1543415}{39995241} & \frac{8878930}{39995241} & -\frac{15824230}{13331747} & 0 & 0 & 1 \end{pmatrix}, N_1^{(3)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1184599}{13531825} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4928131}{196720595} \\ 0 & 0 & 0 & 0 & 0 & -\frac{662506}{44681305} \\ 0 & 0 & 0 & 0 & 0 & \frac{3369}{1025519} \\ 0 & 0 & 0 & 0 & 0 & \frac{227185}{65633216} \\ 0 & 0 & 0 & 0 & 0 & \frac{10918}{3076557} \end{pmatrix},$$

$$G_2^{(3)}=\begin{pmatrix} -\frac{26663494}{40595475} & 0 & 0 & \frac{19370614}{40595475} & 0 & 0 \\ 0 & -\frac{239971446}{196720595} & 0 & -\frac{90831684}{196720595} & 0 & 0 \\ 0 & 0 & \frac{79990482}{44681305} & -\frac{41632092}{44681305} & 0 & 0 \\ 0 & 0 & 0 & \frac{11363100}{13331747} & 0 & 0 \\ 0 & 0 & 0 & \frac{143741115}{106653976} & 0 & 0 \\ 0 & 0 & 0 & \frac{24590860}{13331747} & 0 & 0 \end{pmatrix}, E_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & -\frac{6611194}{8119095} & \frac{3940736}{8119095} & 0 \\ 0 & 0 & 0 & \frac{395872742}{590161785} & -\frac{214095616}{590161785} & 0 \\ 0 & 0 & 0 & \frac{43795628}{44681305} & -\frac{19560448}{44681305} & 0 \\ 0 & 0 & 0 & -\frac{5552072}{13331747} & \frac{1828864}{13331747} & 0 \\ 0 & 0 & 0 & -\frac{277116945}{853231808} & \frac{19198095}{106653976} & 0 \\ 0 & 0 & 0 & \frac{25957700}{119985723} & \frac{52145920}{119985723} & 0 \end{pmatrix},$$

and $$R_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{4442006}{40595475} \\ 0 & 0 & 0 & 0 & 0 & \frac{45955496}{590161785} \\ 0 & 0 & 0 & 0 & 0 & \frac{3878126}{44681305} \\ 0 & 0 & 0 & 0 & 0 & -\frac{337472}{13331747} \\ 0 & 0 & 0 & 0 & 0 & -\frac{24813285}{853231808} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1560500}{119985723} \end{pmatrix}$$

Let the first characteristic polynomial be stated as

$$\begin{aligned} P(u) &= \det\left(uN_2^{(3)} - N_1^{(3)}\right) \\ &= \left|uN_2^{(3)} - N_1^{(3)}\right| = 0 \end{aligned} \tag{7}$$

Now we have,

$$P(u)=\left|u\begin{pmatrix} 1 & -\frac{28186487}{13531825} & \frac{13470063}{13531825} & 0 & 0 & 0 \\ \frac{74433464}{196720595} & 1 & -\frac{266225928}{196720595} & 0 & 0 & 0 \\ \frac{7647927}{44681305} & -\frac{51666726}{44681305} & 1 & 0 & 0 & 0 \\ -\frac{479168}{13331747} & \frac{2795592}{13331747} & -\frac{15691968}{13331747} & 1 & 0 & 0 \\ -\frac{32121945}{853231808} & \frac{185302467}{853231808} & -\frac{1009365735}{853231808} & 0 & 1 & 0 \\ -\frac{1543415}{39995241} & \frac{8878930}{39995241} & -\frac{15824230}{13331747} & 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1184599}{13531825} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4928131}{196720595} \\ 0 & 0 & 0 & 0 & 0 & -\frac{662506}{44681305} \\ 0 & 0 & 0 & 0 & 0 & \frac{3369}{1025519} \\ 0 & 0 & 0 & 0 & 0 & \frac{227185}{65633216} \\ 0 & 0 & 0 & 0 & 0 & \frac{10918}{3076557} \end{pmatrix}\right|$$

$$=\left|\begin{pmatrix} u & -\frac{28186487}{13531825}u & \frac{13470063}{13531825}u & 0 & 0 & 0 \\ \frac{74433464}{196720595}u & u & -\frac{266225928}{196720595}u & 0 & 0 & 0 \\ \frac{7647927}{44681305}u & -\frac{51666726}{44681305}u & u & 0 & 0 & 0 \\ -\frac{479168}{13331747}u & \frac{2795592}{13331747}u & -\frac{15691968}{13331747}u & u & 0 & 0 \\ -\frac{32121945}{853231808}u & \frac{185302467}{853231808}u & -\frac{1009365735}{853231808}u & 0 & u & 0 \\ -\frac{1543415}{39995241}u & \frac{8878930}{39995241}u & -\frac{15824230}{13331747}u & 0 & 0 & u \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1184599}{13531825} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4928131}{196720595} \\ 0 & 0 & 0 & 0 & 0 & -\frac{662506}{44681305} \\ 0 & 0 & 0 & 0 & 0 & \frac{3369}{1025519} \\ 0 & 0 & 0 & 0 & 0 & \frac{227185}{65633216} \\ 0 & 0 & 0 & 0 & 0 & \frac{10918}{3076557} \end{pmatrix}\right|$$

$$\Rightarrow P(u)=\begin{pmatrix} u & -\frac{28186487}{13531825}u & \frac{13470063}{13531825}u & 0 & 0 & -\frac{1184599}{13531825} \\ \frac{74433464}{196720595}u & u & -\frac{266225928}{196720595}u & 0 & 0 & \frac{4928131}{196720595} \\ \frac{7647927}{44681305}u & -\frac{51666726}{44681305}u & u & 0 & 0 & \frac{662506}{44681305} \\ -\frac{479168}{13331747}u & \frac{2795592}{13331747}u & -\frac{15691968}{13331747}u & u & 0 & -\frac{3369}{1025519} \\ -\frac{32121945}{853231808}u & \frac{185302467}{853231808}u & -\frac{1009365735}{853231808}u & 0 & u & -\frac{227185}{65633216} \\ -\frac{1543415}{39995241}u & \frac{8878930}{39995241}u & -\frac{15824230}{13331747}u & 0 & 0 & u-\frac{10918}{3076557} \end{pmatrix}$$

Using Maple (18) software,

$$P(u)=\frac{237490998930113171443 2}{237882254933766577633 75}u^{5}\left(u+1\right)$$

$$\Rightarrow \frac{23749099893011317144432}{23788225493376657763375} u^5 (u+1) = 0$$

$\Rightarrow u_1 = -1, u_2 = 0, u_3 = 0, u_4 = 0, u_5 = 0, u_6 = 0$ .Since $|u_i| < 1,\ i = 1,2,3,4,5,6$ (4) is zero stable.

3.4Convergence

Lambert (1973), studied that for a numerical method to converge, it must be consistent and zero-stable. The proposed method satisfies these two conditions as investigated above,therefore the method is convergent.

3.5 Region of Absolute Stability

The W - and Z - regions of absolute stability of the method for discrete schemes (2), (3) and (4) are plotted and presented in Fig. 1 to 4 below using Maple 18 and MATLAB software.

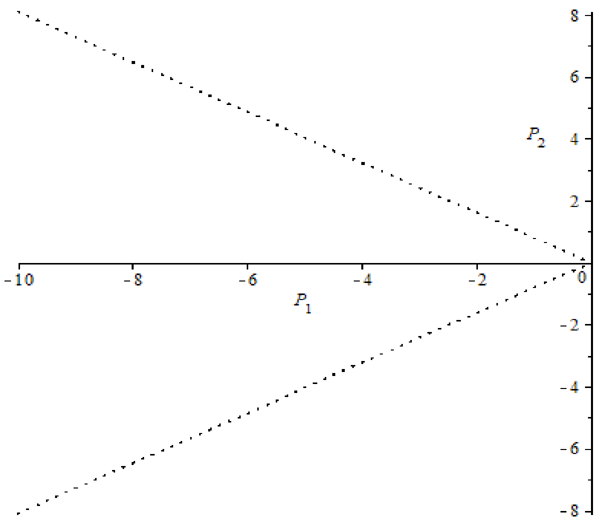

Fig. 1 Region of W -stability in (2)

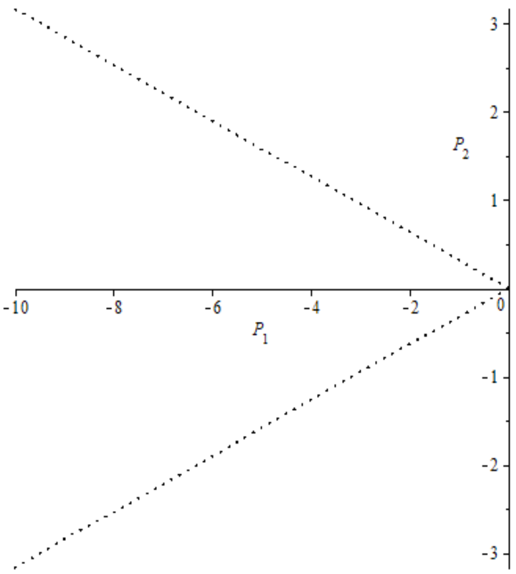

Fig. 3 Region of W -stabilityin (4)

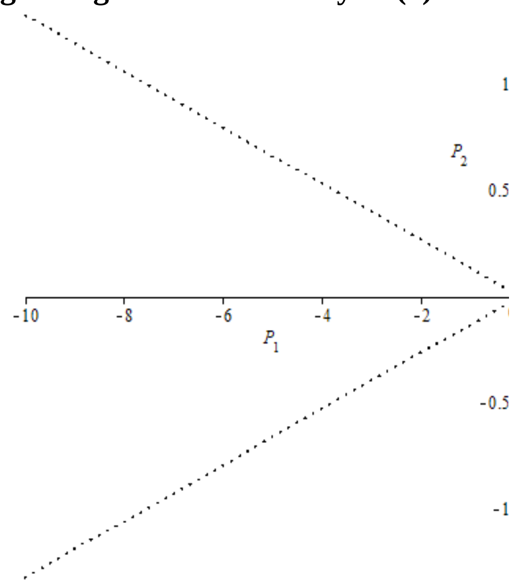

Fig. 2 Region of W -stability in (3)

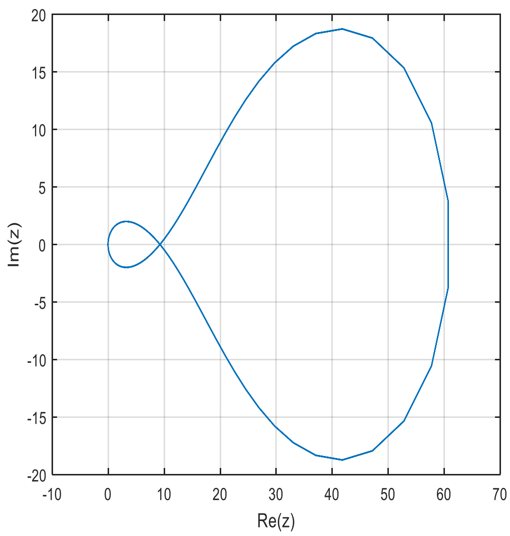

Fig. 4 Region of Z -stability in (2)

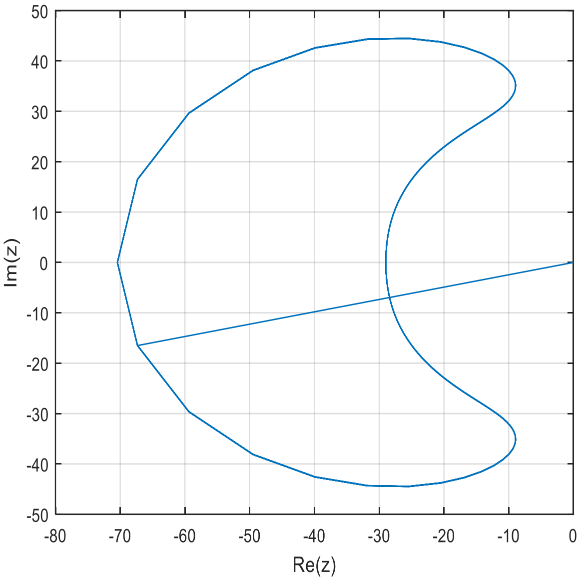

Fig. 5 Region of Z-stability in (3)

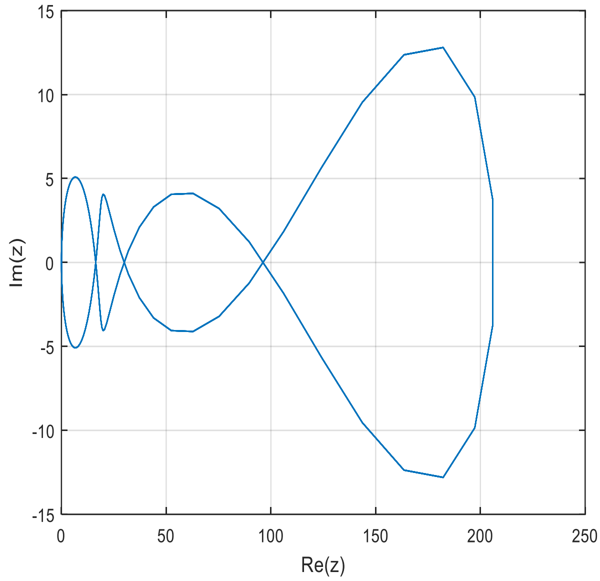

Fig. 6 Region of Z-stabilityin (4)

The W-stability regions in Fig. 1 to 3 lie inside the open-ended region while the Z-stability regions in Fig. 4 to 6 lie inside the enclosed region. Therefore, the region of absolute stability of the proposed method is satisfied.

4.0 Evaluations of the Delay Term $(t+\tau)$and the Noise Term $d\alpha(t)$

One major focus in the application of numerical methods in solving any modeled system is its performance which reveals the accuracy, efficiency and CPU time taken to arrive at the desired results. Chibuisi, Osu, Granados & Basimanebotlhe (2022), Chibuisi, Osu, Olunkwa, Ihedioha, & Amaraihu (2020), Chibuisi, *et al.* (2021) used the mathematical expression formulated by Sirisena and Yakubu (2019) for the evaluation of delay terms in solving some first-order delay differential equations and obtained less accurate results with longer CPU time in computation. For good performance, accuracy, efficiency, faster evaluations with lesser CPU time and diversity of evaluations of delay term and noise term in obtaining numerical solutions of other classes of DDEs such as Stochastic delay differential equations, Advanced Stochastic Delay Differential Equations, Riccati delay differential equations, Partial delay differential equations and Stochastic partial delay differential equations different from the ones in the literature, this study applied three new theorems of mathematical expressions developed below:
In the sequel, one states;

Theorem 1
Let the current state and future part of the drift and stochastic coefficients of (1) be represented as $\lambda$and Φ, then the corresponding value of the functions $y(t+\tau)$and $y(t+\tau)d\alpha(t)$ with an accurate formula for the evaluation of the delay term $(t+\tau)$is given as:

$$\delta_{n+b}(t)=\frac{n}{w}\left((wz+(n+b+e-1)h)\right), w\neq 0.(8)$$

where $e\in(-k,k), k$ is a step number $e=\frac{\tau}{h}\in\mathbb{Z}, \tau=eh, \tau$ is the delay term, $n=0,1,2,\ldots,N-1$and N are the number of solutions in the giving interval which is implemented to approximate the delay term $(t+\tau)$at the point $t=t_n+\tau$ using the values of δ_{n+b} at $t_n+\tau>0$ whenever $t_n+\tau>0$ where$\delta_{n+b}(t)$ is the approximation to $y(t_n+\tau)$.

Proof:
The expression in (8) is formulated using the idea of the sum of arithmetic progression (AP) as developed by Grynkiewicz *et al*. (2013) of the form:

$$\delta_n = \frac{n}{w}(wz + (n-1)h).(9)$$

Incorporating the delay term in (9), one obtains equation (8) as required.
The results of the above expression in (8) are obtained using Maple 18 with $n = 0,1,2,3,\dots,N-1$ and incorporated into some examples of ASDDE before its evaluation at constant step size h to obtain the numerical solutions of $dy(t)$.

Theorem 2
Let $\alpha(t)$ be a normalized Brownian Motion Process for hyperbolic equivalence of Euler's exponential function with the mean $\mu$and the volatility σ given as $N(0,1)$. Then the discrete noise term $d\alpha(t)$ is given as:

$$d\alpha(t) = \frac{1}{\sqrt{2\pi}}\left(e^{\frac{-3t^2}{2}} - te^{\frac{-t^2-2}{2}}\right) \qquad for\ 0 \le t \le 30(10)$$

Proof:
Using the idea of normalized Brownian Motion Process for hyperbolic function, the continuous time stochastic process is given as,

$$\alpha(t) = \frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}sinht \qquad for\ 0 \le t \le 30(11)$$

Then by differentiating and discretizing (11)using hyperbolic equivalence of Euler's exponential function, one arrived at (10) as required.

Theorem 3
Let theorem 3.3.2 exists, then the modified discrete noise term, $d\alpha(t)$, using Iterative Adomian Decomposition Method (IADM) is given as:

$$d\alpha(t) = \frac{A_0}{\sqrt{2\pi}} + \sum_{h=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{A_h(e^t + e^{-t})}{2}, \qquad for\ 0 \le t \le 30 \qquad (12)$$

Proof:
Ogundile & Edeki (2021) obtained the continuous noise term $d\alpha(t)$ as a derivative equivalence of normalized induced Brownian Motion $\alpha(t)$ developed by Bibi & Merahi (2018) for the analytical approximate solution of the linear Stochastic Differential Equations (SDE) using Iterative Adomian Decomposition Method (IADM). The normalized induced Brownian motion and the continuous noise term developed by these two scholars are presented as:

$$\alpha(t) = \frac{A_0}{\sqrt{2\pi}}t + \sum_{h=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{A_h}{h}sinh(t) \qquad for\ t \in [0,\pi] \qquad (13)$$

Differentiating and discretizing (13) using hyperbolic equivalence of Euler's exponential function one arrived at (12) as required.
where $A_h = 0,1,2,3,\dots,A_{h+1},\forall, h = 1,2,3,\dots,30$ are randomly generated values within the time interval $0 \le t \le 30$. If one further assumes $V_0 = 0$, by Riemann-Stieltjes theorem in Stieltjes (1894) ,then (10) and (12) become

$$\begin{cases} D_h(t) = \sqrt{\frac{2}{\pi h}}\int_0^t A_h sinht\ dt \\ E_e(t) = \sqrt{\frac{2}{\pi}}\int_0^t A_h cosht\ dt \end{cases} \qquad (14)$$

Lemma 1: Let the density of a Brownian bridge expressed by Beghin, Orsingher (1999) exist:

$$f(x) = \frac{1}{t(1-t)\sqrt{2\pi}} e^{-\frac{v^2}{2t(1-t)^2}}, \text{ where } \mu = 0 \text{ and } \sigma = t(1-t) \quad (15)$$

Then, the function

$$F(t) := \int_0^1 \frac{1}{\sqrt{t(1-t)}} exp\left\{-\frac{(v-t)^2}{2t(1-t)}\right\} dt \quad (16)$$

is constant for $v \in [0,1]$ given as

$$F(t) = \int_{-\infty}^{\infty} (t^2+1)^{-1} e^{-\frac{t^2}{2}} He_0(t) dt for v \in [0,1],$$

where He_0 is the Hermite polynomial developed by Beghin et al (1999).
Remark: Equation (15) is equivalent to the density function derived in Pao-Liu (2006).
Sketch of Proof:
Put $t = cos^2 v$. Then

$$F(t) := 2\int_0^{\frac{\pi}{2}} exp\left\{-\frac{(cos^2 v - t)^2}{2sin^2 v cos^2 v}\right\} dv = \int_0^{\pi} exp\left\{-\frac{(cosv - 2t + 1)^2}{2sin^2 v}\right\} dv,$$

hence

$$F(t) = F(1-t), v\epsilon[0,1] and F\left(1/2\right) = F(1) = F(0). \quad (17)$$

Following proposition 1 developed by Osu, Chibuisi *et al*. (2021), the proof is completed.
Theorem 4: Let lemma 1 and equation (14) exist, such that

$$D_0 = \frac{1}{\sqrt{2\pi}} F(t), D_h(t) = \sqrt{\frac{2}{\pi}} \int_0^t A_h cos\ ht\ dt = E_h(t) = \sqrt{\frac{2}{\pi h}} \int_0^t A_h sin\ ht\ dt, h = 1,2, \ldots$$

(18)
Then $D_0, D_h(2\pi), M_e(2\pi), h\epsilon N \cup \{0\}, e \in N$ are mutually independent random variables having normal distribution $N(0,1)$.
Proof: Following the proof of asymptotics of solutions to semi-linear stochastic wave equations by Pao-Liu (2006), Let $c, b \in R; h, e \in N$ be arbitrary. For mutually independent events, we have $exp\left[t\left(cD_h(t) + bE_e(t)\right)\right] =$

$$1 + bc\int_0^t exp\left[t\left(cE_e(t) + bD_h(t)\right)\right] dE_e(t) + bc\int_0^t exp\left[t\left(eE_h(t) + bD_h(t)\right)\right] dD_h(t) - \frac{1}{2} cb \int_0^t exp\left[t\left(cE_e(t) + eD_h(t)\right)\right] d[E_e, D_h] - \frac{1}{2} c^2 \int_0^t exp\left[t\left(cE_e(t) + bD_h(t)\right)\right] d[E_e] - \frac{1}{2} b^2 \int_0^t exp\left[t\left(cE_e(t) + bD_h(t)\right)\right] d[D_h], \forall t \in [0,\infty) \quad (19)$$

We observe that the second and third term on the right-hand side are continuous martingales; thus, taking expectations on both sides and denoting $exp\left[t\left(cE_e(t) + bD_h(t)\right)\right]$ by $f(t)$, we get

$$M[f(t)] = 1 - \frac{1}{2\pi} cbM\left[\int_0^t f(t) sinhtcoshtdt\right] - \frac{1}{2\pi} c^2 M\left[\int_0^t f(t) sin^2 htdt\right] - \frac{1}{2\pi} b^2 M\left[\int_0^t f(t) cos^2 htdt\right]. \quad (20)$$

By Fubini's Theorem in Fubini(1907), we now have

$$M[f(t)] = 1 - \frac{1}{2\pi} cb \int_0^t M[f(t)] \sin ht \cos ht\ dt - \frac{1}{2\pi} c^2 \int_0^t M[f(t)] sin^2 htdt - \frac{1}{2\pi} b^2 \int_0^t M[f(t)] cos^2 htdt, \quad (21)$$

So we have that $M[f(t)]$ is the solution of the differential equation of the form:

$$\frac{dM[f(t)]}{dt} = -\frac{1}{2\pi} M[f(t)](cb\ sinht\ cosht + c^2 sin^2 ht + b^2 cos^2 ht), M[f(0)] = 1. \quad (22)$$

Thus we have:

$$M[f(t)] = \exp\,[-\frac{1}{2\pi}\int_0^t (cb\sin ht\cos ht + c^2 sin^2 ht + b^2\, cos\; ht)dt]$$

which implies

$$M[f(2\pi)] = \exp\,[-\frac{1}{2\pi}\int_0^{2\pi} (c^2 sin^2 ht + b^2 \cos ht)dt].$$

$= \exp\,[-\frac{1}{2\pi} sinc^2 + b^2 \int_0^{2\pi} sinht + \cos ht\; dt]$.(23)

From the above, we immediately have:

$M\left[exp[i(cE_e + bD_h)]\right] = M\left[exp[icE_e]\right]M\left[exp[ibD_h]\right].$ (24)

Evaluating the integrals in (23) and through the existence of lemma 1, we arrived at (18) as required.

5.0 Numerical Implementation and Computations

To obtain the numerical solutions ofASDDE, the developed and proved theorems (8), (10) and (12) for the evaluations delay term and the noise term and the discrete schemes (2), (3) and (4) of the proposed method are incorporated into some examples of ASDDE before its numerical evaluations at fixed step size $h = 0.01$ using Maple 18 software.

Numerical Examples

Example 1

$$dy(t) = 1000\left(y(t) + y\left(t + \left(In(1000+1)\right)\right)\right)dt + \left(y(t) + y\left(t + \left(In(1000+1)\right)\right)\right)d\psi(t), 0 \le t \le 30$$

$$y(t) = \mathrm{e}^{-\mathrm{t}}, \mathrm{t} \ge 0$$

Exact solution $y(t) = \mathrm{e}^{-\mathrm{t}}$ in [14]

Example 2

$$dy(t) = cos(t)\left((y(y(t)+2))\right)dt + (y(y(t)+2))d\psi(t), 0 \le t \le 30$$

$$y(t) = 1, t \ge 0$$

Exact Solution $y(t) = 1 + \sin\,(t)$ in [14]

6.0 Result and Discussion

The absolute random errors obtained after the numerical evaluations of the above examples of ASDDE using the proposed method with the developed theorems are computed in Tables 1 to 4:

Table 1: Absolute Random Errors of Example 1 with the incorporation of Theorem 1 and Theorem 2 using the HESDBBBDFM for Step Numbers *k* = 2, 3 & 4.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error	K = 4 Absolute Random Error
1	0.239930006	0.459930033	0.150070004
2	0.296486392	0.526486408	0.326486365
3	0.281692246	0.562692304	0.722514223

4	0.082255943	0.208783257	0.402610853
5	0.07294605	0.551958861	0.807561827
6	0.115258389	0.298238195	0.343387214
7	0.036221929	0.405857287	0.547684386
8	0.55828192	0.766985809	0.863837129
9	0.003493028	0.100733059	0.041948729
10	0.01684601	0.198687875	0.309396558
11	0.139727785	0.798886676	0.888579878
12	0.079385234	0.310901796	0.403527771
13	0.169020678	0.307807363	0.175262798
14	0.052486742	0.599373799	0.588929341
15	0.237673552	0.399260981	0.337607702
16	0.031227799	0.051321825	0.145776804
17	0.387337157	0.595451231	0.784788279
18	0.298905716	0.536227726	0.524724235
19	0.010677768	0.067155212	0.181026249
20	0.458203393	0.702452528	0.795281814
21	0.2509112	0.589441325	0.728966323
22	0.447482168	0.552575213	0.645386505
23	0.137031209	0.311223277	0.400238051
24	0.325265896	0.528719939	0.6223799
25	0.023973601	0.436140678	0.559853295
26	0.305841013	0.170294881	0.650119523
27	0.131047404	0.59618851	0.234893492
28	0.156696336	0.154681628	0.246656543
29	0.12648207	0.29422961	0.320938941
30	0.152917068	0.43971115	0.676373703

CPU time of HESDBBDFM for k = 2 is 0.002s, k = 3 is 0.003 and k = 4 is 0.004s

Table 2: Absolute Random Errors of Example 2 with the incorporation of Theorem 1 and Theorem 2 using the HESDBBDFM for Step Numbers *k* = 2, 3 & 4.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error	K = 4 Absolute Random Error
1	0.322170381	0.522170391	0.622170394
2	0.444863395	0.64486339	0.744863391
3	0.52065847	0.820658466	0.920658465
4	0.642473835	0.742473801	0.842473802
5	0.44449147	0.644491492	0.744491476
6	0.33928579	0.639285771	0.739285782
7	0.431482072	0.531481947	0.63148194
8	0.222780846	0.422780894	0.622780882
9	0.313807802	0.513807777	0.613807778
10	0.404792038	0.604792017	0.790438341
11	0.395817435	0.595817436	0.795817437

12	0.586914319	0.7869143	0.886914297
13	0.278093238	0.478093157	0.67809317
14	0.369357386	0.569357413	0.769357405
15	0.360707677	0.460707663	0.660707662
16	0.252143701	0.552143669	0.752143676
17	0.243664767	0.543664776	0.643664776
18	0.435270194	0.535270196	0.735270197
19	0.426959154	0.626959128	0.726959128
20	0.518730741	0.718730752	0.91873075
21	0.310584255	0.581058425	0.710584246
22	0.302518805	0.502518788	0.602518798
23	0.294533606	0.594533611	0.894533603
24	0.486627871	0.686627866	0.886627861
25	0.27880083	0.578800776	0.678800786
26	0.371051575	0.47105159	0.671051587
27	0.363379503	0.463379492	0.563379495
28	0.255783744	0.455783727	0.555783741
29	0.448263564	0.548263584	0.648263569
30	0.240818211	0.44081821	0.640818218

CPU time of HESDBBDFM for k = 2 is 0.003s, k = 3 is 0.005 and k = 4 is 0.006s

Table 3: Absolute Random Errors of Example 1 with the incorporation of Theorem 1 and Theorem 3 using the HESDBBDFM for Step Numbers *k* = 2, 3 & 4.

t	**K = 2 Absolute Random Error**	**K = 3 Absolute Random Error**	**K = 4 Absolute Random Error**
1	0.639930006	0.359930033	0.149929996
2	0.696486392	0.326486408	0.126486365
3	0.381692246	0.262692304	0.022514223
4	0.582255943	0.308783257	0.202610853
5	0.77294605	0.451958861	0.307561827
6	0.484741611	0.098238195	0.056612786
7	0.363778071	0.205857287	0.147684386
8	0.75828192	0.466985809	0.263837129
9	0.306506972	0.200733059	0.008051271
10	0.38315399	0.098687875	0.009396558
11	0.839727785	0.298886676	0.188579878
12	0.379385234	0.310901796	0.103527771
13	0.269020678	0.107807363	0.024737202
14	0.752486742	0.599373799	0.588929341
15	0.537673552	0.399260981	0.337607702
16	0.331227799	0.051321825	0.054223196
17	0.387337157	0.295451231	0.284788279
18	0.598905716	0.436227726	0.224724235
19	0.489322232	0.032844788	0.081026249

20	0.758203393	0.502452528	0.295281814
21	0.5509112	0.389441325	0.328966323
22	0.447482168	0.152575213	0.145386505
23	0.437031209	0.211223277	0.200238051
24	0.625265896	0.428719939	0.2223799
25	0.476026399	0.136140678	0.059853295
26	0.305841013	0.170294881	0.050119523
27	0.831047404	0.49618851	0.134893492
28	0.156696336	0.145318372	0.053343457
29	0.52648207	0.29422961	0.120938941
30	0.752917071	0.539711152	0.376373702

CPU time of HESDBBDFM for k = 2 is 0.6s, k = 3 is 0.5 and k = 4 is 0.3s

Table 4: Absolute Random Errors of Example 2 with the incorporation of Theorem 1 and Theorem 3 using the HESDBBDFM for Step Numbers *k* = 2, 3 & 4.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error	K = 4 Absolute Random Error
1	0.423661541	0.323661541	0.393661542
2	0.647475802	0.447475802	0.577475804
3	0.223847043	0.124025203	0.024025205
4	0.645875995	0.445703455	0.346230927
5	0.846885974	0.847227504	0.147240909
6	0.540323961	0.441174873	0.340678893
7	0.2273909	0.130217331	0.052296654
8	0.817185954	0.481501769	0.32174099
9	0.802549366	0.805262432	0.805792472
10	0.787579445	0.389287846	0.290822549
11	0.770393687	0.473086825	0.275607407
12	0.76303901	0.575663892	0.460181647
13	0.333901765	0.237556931	0.141044395
14	0.714677259	0.41833243	0.232181239
15	0.694679958	0.429902378	0.302514
16	0.674680349	0.479225141	0.283131645
17	0.455609109	0.34594676	0.100060414
18	0.636630328	0.539759571	0.424508148
19	0.620903356	0.324032596	0.226199579
20	0.305363465	0.208492709	0.107418532
21	0.595689353	0.51931434	0.497744421
22	0.586278882	0.387426545	0.188333953
23	0.518498202	0.281997794	0.079189243
24	0.58398466	0.476858377	0.370312355
25	0.292050426	0.284924132	0.178378107
26	0.604038977	0.593263481	0.486717466
27	0.516482932	0.301187456	0.295328547

28	0.632752377	0.420587497	0.104209415
29	0.552694935	0.339404192	0.224151974
30	0.522657934	0.358320414	0.114114966

CPU time of HESDBBDFM for k = 2 is 0.7s, k = 3 is 0.6s and k = 4 is 0.4s

6.1 Graphical Presentation of Absolute Random Errors (AREs)

The graphs below are the Absolute Random Error Results of Examples1 and 2 above in tables 1 to 4plotted with R and R – studio software, and are presented as;

6.1.1 Graphical Presentations of AREs after the Incorporations of the Theorems 1 and 2 for Examples 1 and 2

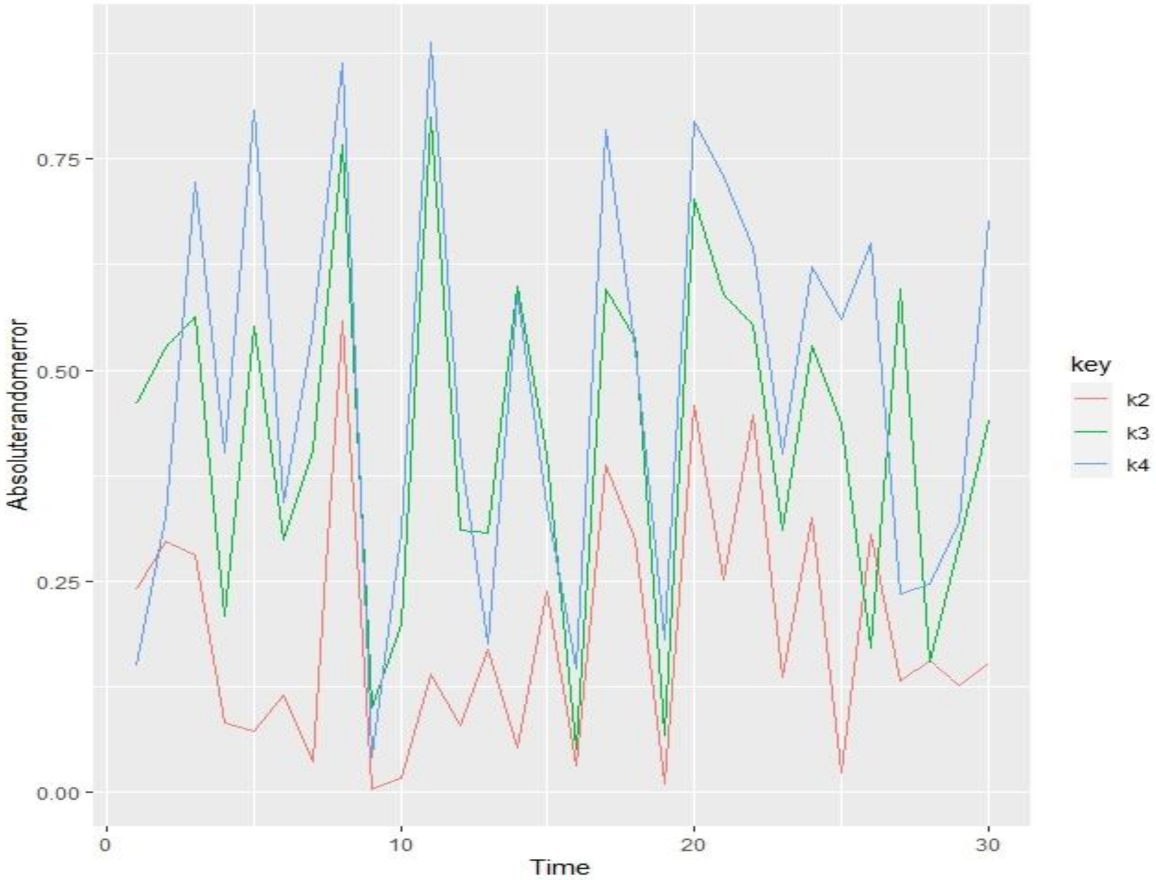

Fig.7:Absolute Random Error Results for Example 1 using HESDBBDFM (as seen in the colors) against Time of future delay in days. The coloured lines represent the behavior or performance of the method for step numbers $k = 2, 3\ and\ 4$ with different Absolute Random Errors.

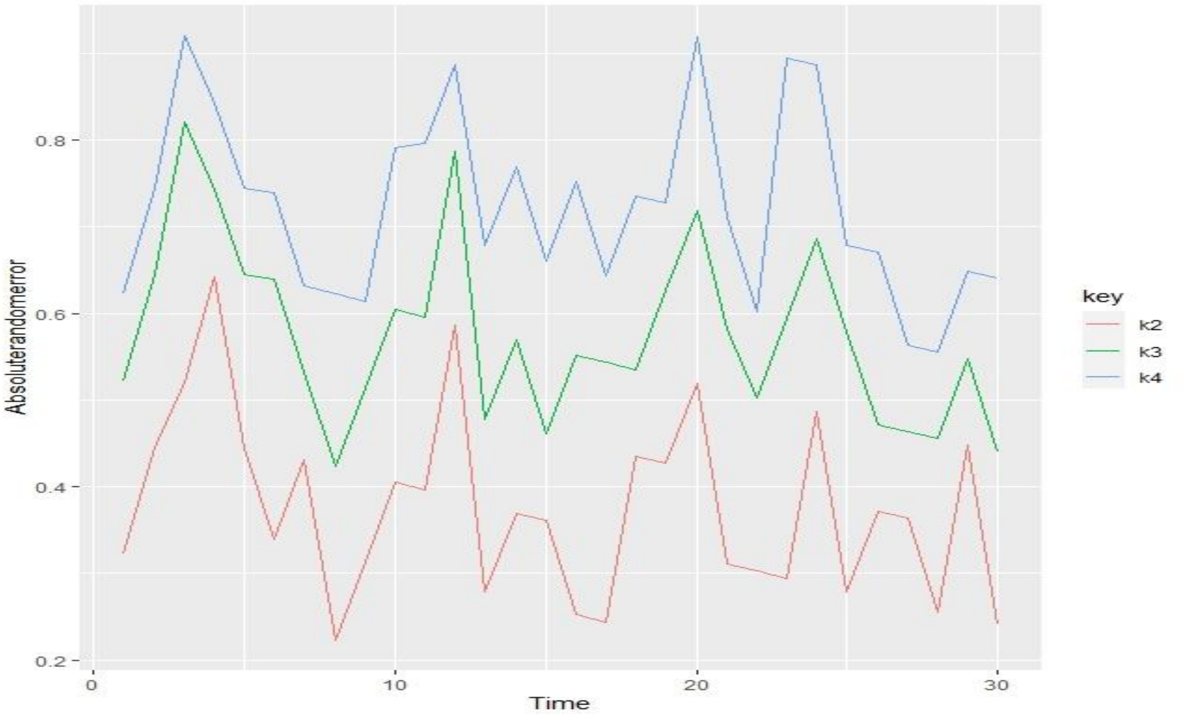

Fig. 8:Absolute Random Error Results for Example 2 using HESDBBDFM (as seen in the colours) against Time of future delay in days. The coloured lines represent the behaviour or performance of the method for step numbers $k = 2, 3\ and\ 4$ with different Absolute Random Errors.

6.1.2 Graphical Presentations of the AREs after the Incorporations of the Theorems 1 and 3 for Examples 1 and 2

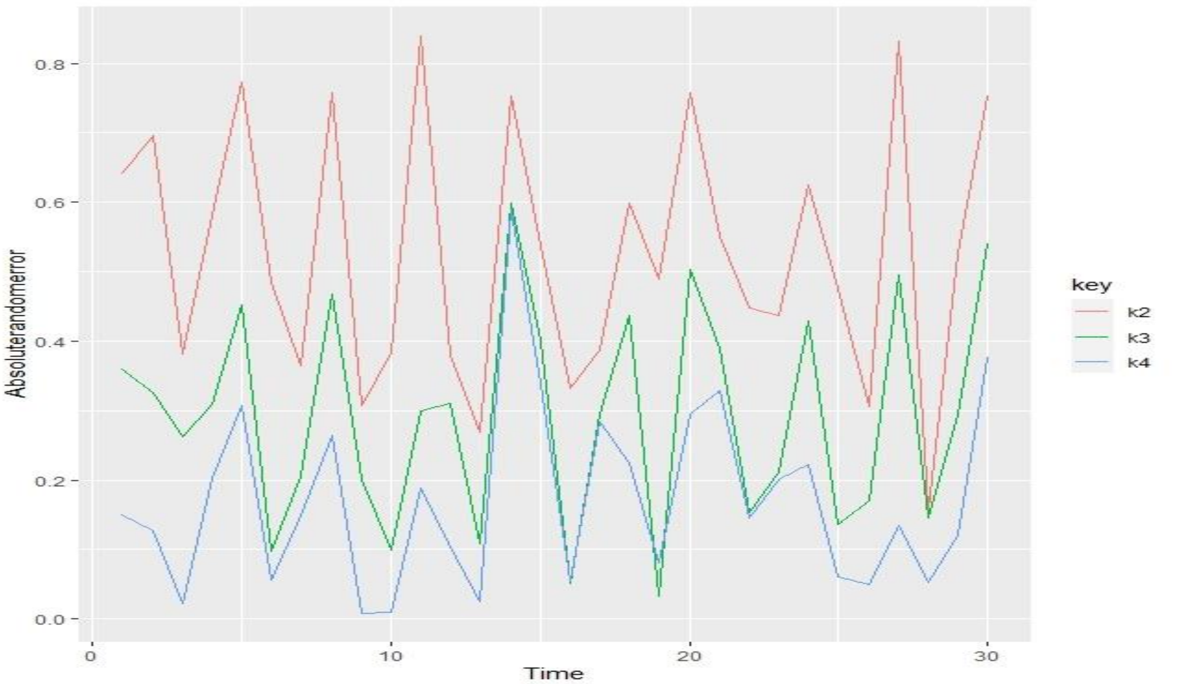

Fig. 9:Absolute Random Error Results for Example 1 using HESDBBDFM (as seen in the colors) against Time of future delay in days. The coloured lines represent the behaviour or performance of the method for step numbers $k = 2, 3\ and\ 4$ with different Absolute Random Errors.

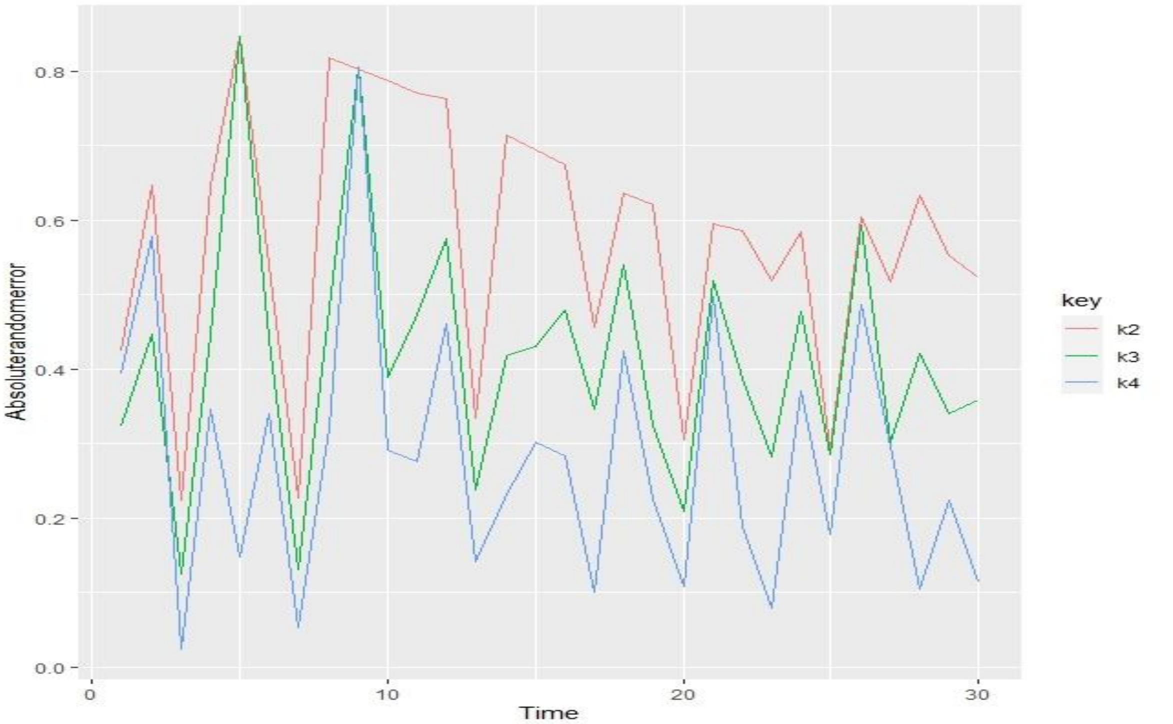

Fig. 10:Absolute Random Error Results for Example 2 using HESDBBDFM (as seen in the colors) against Time of future delay in days. The coloured lines represent the behaviour or performance of the method for step numbers $k = 2, 3$ *and* 4 with different Absolute Random Errors.

7. 0 Comparison of Results

In order to determine the accuracy, efficiency and advantage of our method HESDBBDFM, we compared the Minimum Absolute Random Errors (MAREs) of our method with other existing methods in Akhtari et al (2015), Wang et al (2011) and Bahar (2019)below:

TABLE 5: Comparison between the Minimum Absolute Random Errors (MAREs) of HESDBBDFM after the incorporations of theorem 3.3.1 with theorems 3.3.2 and 3.3.3 for k = 2, 3 and 4 with Akhtari et al (2015), Wang et al (2011) and Bahar (2019)at constant step size h = 0.01 for Example 1.

Numerical Method	COMPARED MAREs with Akhtari et al (2015), Wang et al (2011) & Bahar (2019)
HESDBBDFM MARE for k = 2 by theorems 1 and 2	3.49E-03
HESDBBDFM MARE for k = 3 by theorems 1 and 2	5.13E-02
HESDBBDFM MARE for k = 4 by theorems 1 and 2	4.19E-02
HESDBBDFM MARE for k = 2 by theorems 1 and 3	1.57E-01
HESDBBDFM MARE for k = 3 by theorems 1 and 3	3.28E-02
HESDBBDFM MARE for k = 4 by theorems 1 and 3	8.05E-03
CSSEMM MARE for k = 2 Akhtari et al (2015)	4.76E-01
CSSEMM MARE for k = 3 Akhtari et al (2015)	9.17E-01
CSSEMM MARE for k = 4 Akhtari et al (2015)	1.62E-01
EMM MARE for k = 2 Wang et al (2011)	1.84E+00

EMM MARE for k = 3 Wang et al (2011)	2.47E-01
EMM MARE for k = 4 Wang et al (2011)	9.73E-01
BSM MARE for k = 2 Bahar (2019)	7.04E-01
BSM MARE for k = 3 Bahar (2019)	7.05E-01
BSM MARE for k = 4 Bahar (2019)	7.06E-01

TABLE 6: Comparison between the Minimum Absolute Random Errors (MAREs) of HESDBBDFM after the incorporations of the evaluated theorem 3.3.1 with theorems 3.3.2 and 3.3.3 for k = 2, 3 and 4 with Akhtari et al (2015), Wang et al (2011) and Bahar (2019) at constant step size h = 0.01 for Example 2.

Numerical Method	**COMPARED MAREs with Akhtari et al (2015), Wang et al (2011) & Bahar (2019)**
HESDBBDFM MARE for k = 2 by theorems 1 and 2	2.23E-03
HESDBBDFM MARE for k = 3 by theorems 1 and 2	4.23E-02
HESDBBDFM MARE for k = 4 by theorems 1 and 2	5.56E-02
HESDBBDFM MARE for k = 2 by theorems 1 and 3	2.24E-01
HESDBBDFM MARE for k = 3 by theorems 1 and 3	1.24E-01
HESDBBDFM MARE for k = 4 by theorems 1 and 3	2.40E-02
CSSEMM MARE for k = 2 Akhtari et al (2015)	4.76E-01
CSSEMM MARE for k = 3 Akhtari et al (2015)	9.17E-01
CSSEMM MARE for k = 4 Akhtari et al (2015)	1.62E-01
EMM MARE for k = 2 Wang et al (2011)	1.84E+00
EMM MARE for k = 3 Wang et al (2011)	2.47E-01
EMM MARE for k = 4 Wang et al (2011)	9.73E-01
BSM MARE for k = 2 Bahar (2019)	7.04E-01
BSM MARE for k = 3 Bahar (2019)	7.05E-01
BSM MARE for k = 4 Bahar (2019)	7.06E-01

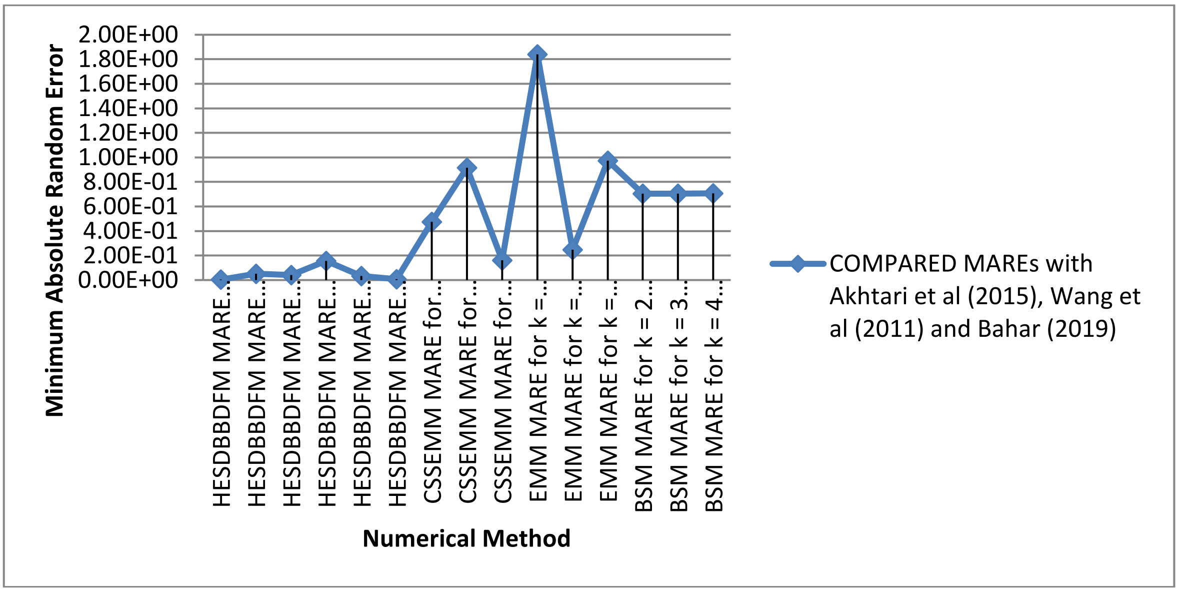

Fig. 11:Compared MAREs of HESDBBDFM withfor Example 1

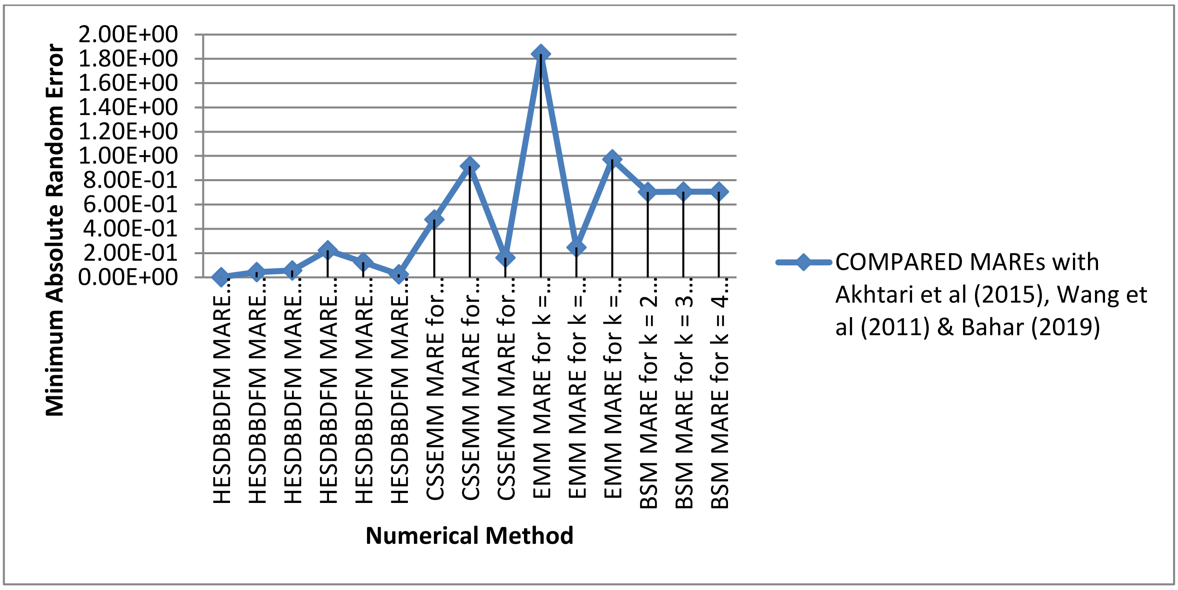

Fig.12:Compared MAREs of HESDBBDFM withfor Example 2

7.1 Graphical Presentation for Compared Results

The results computed in the tables and presented graphically above have revealed the uncertainties caused by the removal of fuel subsidies by the current Nigerian government which its adverse effects directly affect the petroleum marketers, fuel dealers, transport operators, production companies and marketers of produced products and the general public thereby inflicting sufferings on the masses through inflation.

8. 0 Conclusion

After the analysis and evaluations, this study proffers solutions to the Nigerian government, companies and organizations on the best ways to reduce the adverse effects of uncertainties caused by the removal of fuel subsidies to improve the economy of the country and people's standard of living through the numerical solution of some examples of ASDDE. This study has also demonstrated that HESDBBDFM for step numbers k = 2, 3 and 4is suitable for solving some ASDDE numerically with the newly developed theorems for evaluations of the delay term and the noise term. As observed in Tables 1 to 4 and Figs. 7 to 10, the numerical results of the discrete schemes of the lower step number $k=2$ of HESDBBDFM performed slightly better and faster than the higher step numbers $k=3$ and 4 by producing the Least Minimum Absolute Random Error (LMARE). In comparing the numerical results of this method with other existing methods in literature as shown in tables 5 to 6 and Figs. 11 to 12, the newly developed theorems 3.3.1 and 3.3.2 for the evaluations of the delay term and the noise term in solving some ASDDEs with the discrete schemes of HESDBBDFM give better results by producing the Least Minimum Absolute Random Error (LMARE) in a Lower Computational Processing Unit Time (LCPUT) faster than the theorems of 3.3.1 and 3.3.3and other existing methods. The lower the Absolute Random Error (ARE), the lower the uncertainties and the lower the uncertainties the better the economy and people's standard of living. Further studies should be carried-out for step numbers $k=5,6,7,...$ on the numerical solutions of ASDDE using HESDBBDFM with its applications in finance. To avoid or drastically reduce the adverse effects of uncertainties created by the fuel subsidy removal, this paper recommends that:

1. the government of Nigeria should put up strategic measures on the ground by ensuring the rehabilitation and functioning of the old refineries.
2. the government should ensure 100% increment ofworkers' salaries.
3. the government and private companies should put-up policies that will discourage inflation throughdiversification and development of other economies.
4. the government should establish a well-designed communication channelto note the negative impact of fuel subsidies and the benefits from its removal.
5. the government should strike a balance between market realities and the welfare of the general public through compensation.
6. the government should carry-out a feasibility study before embarking on any economic and financial policy.
7. the government should have a long-term implementation plan and avoid a fire-brigade approach in its financial policy decisions.

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Compliance with Ethical Standards Declarations

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data public.

Competing interests

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Authors' Contributions

This study was carried out in collaboration among the authors. Authors Chigozie chibuisi and Bright O. Osu designed the study, carried-out the numerical analysis, investigated the basic properties and the first draft of the manuscript. Authors Kevin Ndubuisi C. Njoku and Chukwuka Fernando Chikwe conducted the analyses of the study. Authors Chigozie chibuisi and Bright O. Osu handled the literature reviews. All authors read and approved the final manuscript.