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Difference Synchronization of Fractional Order Chaotic Systems Via Active Control

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Abstract: The integration of memristors and meminductors into fractional-order chaotic systems has opened up new avenues for exploring complex dynamics. This research investigates difference synchronization in memristive and meminductive fractional-order chaotic systems evolving from diverse initial conditions. Active control techniques are employed to achieve difference synchronization among three such systems. Numerical simulations validate the effectiveness of the active control techniques. This study contributes to the understanding of synchronization in complex systems and offers insights into potential applications.

Keywords: *Memristors and meminductors; fractional-order chaotic systems; difference synchronization; active control; numerical simulations.*

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1.0 Introduction

Synchronization expresses a notion of strong correlations between coupled systems. In its most elementary and intuitive form, synchronization refers to the tendency to have the same dynamic behaviour. The synchronization of chaotic systems is the tendency of coupled systems to undergo closely related motions (Pecora and Carroll, 1990). It can also be seen as a way to couple chaotic systems to let them achieve identical dynamics asymptotically with time (Hegazi et al, 2013). Types of chaos synchronization developed by researchers include complete synchronization (Razminia and Baleanu, 2013), generalized synchronization (Yang et al, 2016), phase synchronization (Yu et al, 2011), lag synchronization (Sourav et al, 2012), anti-phase synchronization (Taghvafard and Erjaee, 2011), and projective synchronization (Li et al, 2006). In the course of further research works, some synchronization schemes such as difference synchronization (Dongmo et al, 2018), combination synchronization (Ojo et al, 2022), combination–combination (or dual combination) synchronization (Ojo et al, 2016; Junwei et al, 2015), compound-combination synchronization (Ojo et al, 2015), and double compound synchronization (Al Themairi et al, 2022) have been discovered. Moreover, in secure communication, it is realized that combination synchronization can be used to split signals into several parts, each part loaded in different chaotic drive systems, which provides better security to the transmitted signals.

Fractional-order systems have been shown to exhibit more complex dynamics than integer-order systems, and hence, studying the synchronization of fractional-order chaotic systems gives a better understanding of the complex dynamical behaviour of coupled systems. The study of synchronization of fractional-order chaotic systems is a relatively new and emerging field that has gained a lot of attention in recent years (Borah and Roy, 2017). The exponential growth in research works on the synchronization of fractional order chaotic is due to its potential real-life applications in secure communication, image encryption, chaos-based cryptography, and many more (Serdouk et al, 2023; Tang et al, 2022; Mitkowski et al, 2022; Xinyu et al, 2021; Borah and Roy, 2020; Yang et al, 2016). As a result of the potential applications of synchronization of fractional-order chaotic systems, many synchronization methods, types, and schemes have been developed as it is done for integer order systems (Zhang and Wu, 2020; Wang, 2018; Ogunjo et al, 2017; Song et al, 2017; Zhou and Zhu, 2017; Wang, et al, 2016). Despite several reported papers on the synchronization of fractional order chaotic systems, none have been reported on the difference synchronization of chaotic memristive-meminductive systems to the best of our knowledge.

2. 0 Definition of fractional order differential model

Fractional order refers to a mathematical concept that involves using non-integer exponents or orders in equations. The Grunwald–Letnikov definition of fractional order systems, the fractional order derivative of order q can be written as (Podlubny, 1999).

$$D_t^q f(t) = \lim_{h\to 0} \frac{1}{h^q}\sum_{j=0}^{\infty}(-1)^j \binom{q}{j} f(t-jh)$$

where $0 < q < 1$ and, t is the integration time, h is the time step.

The binomial coefficients can be written in terms of the Gamma function as:

$$\binom{q}{j} = \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)'}$$

The Riemann Liouville definition of fractional derivative is given as:

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_a^t \frac{f(T)}{(t-T)^{q-n+1}}\,dT$$

where $n-1 < q \le n$, $n \epsilon N$, for $q \epsilon (0,1)$.

The Caputo fractional derivatives can be written as:

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(T)}{(t-T)^{q-n+1}}\,dT$$

where $n-1 < q \le n$, $n \epsilon N$, for $q \epsilon (0,1)$. T is the integration variable.

Fractional-order mathematical models have limitless applications in various disciplines such as physics, chemistry, medicine, engineering, economics and many more (Boulaaras et al, 2023; Mitkowski et al, 2022; Valentim et al, 2021; Magin, 2010).

3.0 Description of difference synchronization scheme in fractional order systems

This section gives a generalized mathematical formulation of difference synchronization between two drives and one response fractional order system. Consider the following fractional order systems

$$D_t^{\propto^i} p_i = f_1(p) \quad (1)$$

$$D_t^{\propto^i} w_i = f_2(w) \quad (2)$$

$$D_t^{\propto^i} q_i = f_3(q_i) + U(p,w,q) \quad (3)$$

where $p = (p_1, p_2, , \dots, p_{ni})^T$; $w = (w_1, w_2, , \dots, w_{n2})^T$; $q = (q_1, q_2, , \dots, q_{n3})^T$ are the state variables of the drive systems and the response system. Furthermore, $f_1 : \mathbb{R}^{n1} \to \mathbb{R}^{n3}$; , $f_2 : \mathbb{R}^{n1} \to \mathbb{R}^{n3}$; , $f_3 : \mathbb{R}^{n1} \to \mathbb{R}^{n3}$ are the continuous state vector functions and $U(p,w,q) : \mathbb{R}^{n1} \times \mathbb{R}^{n2} \times \mathbb{R}^{n3} \to \mathbb{R}^{n3}$ are the control functions to be designed via the active control technique.

Definition: if there exit nonzero matrices A, B, C in equations (1), (2), and (3) respectively such that $\lim_{t\to\infty}\|Cq-(Bw-Ap)\|=0$ where $\|.\|$ represent the norm of the matrix

4. 0 Model description

Based on the proposed memristor and meminductor models, a simple chaotic circuit that contains only three- element model where the memristor and meminductor are in parallelwith a capacitor (Borah, 2021):

$$D_t^{\propto^1}x_1=-(ax_2^2-b)x_1-(\gamma x_4+\beta)x_3 D_t^{\alpha^2}x_2=-cx_1-dx_2+ex_1^2x_2$$
$$D_t^{\alpha^3}x_3=x_1$$
$$D_t^{\propto^4}x_4=x_3 \qquad (1)$$

where x_1 = the voltage across the capacitor.
x_2= the state variable of the memristor.
x_4= the state variable of the meminductor.

α = is the bifurcation parameter and in the range. α to [0.94, 1].

The chaotic phase attractor of the memristive-meminductive chaotic systems (1) is depicted in Fig. 1

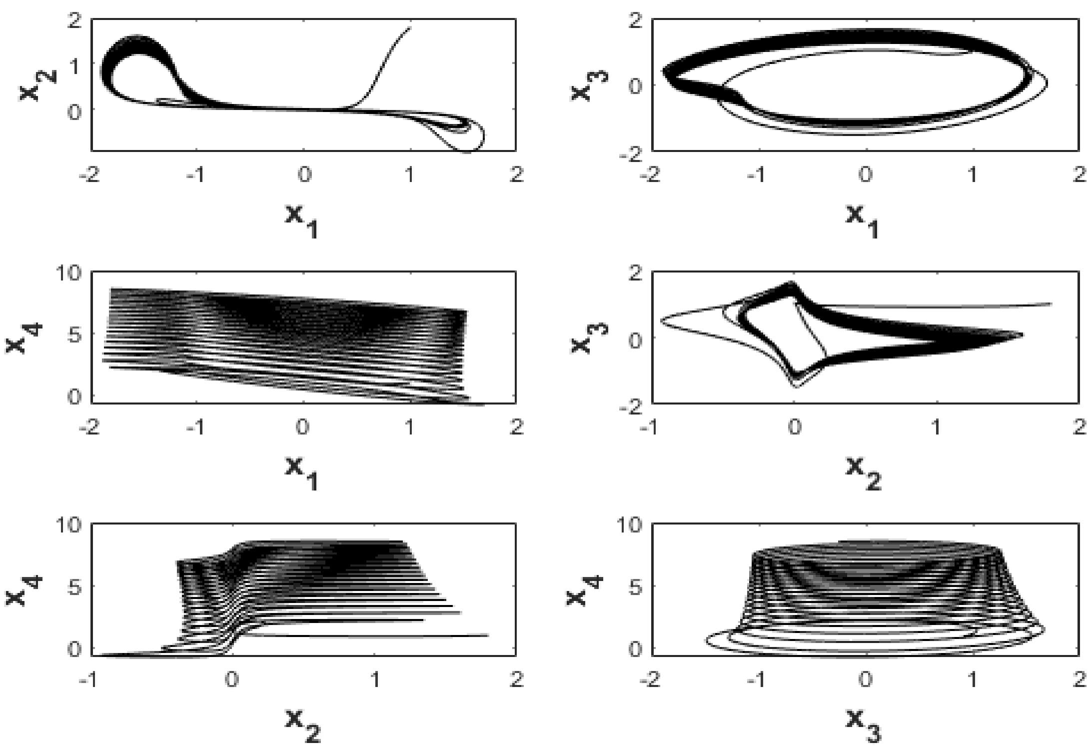

Fig. 1: Visualization display of phase portrait of the chaotic attractor of the memristive-meminductive chaotic systems using system parameter set (a = 0.1, b = 0.4, c = 0.2, d = 0.1, e = 4, γ = 0.1, β =3 with the initial conditions (1, 1.8, 1, 1).

5.0 Difference synchronization of four fractional order memristive-meminductive chaotic systems

The memristive-meminductive chaotic systems (2) and (3) are taken as the drive systems and the memristive-meminductive chaotic system (4) is taken as the response system to achieve difference synchronization.

$$D_t^{\propto^1} x_1 = -(ax_2^2 - b)x_1 - (\gamma x_4 + \beta)x_3$$
$$D_t^{\alpha^2} x_2 = -cx_1 - dx_2 + ex_1^2 x_2$$
$$D_t^{\alpha^3} x_3 = x_1$$
$$D_t^{\propto^4} x_4 = x_3 \quad (2)$$

$$D_t^{\propto^1} y_1 = -(ay_2^2 - b)y_1 - (\gamma y_4 + \beta)y_3$$
$$D_t^{\alpha^2} y_2 = -cy_1 - dy_2 + ey_1^2 y_2$$
$$D_t^{\alpha^3} y_3 = y_1$$
$$D_t^{\propto^4} y_4 = y_3 \quad (3)$$

$$D_t^{\propto^1} z_1 = -(az_2^2 - b)z_1 - (\gamma z_4 + \beta)z_3 + u_1$$
$$D_t^{\alpha^2} z_2 = -cz_1 - dz_2 + ez_1^2 z_2 + u_2$$
$$D_t^{\alpha^3} z_3 = z_1 + u_3$$
$$D_t^{\propto^4} z_4 = z_3 + u_4 \quad (4)$$

where u_1, u_2, u_3, and u_4 are the controllers to be designed.

The active control technique is utilized to realize the difference synchronization of these identical chaotic systems. The error system is defined as follows:

$$e_1 = (x_1 - y_1) - z_1$$
$$e_2 = (x_2 - y_2) - z_2$$
$$e_3 = (x_3 - y_3) - z_3$$
$$e_4 = (x_4 - y_4) - z_4 \quad (5)$$

Substituting equations (2)-(4) into fractional order time derivative of equation (5) yields the following result the error system defined in equation (6)-(9)

$$D_t^{\propto^1} e_1 = D_t^{\propto^1} x_1 - D_t^{\propto^1} y_1 - D_t^{\propto^1} z_1$$
$$D_t^{\propto^1} e_1 = be_1 - \beta e_3 - u_1 + f_1 \quad (6)$$

where, $f_1 = -ax_2^2 x_1 - \gamma x_4 x_3 + ay_2^2 y_1 + \gamma y_4 y_3 + az_2^2 z_1 + \gamma z_4 z_3$

$$D_t^{\propto^2} e_2 = D_t^{\propto^2} x_2 - D_t^{\propto^2} y_2 - D_t^{\propto^2} z_2$$
$$D_t^{\propto^2} e_2 = -ce_1 - de_2 - u_2 + f_2 \quad (7)$$

where $f_2 = ex_1^2 x_2 - ey_1^2 y_2 - ez_1^2 z_2$

$$D_t^{\propto^3} e_3 = D_t^{\propto^3} x_3 - D_t^{\propto^3} y_3 - D_t^{\propto^3} z_3$$
$$D_t^{\propto^3} e_3 = e_1 - u_3 \quad (8)$$
$$D_t^{\propto^4} e_4 = D_t^{\propto^4} x_3 - D_t^{\propto^4} y_3 - D_t^{\propto^4} z_3$$
$$D_t^{\propto^4} e_4 = e_3 - u_4 \quad (9)$$

In order to eliminate the nonlinear terms, the control functions are redefined as follows:

$$u_1 = f_1 - v_1$$
$$u_2 = f_2 - v_2 \quad (10)$$
$$u_3 = -v_3$$
$$u_4 = -v_4$$

where $V = [v_1, v_2, v_3, v_4]^T$ can be defined as $V = (\lambda I - B)e$. Now, λ is the eigenvalues, I is the identity matrix, and B is the coefficients matrix of the error state in equations (6)-(8) that are linear in e_1, e_2, e_3, e_4. So, V is defined as follows:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} b & 0 & -\beta & 0 \\ -c & -d & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \quad (11)$$

Substituting equations (10) into (11)

$$\begin{aligned} u_1 &= f_1 - (\lambda - b)e_1 - \beta e_3 \\ u_2 &= f_2 - ce_1 - (\lambda + d)e_2 \\ u_3 &= e_1 - \lambda e_3 \\ u_4 &= e_3 - \lambda e_4 \end{aligned} \quad (12)$$

To realize stable synchronization, eigenvalues λ is chosen such that the matrix in equation (10) is negative definite. Then, substitute equations (10) into (11) to obtain the desired control functions. The numerical results obtained from this analytical procedure is depicts in the Fig. 2 and 3 below.

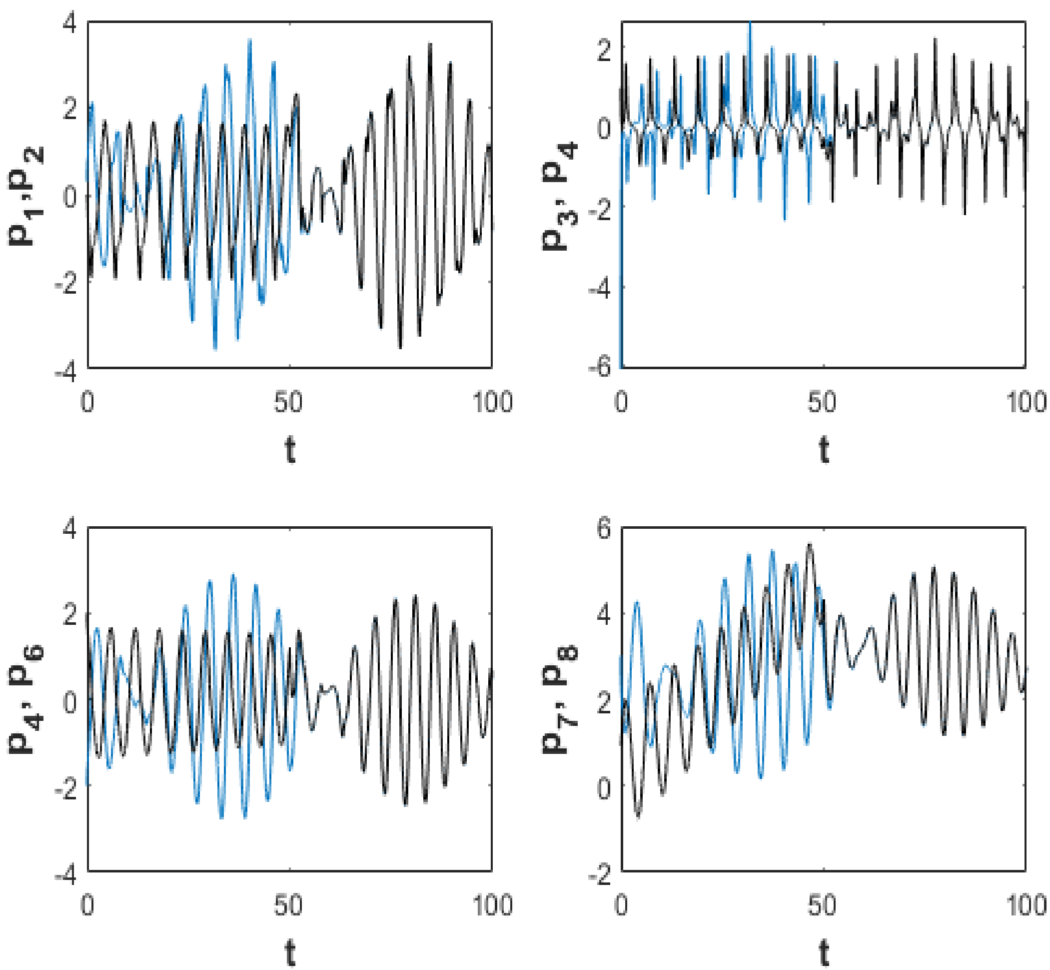

Fig. . 2: Dynamics of the state variables after the control function are applied at $t = 50$.

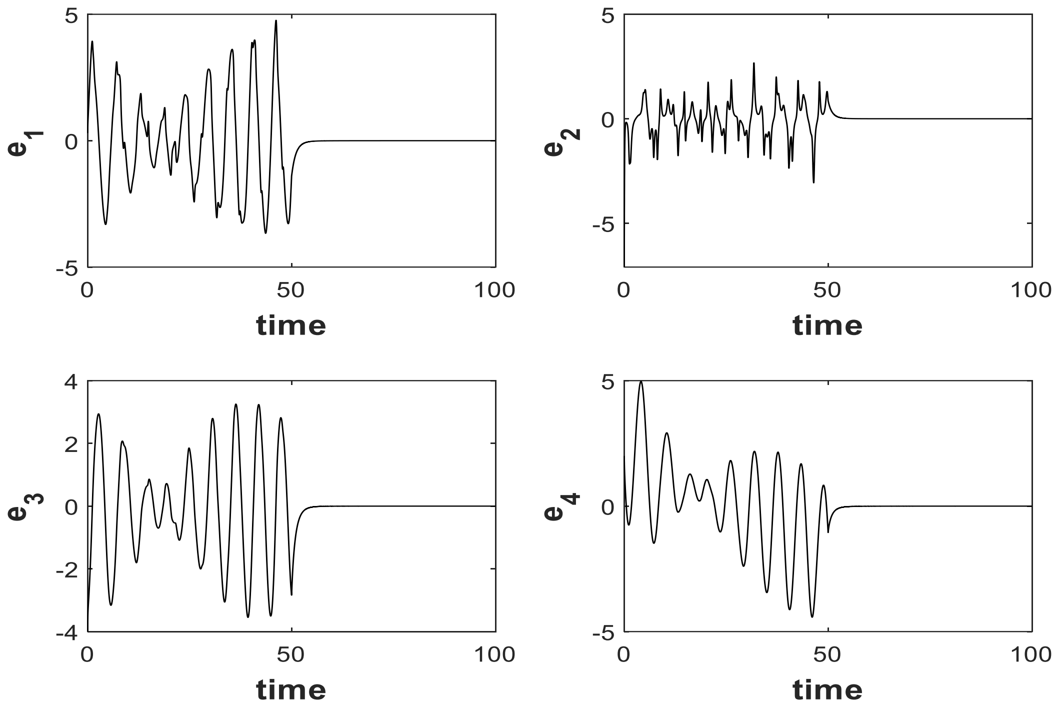

Fig. 3: Dynamics of the error variables when the control is applied at $t = 50$.

The difference synchronization scheme is a newly introduced scheme which has been used to synchronize three integer-order chaotic systems but has not been applied to fractional-order chaotic systems. In light of this, this research work applied the difference synchronization that was formerly developed for integer order nonlinear chaotic systems to fractional order systems and confirmed the effectiveness via numerical simulation. Having derived the analytical criteria via the active control, the analytical criteria were applied to three fractional order memristive-meminductive chaotic systems evolving from different initial conditions, where one is regarded as the drive system and one as the response system. The system parameters are as follow (a = 0.1, b = 0.4, c = 0.2, d = 0.1, e = 4, γ = 0.1, β =3) with the initial conditions (1,1.8, 1,1) and (2, -1,1,2). The numerical simulations result in Fig. 2 shows that the system followed different trajectories when the control functions were deactivated for $0 < t < 50$. However, when the control functions were activated at $50 < t < 100$, both the drive and response systems achieved identical dynamics which is clear evidence of synchronization. Further evidence of difference synchronization is depicted in Fig. 3 where the error dynamics moved chaotically when the control functions were deactivated for $0 < t < 50$ and then stabilized to zero when the control functions were activated for $50 < t < 100$. The analytical and numerical results confirm the achievement of difference synchronization in three fractional memristive-meminductive systems.

6.0 Conclusion

In conclusion, this research demonstrates the feasibility and effectiveness of achieving difference synchronization in memristive and meminductive fractional-order chaotic systems through active control techniques. By extending the concept of difference synchronization to fractional-order systems, this study contributes to the growing body of knowledge on synchronization in complex dynamical systems. The numerical simulations validate the proposed method, showcasing the ability to synchronize chaotic systems evolving from different initial conditions. This research underscores the importance of exploring synchronization phenomena in fractional-order systems and highlights the potential applications of such synchronization in various fields including secure communication and chaos-based cryptography.

In view of the above, we present the following recommendations

(i) **Experimental Validation**: Future research should focus on experimentally validating the proposed synchronization scheme in real memristive-meminductive circuits. Experimental validation will provide more robust evidence of the effectiveness of the proposed approach and its applicability in practical systems.

(ii) **Extension to Multi-System Synchronization**: Investigate the extension of the proposed difference synchronization scheme to synchronize multiple memristive-meminductive systems. Exploring synchronization among multiple chaotic systems can lead to insights into complex network dynamics and their applications in information processing and communication.

(iii) **Robustness Analysis**: Conduct robustness analysis of the synchronization scheme against parameter variations, noise, and external disturbances. Understanding the robustness of synchronization in fractional-order chaotic systems will enhance its reliability and applicability in real-world scenarios.

(iv) **Exploration of Applications**: Further explore potential applications of difference synchronization in fractional-order systems, particularly in secure communication, image encryption

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Compliance with Ethical Standards

Declarations:

The authors declare that they have no conflict of interest.

Data availability:

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable.

Availability of data and materials

The publisher has the right to make the data public.

Competing interests

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Authors' Contributions

AEA conceptualized the study and also partook in the development of the manuscript and review with other authors.