Communication in Physical Sciences, 2024, 11(3): 463-475

Convergence Analysis of Sinc-Collocation Scheme With Composite Trigonometric Function for Fredholm Integral Equations of the Second Kind

Eno John, Promise Asukwo and Nkem Ogbonna

Received: 24 January 2024/Accepted:06 June 2024/Published: 10 June 2024

Abstract: *The paper discusses the convergence of Sinc collocation scheme for the solution of Fredholm integral equation of the second kind. A modified composite trigonometric function is employed as a variable transformation function for this procedure. We first show that the constructed variable transformation function decays exponentially and thus satisfies the conditions for the error bound associated with single exponential transformation functions. Next, the convergence analysis of the scheme showing exponential convergence is discussed. Finally, some numerical examples are presented to illustrate the efficiency and stability of the numerical scheme.*

Keywords: *Fredholm integral equations of the second kind; Composite trigonometric function; Sinc approximation; Collocation method; Convergence analysis*

Eno John
Department of General Studies, Akwa Ibom State Polytechnic, Ikot Osurua
Email;; eno.john@akwaibompoly.edu.ng
Orcid id:

Promise Asukwo
Department of Statistics, Federal Polytechnic, Ukana
Email: promiseasukwo@gmail.com

Nkem Ogbonna
Department of Mathematics, Michael Okpara University of Agriculture, Umudike
Email: Ogbonna.n42@gmail.com

1.0 Introduction

The Fredholm integral equations of the second kind frequently arise in the study of many physical and engineering problems covering fields like quantum mechanics, fluid dynamics, and signal processing. Obtaining the solution of these integral equations efficiently and accurately is crucial for advancing theoretical and applied research. One of the recent and promising approaches to address this challenge is the Sinc-collocation method, which has garnered attention for its potential to deliver high precision with relatively low computational effort.

The Sinc-collocation method is developed using the combination of Sinc function and a variable transformation function such as composite trigonometric functions. The scheme leverages the properties of the Sinc function which is known for its excellent approximation capabilities. This procedure provides an efficient numerical method and offers a robust framework for tackling Fredholm integral equations of the second kind.

Recently, there has been a steady and growing interest in the study of integral equations based on collocation methods. The efficiency of Sinc methods in approximating functions with rapid convergence rates was highlighted by Stenger (2011). This has also served as the basis for the development of collocation schemes with different variable transformation functions and the rise of new research with a focus on collocation points Okayama (2023). New studies have explored the enhancements achievable through composite trigonometric functions John et al (2024), it was also shown to be a flexible and powerful means to represent complex periodic behaviours inherent in many physical systems (Wei & Yang, 2019). Furthermore, the expansion of the scheme into other research areas as witnessed in Zabihi, F. (2024) is promising, thus pointing to a greater involvement of the scheme in the newer area of science and engineering.

The convergence analysis of the Sinc-collocation scheme based on composite trigonometric functions, is a critical area of study. It provides the theoretical and practical validation for the method by showing that the numerical solutions converge to the exact solution with the increase in a number of collocation points. In earlier research, the convergence of the collocation scheme was considered by Maleknejad et al 2011 for Fredholm Integral equations of the second kind based on the tanh function. Both Zarebnia & Rashidinia (2010) and John (2016) studied the convergence of the collocation scheme for Volterra integral equations of the second kind with the tanh function employed as a variable transformation function. Zhang et al. (2022) in their recent investigations have demonstrated the potential of this approach, he also highlighted its applicability to a wide range of problems with varying degrees of complexity.

This paper discusses the convergence properties of the Sinc-collocation scheme with composite trigonometric functions for solving Fredholm integral equations of the second kind. We aim to contribute to the broader understanding and application of these numerical techniques in solving integral equations by building on recent advancements and providing a rigorous but comprehensible analysis of the scheme.

In this paper, the convergence of the collocation scheme

$$\{w_a(x_k) - K_F[w_a](x_k)\}u_{-N-1} + \sum_{j=-N}^{N} \delta_{kj} - hk\left(x_k, t_j\right)\varphi'(jh)u_j + \{w_b(x_k) - K_F[w_b](x_k)\}u_{N+1} = g(x_k) \quad (1)$$

for the approximate solution u of Fredholm linear integral equation of the second kind

$$u(x) = \lambda \int_a^b k(x,t)u(t)dt + g(x), \; a \le x \le b \quad (2)$$

was considered; where $k(x,t)$, $g(x)$ are smooth functions and λ is a scalar.

We will employ in this work a composite trigonometric function, John et al (2024)

$$x = \varphi(t) = sin\left(arctan\left(\frac{e^t}{\sqrt{\alpha}}\right)\right) = \frac{1}{\sqrt{1+\alpha e^{-2t}}}, t \in (-\infty, \infty) \quad (3)$$

as a variable transformation function in the Sinc collocation scheme for the solution of linear Fredholm integral equations of the second kind (2).

2.0 Preliminaries

2.1 Approximation on the Real Line

Let the trapezoidal rule for approximation on the real $\mathbb{R}$ be defined by

$$T_h = h\sum_{j=-\infty}^{\infty} F(jh), h > 0 \quad (4)$$

then for the integral

$$I = \int_{-\infty}^{\infty} F(u)du \quad (5)$$

of the function F(u) is defined on $\mathbb{R}$ as

$$F(u) \approx \sum_{j=-N}^{N} F(jh)S(j,h)(u), \quad u \in \mathbb{R} \quad (6)$$

we have

$$\int_{-\infty}^{\infty} F(u)du \approx \sum_{j=-N}^{N} F(jh)\int_{-\infty}^{\infty} S(j,h)(u)du = h\sum_{j=-N}^{N} F(jh). \qquad (7)$$

The function $S(j,h)(t)$ in (7) is known as Sinc function Stenger (1993) and defined by the formula

$$S(j,h)(t) = S\left(\frac{t}{h} - j\right) = \frac{sin\pi(\frac{t}{h} - j)}{\pi(\frac{t}{h} - j)}, j = 0, \pm 1, \pm 2, \ldots \qquad (8)$$

Furthermore, given that $t_k = kh$,

$$S(j,h)(kh) = \begin{cases} 0, k \neq j \\ 1, k = j \end{cases}.$$

2.2 Approximation on a Finite Interval $\Lambda = (a, b)$

On the finite interval (a, b), equation (3) has the representation

$$x = \varphi(t) = a + \frac{b-a}{\sqrt{1+\alpha e^{-2t}}}, \qquad (9)$$

and satisfies the map $\varphi(-\infty, \infty) \to (a, b)$, with

$$x' = \varphi'(t) = \frac{(b-a)\alpha e^{-2t}}{(1+\alpha e^{-2t})^{3/2}} \qquad (10)$$

and

$$t = \varphi^{-1}(x) = \frac{1}{2}log\left(\frac{\alpha(x-a)^2}{(b+x-2a)(b-x)}\right). \qquad (11)$$

Considering equations (4) and (7) above, the integral

$$I = \int_a^b f(x)dx = \int_{-\infty}^{\infty} f(\varphi(t)\varphi'(t)dt \approx h\sum_{j=-N}^{N} f\big(\varphi(jh)\big)\varphi'(jh). \qquad (12)$$

Also, from (10), we note that

$$\varphi'(t) = \frac{(b-a)\alpha e^{-2t}}{(1+\alpha e^{-2t})^{3/2}} = O(\exp(-2(1-\epsilon)t)) \text{ as } t \to \infty. \qquad (13)$$

$0 < \epsilon < 1$.

Following Mori & Mohammad (2003), we assume that $f(x)$ satisfies

$$f(x) = \begin{cases} O((x)^{\alpha-1}) \; as \; x \to 0 \\ O((1-x)^{\alpha-1}) \; as \; x \to 1 \end{cases} \text{for } \alpha > 0, \qquad (14)$$

and from (13), we have,

$$f\big(\varphi(t)\big)\varphi'(t) = O(\exp(-2(\alpha-\epsilon)|t|)) \text{ as } t \to \pm\infty \qquad (15)$$

Definition 2.1

A function f is said to decay single exponentially with respect to a conformal map φ if there exist positive constants α and C such that

$$|f(\varphi(t))\varphi'(t)| \leq Cexp(-\alpha|t|)$$

for all $t \in \mathbb{R}$ and scalar α.

Hence, we consider the variable transformation function $\varphi(t)$ to be a single exponential function.

2.3 Convergence Theorems for Single Exponential Sinc Approximation

Definition 2.1

Let $d > 0$, then D_d denotes a strip region of width $2d$ defined by $D_d = \{z \in C: |im\, z| < d\}$.

Definition 2.2 Okayama *et al* (2011)

Let α be a positive constant, and let D be a bounded and simply connected domain which satisfies $(a, b) \subset D$. Then L_α(D) denotes the family of functions f that satisfy the following conditions: (i) f is analytic in D; (ii) there exists a constant C such that for all z in D

$$|f(z)| \leq C|Q(z)|^\alpha \tag{16}$$

where the function Q is defined by

$$Q(z) = (z - a)(b - z).$$

For the implementation of the single exponential transformation in the above Definition 2.2, the domain D considered to be the region $\varphi(D_d) = \{z = \varphi(\mu): \mu \in D_d\}$ such that

$$\varphi(D_d) = \left\{z \in C: \left|\arg \frac{1}{2} log\left(\frac{\alpha(x-a)^2}{(b+x-2a)(b-x)}\right)\right| < d\right\}. \tag{17}$$

Theorem 1 Stenger (1993)

Let $f \in L_\alpha\varphi(D_d)$ for d with $0 < d < \pi$, let N be a positive integer and let h be selected by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}}$$

then there is a constant C independent of N, such that

$$\max_{a \leq x \leq b}\left|f(x) - \sum_{j=-N}^{N} f(\varphi(jh)S(j,h)(\{\varphi\}^{-1}(x))\right| \leq C\sqrt{N}e^{-\sqrt{\pi d\alpha N}} . \tag{18}$$

The choice of h is optimal and satisfies (18) based on Sugihara (2002).

Theorem 2 Okayama *et al* (2011)

Let $(fQ) \in L_\alpha\varphi(D_d)$ for d with $0 < d < \pi$, let N be a positive integer and let h be selected by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}}$$

then there is a constant C independent of N, such that

$$\left|\int_a^b f(t)dt - h\sum_{j=-N}^{N} f(\varphi(jh)S(j,h)(\{\varphi\}'(jh))\right| \leq Ce^{-\sqrt{\pi d\alpha N}} \tag{19}$$

Definition 2.3 Okayama *et al* (2011)

Let D be a bounded and simply connected domain, then we denote by $HC(D)$ the family of all functions that are analytic in D and continuous in $\overline{D}$. The function space is complete with the norm $||\cdot||_{HC(D)}$ defined by

$$\|f\|_{HC(D)} = \max_{z\in\overline{D}}|f(z)|. \tag{20}$$

Definition 2.4 Stenger (2000)
Let α be a constant with $0 < \alpha < 1$ and let D be a bounded and simply connected domain such that $(a,b) \subset D$. Then the space $M_\alpha(D)$ consists of all functions f that satisfies the following conditions

(i) $f \in HC(D)$;
(ii) there exists a constant C for all z in D such that

for

$$\rho(z) = \exp\,(\varphi^{-1}(z),$$

$$|f(z) - f(a)| = O(|\rho(z)|^\alpha)\ as\ z \to a,$$
$$|f(z) - f(b)| = O(|\rho(z)|^\alpha)\ as\ z \to b. \tag{21}$$

According to Stenger (1993), the translated function

$$T[f](x) = f(x) - \frac{f(a) + \rho(x)f(b)}{1 + \rho(x)} \in L_\alpha(D) \tag{22}$$

if $f \in M_\alpha(D)$ and the approximation

$$T[f](x) \approx \sum_{j=-N}^{N} T[f](\varphi(jh)S(j,h)(\varphi^{-1}(x)). \tag{23}$$

Combining equations (22) and (23),

$$f(x) \approx P_N[f](x) = \sum_{j=-N}^{N} T[f](\varphi(jh)S(j,h)(\varphi^{-1}(x)) + \frac{f(a) + \rho(x)f(b)}{1 + \rho(x)}. \tag{24}$$

Let the auxiliary basis function be defined by

$$\omega_a(x) = \frac{1}{1 + \rho(x)}, \omega_b(x) = \frac{\rho(x)}{1 + \rho(x)}, \tag{25}$$

then the generalized approximation to $f(x)$ is of the form

$$P_N[f](x) = f(a)\tau_a(x) + \sum_{j=-N}^{N} T[f](\varphi(jh)S(j,h)(\varphi^{-1}(x)) + f(b)\tau_b(x). \tag{26}$$

Theorem 3
Let $f \in M_\alpha(\varphi(D_d))$for d with $0 < d < \pi$, let N be a positive integer and h defined as

$$h = \sqrt{\frac{\pi d}{\alpha N}}$$

then there exists a constant C independent of N such that

$$\|f - P_N f\|_{C([a,b])} \le C\sqrt{N}e^{-\sqrt{\pi d\alpha N}} \tag{27}$$

Remark: this condition holds since φ is a single exponential function.

3.0 The Sinc Collocation Method
3.1 Construction of the Sinc Collocation Scheme

Let $u(x) \in M_\alpha(\varphi(D))$ and $u_N(x)$ be the exact and approximate solution of (1), while $u(x_j)$ and u_j are the exact and approximate solutions at a sinc point x_j respectively.

According to Okayama *et al* (2011),

$$u(x) = u(x_{-N-1})w_a(x) + \sum_{j=-N}^{N} u(x_j)S(j,h)(\{\varphi\}^{-1}(x)) + u(x_{N+1})w_b(x) \tag{28}$$

will satisfy (1) at Sinc points of φ, since it is a linear combination of the sinc functions $S(j,h)(t)$ and the auxiliary basis functions $\omega_a(x)$ and $\omega_b(x)$. Here, the basis functions are considered to be fixed and the collocation points are defined as

$$x_i = \begin{cases} a, & i = -N-1 \\ \varphi(ih), & i = -N \dots N \\ b, & i = N+1. \end{cases} \tag{29}$$

With the collocation points defined as above, we set the approximate solution u of (2) as

$$u_N(x) = u_{-N-1}w_a(x) + \sum_{j=-N}^{N} u_j S(j,h)(\{\varphi\}^{-1}(x)) + u_{N+1}w_b(x) \tag{30}$$

and the integral in (2) becomes

$$\int_a^b k(x,t)u(t)dt$$
$$= K_N[w_a](x)u_{-N-1} + h\sum_{j=-N}^{N} k(x,t_j)\varphi'(jh)u_j + K_N[w_b](x)u_{N+1} + Oe^{-\frac{\pi d}{h}} \tag{31}$$

Noting that $S(j,h)(\{\phi\}^{-1}(x_k) = S(j,h)(\{\phi\}^{-1}(\varphi(kh)) = S(j,h)(kh) = \delta_{kj}$.
with

$$K_N[f](x) = h\sum_{j=-N}^{N} k(x,t_j)f(x_j)\varphi(jh). \tag{32}$$

Using (30) - (32) in (2), we obtain the collocation formula (1) is obtained as $(2N+3) \times (2N+3)$ system of linear equations,

$$(E_n - K_n)u_n = g_n \tag{33}$$

in compact form
with

$$E_n = w_a(x_k) + \sum_{j=-N}^{N} S(j,h)(\{\varphi\}^{-1}(x_k)) + w_b(x_k)$$

$$K_n = K_N[w_a](x_k) + h\sum_{j=-N}^{N} k(x_k,t_j)\varphi'(jh) + K_N[w_b](x_k)$$

$$g_n = [g(a), g(x_{-N}), \dots, g(x_N), g(b)]^T$$

$$u_n = [u_{-N-1}, u_{-N}, \dots u_N, u_{N+1}]^T.$$

By solving the above system of equations for u_n and using the result in (30), we obtain the approximate solution $u_N(x)$ to (2).

3.2 Convergence of Sinc Collocation Method Fredholm Integral Equations of Second Kind

Let $[a, b]$ be a finite interval and $\|.\|_{C[a,b]}$ defines a norm in the interval. Also, let $g, kQ \in C[a, b]$ for $x, t \in [a, b]$. Let $\varphi, \varphi^{-1}(t)$ be defined as in (3) and let N be a positive integer.
We will give the following Lemma.

Lemma 3.1

Let $u(x)$ represent the exact of (2), and let $k(x, .)Q(.) \in C[a, b], k(., t)Q(t) \in C[a, b]$ and $g \in C[a, b]$ for all $x, t \in [a, b]$. If $u(\varphi(t))$ is analytic in the domain D_d, then there exists a constant C_d independent of N such that

$$\|(E_n - K_n)\bar{u}_n - g_n\| \leq C_d \exp -\sqrt{\pi d \alpha N} \tag{34}$$

where

$$h = \sqrt{\frac{\pi d}{\alpha N}}$$

and

$$\bar{u}_n = (u(x_{-N-1}), \dots, u(x_{N+1}))^T \tag{35}$$

with

$u(x_j) and\ u_j$ being the exact and approximate solution of (2) respectively at the collocation points $x_j = \varphi(jh)$.

Proof:

From (28) and (1), let the rth component of the vector $(E_n - K_n)\bar{u}_n - g_n$be given by

$$|r_k| = \|[(E_n - K_n)\bar{u}_n - g_n]\| \tag{36}$$

$$= |u(x_{-N-1})w_a(x_k) + \sum_{j=-N}^{N} u(x_j)S(j,h)(\{\varphi\}^{-1}(x_k)) + u(x_{N+1})w_b(x_k)$$

$$-\{K_N[w_a](x_k)u(x_{-N-1}) + h\sum_{j=-N}^{N} k(x_k, t_j)\varphi'(jh)u(t_j) + K_N[w_b](x_k)u(x_{N+1})\} - g_n(x_k)|$$

$$\leq \left|K_N[w_a](x_k)u(x_{-N-1}) + h\sum_{j=-N}^{N} k(x_k, t_j)\varphi'(jh)u(t_j) + K_N[w_b](x_k)u(x_{N+1})\right|$$

$$+ Oe^{-\frac{\pi d}{h}}$$

$$\leq C \exp\left(-\pi d. \sqrt{\alpha N/_{\pi d}}\right)$$

$$= C_d \exp(-\sqrt{\pi d \alpha N})$$

Thus,

$$\|(E_n - V_n)\bar{u}_n - g_n\| = \left(\sum_{k=-N}^{N} |r_k|^2\right)^{1/2}$$

$$\leq C_d \exp -\sqrt{\pi d \alpha N}. \tag{37}$$

Using Lemma 3.1, the bound on the difference $u(x) - u_N(x)$ of exact solution and the approximate solution can be estimated using the max norm. This is demonstrated in the theorem below.

Theorem 1

Assuming the conditions of Lemma 3.1 is satisfied, let $u(x)$ *and* $u_N(x)$ be the exact solution and approximate solution of (2) respectively, then for h and N satisfying (35), there exist constants C_2 and C_3 independent of N such that

$$\max_{x\in(a,b)} |u(x) - u_N(x)| \leq (C_2 + A_N C_3) \exp -\sqrt{\pi d\alpha N} \quad (38)$$

Proof:

Let us first define the analytic solution $P_N(x)$ of equation (2) at Sinc points $x = x_j$. We have from (28) and (2),

$$\{w_a(x_k) - K_N[w_a](x_k)\}u(x_{-N-1}) + \sum_{j=-N}^{N} \delta_{kj} - hk(x_k, t_j)\,\varphi'(jh)u(t_j) + \{w_b(x_k) - K_N[w_b](x_k)\}u(x_{N+1}) = g(x_k) \quad (39)$$

Now, by triangle inequality,

$$|u(x) - u_N(x)| \leq |u(x) - P_N(x)| + |P_N(x) - u_N(x)| \quad (40)$$

By Lemma 3.1 and equation (34)

$$|u(x) - P_N(x)| \leq \left| V_N[w_a](x_k)u(x_{-N-1}) + h\sum_{j=-N}^{N} k(x_k, t_j)\varphi'(jh)u(t_j) + V_N[w_b](x_k)u(x_{N+1}) \right| + Oe^{-\frac{\pi d}{h}} \leq C_2 \exp -\sqrt{\pi d\alpha N} \quad (41)$$

by (37).

For the second term on the right of (40), we have,

$$|P_N(x) - u_N(x)| = |\{w_a(x_k) - K_N[w_a](x_k)\}u(x_{-N-1}) + \sum_{j=-N}^{N} \delta_{kj} - hk(x_k, t_j)\,\varphi'(jh)u(t_j) + \{w_b(x_k) - K_N[w_b](x_k)\}u(x_{N+1})$$
$$-\{\{w_a(x_k) - K_N[w_a](x_k)\}u_{-N-1} + \sum_{j=-N}^{N} \delta_{kj} - hk(x_k, t_j)\varphi'(jh)u_j + \{w_b(x_k) - K_N[w_b](x_k)\}u_{N+1}\}|$$
$$\leq \sum_{j=-N-1}^{N+1} |hk(x_k, t_j)\varphi'||u(t_j) - u_j|.$$

Assuming that for $M \in R^+$,

$$h\left(\left|\sum_{j=-N-1}^{N+1} |k(x_k, t_j)\varphi'(jhJ(j,h)(x_k)|\right|^2\right)^{1/2} \leq M \quad (42)$$

holds uniformly for $x \in (a, b)$ based on the assumptions of the kernel (Lemma 3.1), then by Schwartz inequality,

$$|P_N(x) - u_N(x)| \leq h\left(\left|\sum_{j=-N-1}^{N+1} |k(x_k, t_j)\varphi'(jh)|\right|^2\right)^{1/2}\left(\left|\sum_{j=-N-1}^{N+1} |u(t_j) - u_j|\right|^2\right)^{1/2}$$

$$\leq M\|\bar{u}_n - u_n\|.$$

We note that by (33),

$$u_n = (E_n - K_n)^{-1} g_n.$$

Hence,

$$\|\bar{u}_n - u_n\| = \|\bar{u}_n - (E_n - K_n)^{-1} g_n\| \leq \|(E_n - K_n)^{-1}\| \|(E_n - K_n)\bar{u}_n - g_n\|$$

$$\leq C_3 A_N \exp -\sqrt{\pi d \alpha N} \qquad (43)$$

where $A_N = \|(E_n - K_n)^{-1}\|$ and C_3 is independent of N.

Combining (41) and (42), we have,

$$\max_{x \in (a,b)} |u(x) - u_N(x)| \leq (C_2 + A_N C_3) \exp -\sqrt{\pi d \alpha N} \qquad (44)$$

which completes the proof.

4.0 Numerical Results

We present some examples in this section to illustrate the implementation of the Sinc collocation scheme (1). In the following calculations, $\lambda = 1$ from (2) and following Sugihara (2002), $\alpha = 1, d = \frac{\pi}{2}$ with the step size

$$h = \sqrt{\frac{\pi d}{\alpha N}} = \frac{\pi}{\sqrt{2N}}.$$

The maximum absolute error between the exact solution $u(x)$and the approximate solution $u_N(x)$ at sinc points x_k given by $|E_N(h(\varphi))|$ concerning L_∞ norm is given by

$$|E_N(h(\varphi))| = \max_{k = -N-1, -N, \dots N, N+1} |u(x_k) - u_N(x_k)|. \qquad (45)$$

The numerical stability of the scheme was monitored using the condition number $\kappa(Z)$ of the coefficient matrix $Z = E_n - K_n$ of the system (36) based on infinity norm and defined by

$$\kappa(Z) = \|Z\|_\infty \|Z^{-1}\|_\infty. \qquad (46)$$

The computations were carried out using MATLAB® software.

Example 1

$$u(x) = -\int_0^1 e^{xt} u(t) dt + x e^x + \frac{x e^{x+1} + 1}{(x+1)^2}$$

with exact solution $u(x) = xe^x$, Wazwaz (2011).

Example 2

$$u(x) = \int_0^1 x \tan^{-1} t\, u(t) dt + \frac{1}{1+x^2} - \frac{\pi^2}{32} x$$

with

$$\frac{1}{1+x^2}$$

as exact solution, Wazwaz (2011).

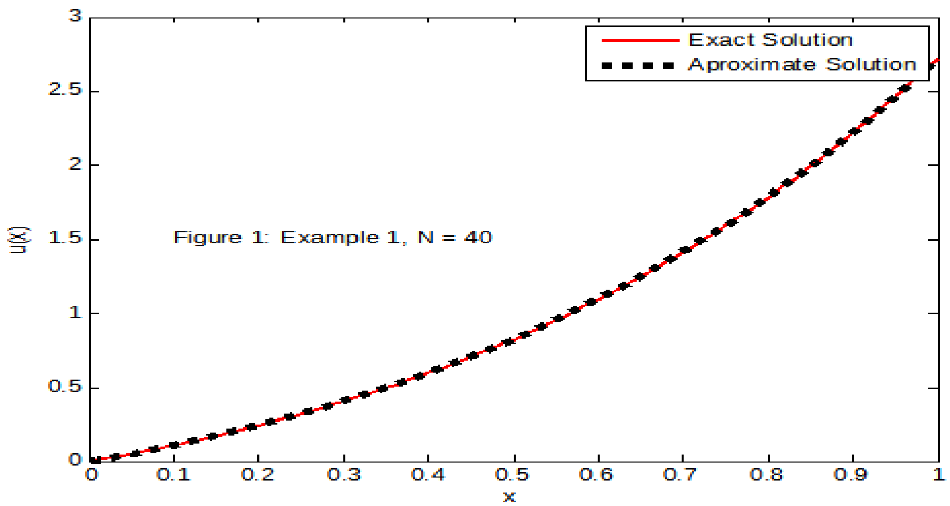
Table 1: Maximum error and condition number for Example 1

N	$\lvert E_N(h(\varphi))\rvert$	$\kappa(Z)$
10	1.2713×10^{-4}	11.3×10^{-0}
20	8.3677×10^{-7}	11.3×10^{-0}
30	4.5933×10^{-9}	$11.\ 3\times 10^{-0}$

40	5.0328×10^{-10}	11.3×10^{-0}
50	1.3064×10^{-11}	11.3×10^{-0}

Table 2: Maximum error and condition number for Example 2

N	$\lvert E_N(h(\varphi))\rvert$	$\kappa(Z)$
10	1.3232×10^{-5}	4.96×10^{-0}
20	3.5035×10^{-8}	4.97×10^{-0}
30	1.4070×10^{-10}	4.97×10^{-0}
40	7.7305×10^{-13}	4.97×10^{-0}
50	1.3178×10^{-13}	4.97×10^{-0}



Fif. 1: Exact and approximate solution for Example 1

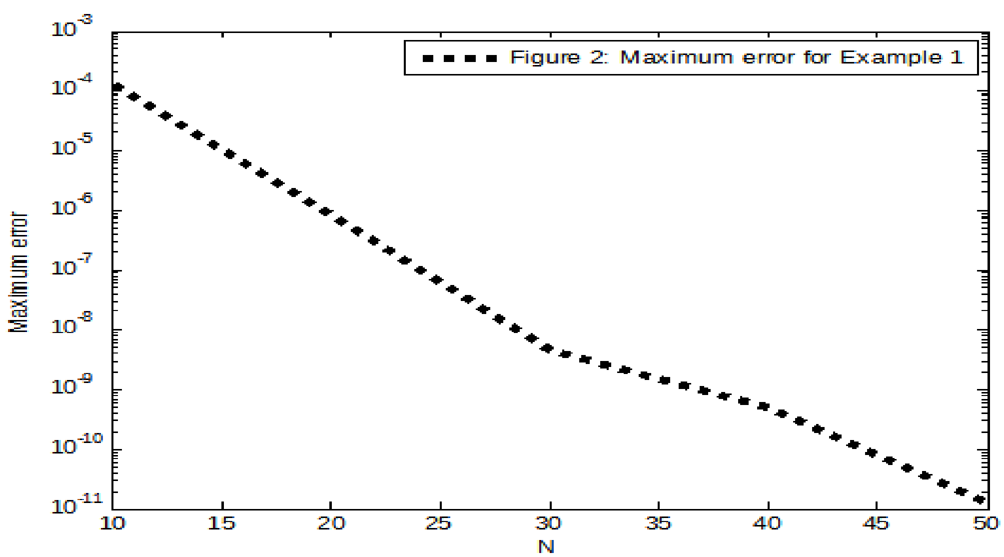


Fig. 2: Maximum absolute error for Example 1

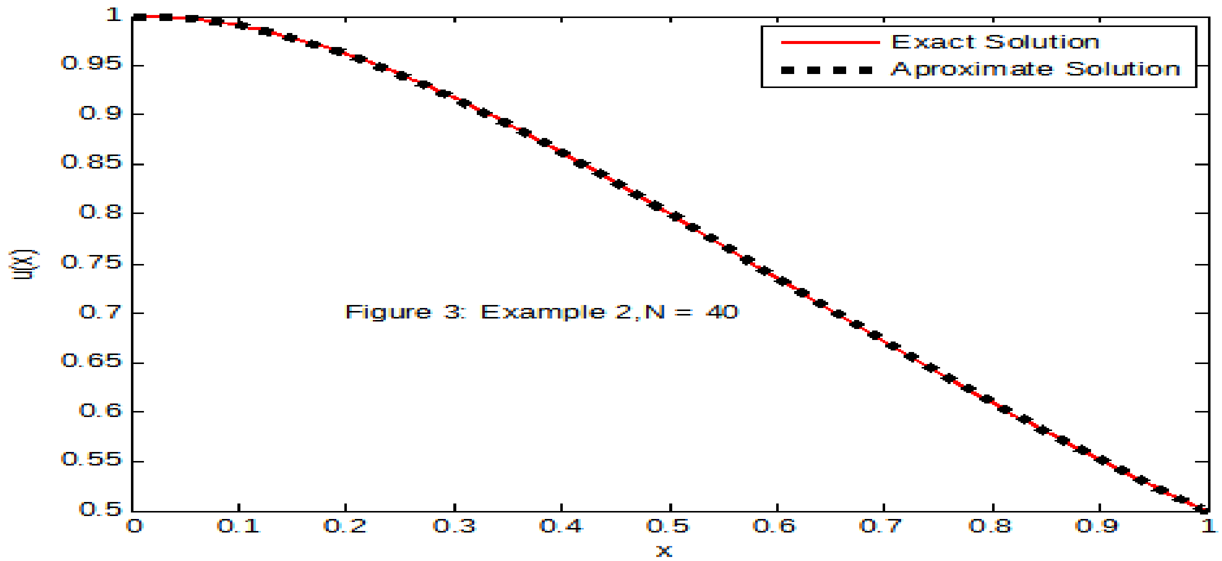


Fig. 3: Exact and approximate solution for example 2

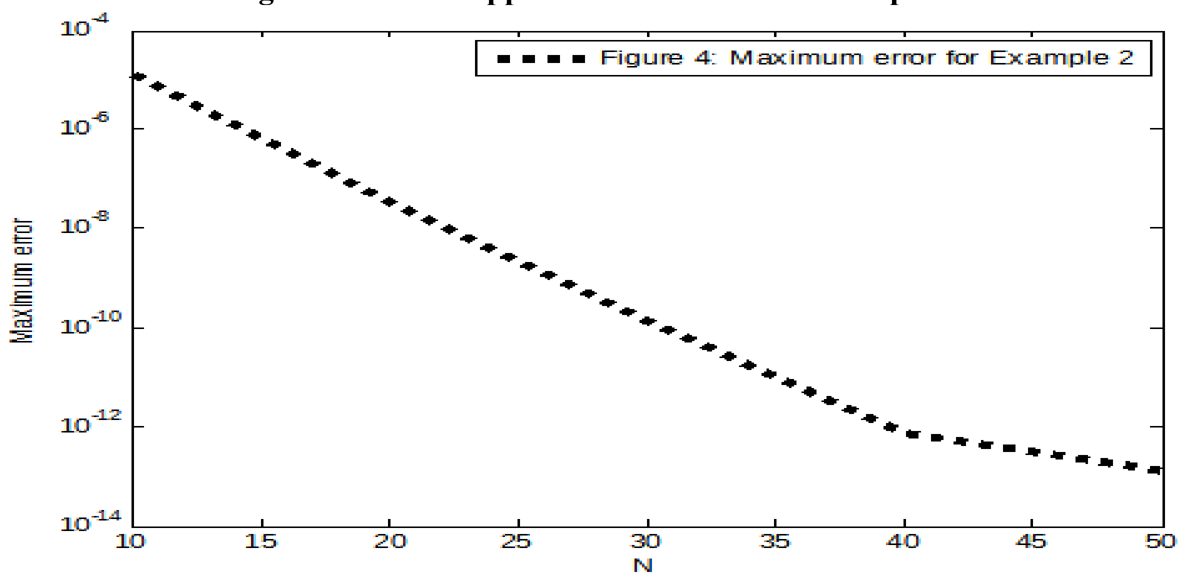


Fig. 4: Maximum absolute error for example 2

5.0 Conclusion

Our work in this paper was to demonstrate the convergence of Sinc collocation method with composite trigonometric function for the solution of Fredholm integral equations of the second kind. The theoretical results showed improved convergence of where (N) is the number of function evaluations. The research provided a rigorous convergence analysis with the help of recent advancements in that field of study which also contributed to a deeper understanding and application of the scheme.

The foundation of the analysis relied on the already established results of single exponential Sinc approximation using collocation points. The theoretical result of the scheme as seen in the formulated theorem showed that the error between the exact solution and approximate solution converges exponentially in the order $O(\exp -C\sqrt{N}$ concerning the increase in collocation points (N) and a constant C under specific conditions.

Furthermore, the results of the numerical examples as seen in Tables 1 and 2 as well as Figs. 1 to 4 showcased the efficiency of the method by demonstrating a decrease in the maximum absolute error between the exact and approximate solutions as the number of collocation points increased. The stability of the scheme was also studied by monitoring the condition numbers of the coefficient matrix of the linear system of equations arising from the collocation scheme. This is also reported in Tables 1 and Table 2 of the results for numerical illustrations.

In summary, a theoretical framework and numerical illustrations to support the convergence and effectiveness of the Sinc-collocation method with composite trigonometric functions for solving Fredholm integral equations of the second kind was presented in this work. The efficiency of the method is demonstrated to encourage researchers seeking to apply the method to similar equations.

In subsequent work, we hope to extend the analysis to consider a broader class of Fredholm integral equations and also explore adaptive collocation strategies to optimize the convergence rate; an investigation into the computational cost of the method as compared to alternative approaches would to give valuable insights for practical applications will also be considered.

6.0 References

John, E, Promise, A. & Nkem. O. (2024). Solution of Fredholm integral equations of second kind using a composite trigonometric function, GPH *International Journal of Mathematics*, 7, 3, pp. 108 - 121.

John, E. D. (2016). Analysis of convergence of the solution of Volterra integral equations by Sinc collocation method, *Journal of Chemical, Mechanical and Engineering Practice, International Perspective*, 6, 1, 2, pp. 16 – 27.

Maleknejad, K., Mollopourasl, M. & Alizadeh, M. (2011). Convergence analysis numerical solution of Fredholm integral equation by Sinc approximations, *Communications in Nonlinear Science and Numerical Simulation*, 16, 6, pp. 2478-2485.

Muhammad, M. & Mori, M. (2003). Double exponential formulas for numerical indefinite integration, *Journal of Computational and Applied Mathematics*, 161, pp. 431-448.

Okayama, T. (2023). Sinc collocation method with constant collocation points for Fredholm integral equation of the second kind, Dolomite Research *Notes on Approximation,* 16(, 3, pp. 67 – 74. doi: 10.14658/PUPJ-DRNA-2023-3-9

Okayama, T., Matsuo, T. & Sugihara, M. (2011). Improvement of Sinc- collocation method for Fredholm integral equations of the second kind, *BIT Numer. Math.* 51, pp.339-366.

Stenger, F. (2011). *Handbook of Sinc Numerical Methods*. CRC Press.

Stenger, F. (1993). *Numerical methods based on Sinc and analytic functions*, Springer-Verlag, New York.

Sugihara, M. (2002). Near optimality of Sinc approximations. *Mathematics of Computation*, 72, 242, pp. 767-786.

Wazwaz, A. (2011). *Linear and nonlinear integral equations, methods and applications*, Higher education press, Beijing and Springer-Verlag Heidelberg.

Wei, J., & Yang, L. (2019). Numerical solution of integral equations using composite trigonometric functions. *Journal of Computational and Applied Mathematics*, 350,, pp. 12-123.

Zabihi, F. (2024). The use of Sinc-collocation method for solving steady–state concentrations of carbon dioxide absorbed into phenyl glycidyl ether. *Computational*

Methods for Differential Equations, . doi: 10.22034/cmde.2024.55413.2304.

Zarebnia, M. and Rashidinia, J. (2010). Convergence of the Sinc method applied to Volterra integral equations, Applications and Applied Mathematics, 5, 1, pp. 198-216.

Zhang, X., Li, H., & Sun, J. (2022). Improved Sinc-collocation methods for integral equations with applications. *Applied Numerical Mathematics*, 175, pp. 233-247.

Compliance with Ethical Standards Declarations

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public

Availability of data and materials

The publisher has the right to make the data public.

Competing interests

The authors declared no conflict of interest.

Funding

The authors declared no source of funding.

Authors' contributions

Eno John: Article developer and corresponding author. Promise Asukwo: Data checking and proofreading; Ogbonna Nkem: Supervisor