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On Investment Model for a CARA Pension Scheme Member with Return of Contributions Clause for Mortgage Housing Scheme

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Abstract: *This paper investigates a pension scheme member's (PSM) portfolio in a defined contributory (DC) pension plan with a return of premium for mortgage scheme and charge on balance(CB) for time-consistent utility. A portfolio with one risk-free asset and two risky assets which are modelled using the geometric Brownian motion (GBM) process is considered, such that the instantaneous volatilities of the risky assets form a 2×2 positive definite matrix $m = \{m_{ab}\}_{2X2.}$ with mm^T. The PSMs interested in the mortgage housing scheme are modelled using Abraham De Moivre's force function, and the optimization problem is obtained using a dynamic programming approach. Using Legendre transformation and dual theory with variable change technique, the optimal value function (OVF), and optimal investment distribution (OID) are obtained for a PSM with utility function exhibiting constant absolute risk averse (CARA). Furthermore, some numerical analyses were carried out and was observed that factors such as risk-free interest rate, risk-averse coefficient, entry age of PSM, initial wealth and CB were critical in developing an OID with this kind of return clause.*

Keywords*: Optimal investment distribution, pension scheme member, Ito's lemma, Mortgage Housing Scheme, Return Clause,*

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1.0 Introduction

The pension scheme before 2004, was largely subject to dispute because of the cumbersome nature of administering and management of retirement benefits and was entirely analog. There were usually long queues for the verification of pensioners due to the procedure used which was the manual way of computation of members' benefits and requests for old files, documents, and credentials. However, pension reform has made the scheme seamless with the help of technology. This has helped instil sincerity and stimulated hope in the scheme. According to (Antolin *et al.*, 2010), the function of the pension scheme over the years in preparing for a member's retirement cannot be exaggerated, in the old-age retirement scheme has allowed members of the scheme to have a certain level of sustainability after their years in active service. Although, in most cases, this scheme works efficiently and effectively in countries with stable economic policies, the case is not so for countries with unsteady economic policies where the political class influences the workability of the pension scheme and this has led to pension managers not living up to it expectations of their members and as such do not pay their members their retirement benefits as at when due and this has left many of his members on the street barging and some out of frustrations lose their lives in the process of pursuing what rightfully belongs to them. More concerned are members who lose their lives during active service tend to lose everything after contributing to the pension scheme.

Presently, there exist two types of pension plans in which members can be involved; these include the defined benefit (DB) pension plan and the defined contribution pension (DC) plan. In the DB plan, eligible employees have guaranteed income for life when they retire. Employers guarantee a specific retirement benefit amount for each participant based on factors such as age, years in service, salary histories of the members, grade level etc. Though most members appreciate this plan since the employers bear the financial burden, it has over the years created disagreements and interruptions in execution after retirement and this has led to the introduction of a different plan known as the defined contribution pension plan which is mostly members dependent; in this scheme, employees contribute a fixed proportion of their earnings to a unique account known as the Retirement Savings Account (RSA) with a Pension Fund Administrator (PFA) of their choice.

However, there has been a major migration from the DB plan to the DC plan because it is more realistic, dependable and sustainable than the DB plan since members are fully involved in the contributions and investment process (Gao, 2008; Gao,2009 Zhang and Rong, 2013;Othusitse and Xiaoping, 2015; Osu et al, 2017). Although the DC plan looks more attractive and encouraging than the DB plan, it requires members and fund administrators to have sound investment knowledge on how to invest in the financial market for optimal returns. These assets include but are not limited to fixed deposits (risk-free assets) (Li et al, 2013; Zhang and Rong, 2013, Muller, 2014; Muller, 2018), bonds (risk-less or risky asset) (Zhang and Rong, 2013; He and Liang, 2013; Li et al, 2017) depending on the type of bond involved and the stock (Li et al, 2017, Li et al, 2013, Njoku et al, 2019; Akpanibah and Ini, 2020) or loan (Muller, 2014; Muller, 2018; Akpanibah et al, 2019) which is always known to be a risky asset. it is necessary to note that, investment in these risky assets isalways volatile, hence it is necessary for members and PFAs to under study the best conceivable way of investing the accumulated funds for an optimum return. Hence this leads to the study of OID by financial institutions/PFAs which describe the percentage of members' wealth to be invested between the various assets involved to produce optimum profit with minimal risk (Wu and Zeng, 2015). There have been some works on OID under different assumptions which include but are not limited to (Akpanibah& Ini, 2021) who determined an investor's investment distributions with a stochastic interest rate under the CEV model and the Ornstein-Uhlenbeck process. (Ihedioha et al 2020) who studied the OIP for an investor under the M-CEV and Ornstein-Uhlenbeck models; they showed that the investor's OID when the Brownian motions (BM) do not correlate is less than the OID when the Brownian motions correlate. (Dawei and Jingyi, 2014) studied OID with multiple contributors. also studied ODP in defined contribution pension plans under loss aversion. (Eun and Jai 2012) study investment Strategies for HARA utility under the CEV model.

It is worthy of note that the best-known aspect of pensions in Nigeria is the payment of post-retirement financial benefits. However, there are other aspects of the reform that the PensionCommission has been developing with policy instruments. One such is the Residential Mortgage Scheme. Section 89 (2) of the PRA 2014 provides that: "A PFA may, subject to guidelines issued by the Commission, apply for 25% of the pension assets in the retirement savings account towards payment of equity contribution for payment of residential mortgage by a holder of RSA". Section 16(2)(d) of the 1999 Constitution (as amended) stipulates that "the State shall direct its policy towards ensuring that suitable and adequate shelter is provided for all citizens in line with the fundamental objectives and directive principles of State Policy". Pension is borne out of social security. Access to housing is also borne out of social security and shelter is a basic human need.

Owning a home is a big deal anywhere in the world but having the means to fulfil the desire is often a challenge, particularly in countries where lenders tend to focus on only high-net-worth individuals (Barr et al, 2020). Pension Commission issued the "guidelines on accessing RSA balance towards payment of Equity contribution for a residential mortgage by RSA holders". As this may sound like a piece of wonderful news to the RSA holders, it poses a new challenge to the PFAs to return 25% equity contribution for residential mortgages. Pulling out such funds and how the PFAs will be able to manage the investments and returns of retirement benefits to its members form the basis of this research. Hence, we focused on developing an investment strategy for the PFAs that will enable a return of premium clause on housing mortgages. By this type of clause, plan members can withdraw a quarter of the contributions she/he contributed to acquire a residential mortgage.

In recent times, some authors such as (He and Liang, 2013: Sheng and Rong, 2014, Li et al, 2017: Akpanibah and Osu, 2019: Akpanibah et al, 2020, Wang et al, 2018, Chai et al, 2021, Azor et al, 2023) have studied the investment strategies for the DC pension scheme with a refund of contributions clause under different assumptions. In their work, they considered return clauses based on mortality but in this research, we are interested in developing investment strategies with return clauses based on mortgage housing schemes for PSM with utility function exhibiting CARA and studying the effect of some parameters influencing the investment decision.

2.0 Model Formulations and Methodology

2.1 Abraham De Moivre Force Function

In this section, we discuss a pension fund system with a return clause of premium based on a mortgage housing scheme where the rate of PSM involvement is modeled using the Abraham De Moivre force function. Suppose c is the PPM monthly contribution at time t, e_0, the entry age of PPM during the accumulation period, T the accumulation phase period such that $e_0 + T$ is the retirement age of the PSM. $\aleph_{\frac{1}{t}, e_0+t}$ is the fraction of members interested in the mortgagehousing scheme for time t to $t + \frac{1}{t}$, tc is the accumulated contributions at time t, $0.25tc\aleph_{\frac{1}{t}, e_0+t}$ is the returned contributions paid toward the mortgage housing scheme.

From the work of He and Liang, 2013; Sheng and Rong and Li et al, 2017; Akpanibah et al. 2020, we have

$$\begin{cases} \dfrac{\aleph_{\frac{1}{t}, e_0+t}}{1-\aleph_{\frac{1}{t}, e_0+t}} = \emptyset(e_0 + t)dt,, \\ \aleph_{\frac{1}{t}, e_0+t} = \emptyset(e_0 + t)dt, \end{cases} \quad (1)$$

where$\aleph(t)$ is the force function and e is the maximal age of the life table. From (Kohler and Kohler, 2000; He and Liang, 2013), the Abraham De Moivre force function formula is given as

$$\emptyset(t) = \frac{1}{(e-t)} \quad 0 \le t < e \quad (2)$$

This implies that

$$\emptyset(e_0 + t) = \frac{1}{(e-e_0-t)} \quad (3)$$

2.2 PSM's Portfolio with Administrative Charges

In this section, we shall consider a financial market which is continuously open for an interval $t \in [0, T]$ where $T > 0$, is the expiration date of the investment. Also, we shall consider a portfolio with one risk-free asset and two risky assets. Let $\{\mathcal{W}_1(t), \mathcal{W}_2(t), t \ge 0\}$ be two standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is a real space $\mathcal{P}$is a probability measure and $\mathcal{F}$ is the filtration which represents the information generated by the two Brownian motions. Let $\mathcal{N}_t^0(t)$ denote the price of the risk-free asset at time t and the model is given as follows

$$\begin{cases} \dfrac{d\mathcal{N}_t^0(t)}{\mathcal{N}_t^0(t)} = r dt \\ \mathcal{N}_t^0(t) = 0 \end{cases} \quad (4)$$

Where $r(t)$ is the predetermined interest such that $r > 0$

Let $\mathcal{N}_t^1(t)$ and $\mathcal{N}_t^2(t)$ be the prices of stock 1 and stock 2 respectively which are modeled by the geometric Brownian motions model and the dynamics of the stock market prices are given by stochastic differential equations at $t \ge 0$ as follows

$$\begin{cases} \dfrac{d\mathcal{N}_t^1(t)}{\mathcal{N}_t^1(t)} = \ell_1 dt + m_{11} d\mathcal{W}_1 + m_{12} d\mathcal{W}_2 \\ \mathcal{N}_t^1(0) = 1 \end{cases} \quad (5)$$

$$\begin{cases} \dfrac{d\mathcal{N}_t^2(t)}{d\mathcal{N}_t^2(t)} = \ell_2 dt + m_{21} d\mathcal{W}_1 + m_{22} d\mathcal{W}_2 \\ \mathcal{N}_t^2(0) = 1 \end{cases} \quad (6)$$

where ℓ_1 and ℓ_2 represent the expected appreciation rate of stocks 1 and 2 respectively. $m_{11}, m_{12}, m_{21}, m_{22}$represent the instantaneous volatilities that form a $2 X 2$ matrix $m = \{m_{ab}\}_{2X2.}$see (Zhao and Rong, 2011).

Furthermore, we consider a case where the PSM is allowed to invest in one risk-free asset and two risky assets. Let $\mathcal{B}$ be the administrative charges which is determined based on the value of the stock by the pension fund administrators Lai et al, (2021). Corresponding to the optimal investment strategy $k = (k_0, k_{1,} k_2)$, where $k_0 = 1 - k_1 - k_2$ and the accumulation phase period $\left[t, t + \frac{1}{t}\right]$, the stochastic differential equation for the wealth function similar to (He and Liang, 2013; Sheng and Rong, 2014; Akpanibah and Osu 2020) is given as:

$$X\left(t+\frac{1}{t}\right)=\begin{bmatrix} X(t)\begin{pmatrix} \left(1-k_1(t)-k_2(t)\right)\frac{\mathcal{N}^0_{t+\frac{1}{t}}}{\mathcal{N}^0_t} \\ +k_1(t)\frac{\mathcal{N}^1_{t+\frac{1}{t}}}{\mathcal{N}^1_t}+k_2(t)\frac{\mathcal{N}^2_{t+\frac{1}{t}}}{\mathcal{N}^2_t} \\ \left(1-0.25tc\aleph_{\frac{1}{t},e_0+t}\right) \\ \left(1-k_1(t)-k_2(t)\right)\frac{\mathcal{N}^0_{t+\frac{1}{t}}}{\mathcal{N}^0_t} \end{pmatrix} \\ -\mathcal{B}X(t)\frac{1}{t}-0.25tc\aleph_{\frac{1}{t},e_0+t}+tc\frac{1}{t} \end{bmatrix} \quad (7)$$

$$X\left(t+\frac{1}{t}\right)=$$

$$\begin{bmatrix} X(t)\begin{pmatrix} \left(1-k_1(t)-k_2(t)\right)\begin{pmatrix}\frac{\mathcal{N}^0_{t+\frac{1}{t}}-\mathcal{N}^0_t}{\mathcal{N}^0_t} \\ +\frac{\mathcal{N}^0_t}{\mathcal{N}^0_t}\end{pmatrix} \\ +k_1(t)\left(\frac{\mathcal{N}^1_{t+\frac{1}{t}}-\mathcal{N}^1_t}{\mathcal{N}^1_t}+\frac{\mathcal{N}^1_t}{\mathcal{N}^1_t}\right) \\ +k_2(t)\left(\frac{\mathcal{N}^2_{t+\frac{1}{t}}-\mathcal{N}^2_t(t)}{\mathcal{N}^2_t(t)}+\frac{\mathcal{N}^2_t(t)}{\mathcal{N}^2_t(t)}\right) \\ \left(1-0.25tc\aleph_{\frac{1}{t},e_0+t}\right) \\ \left(1-k_1(t)-k_2(t)\right)\begin{pmatrix}\frac{\mathcal{N}^0_{t+\frac{1}{t}}-\mathcal{N}^0_t}{\mathcal{N}^0_t} \\ +\frac{\mathcal{N}^0_t}{\mathcal{N}^0_t}\end{pmatrix} \end{pmatrix} \\ -\mathcal{B}X(t)\frac{1}{t}-0.25tc\aleph_{\frac{1}{t},e_0+t}+tc\frac{1}{t} \end{bmatrix} \quad (8)$$

Simplifying (8), we have

$$X\left(t+\frac{1}{t}\right)-X(t)=$$

$$\begin{bmatrix} X(t)\begin{pmatrix} \left(1-k_1(t)-k_2(t)\right) \\ \left(1-0.25tc\aleph_{\frac{1}{t},e_0+t}\right) \\ +\left(2-0.25tc\aleph_{\frac{1}{t},e_0+t}\right)\left(\frac{\mathcal{N}^0_{t+\frac{1}{t}}-\mathcal{N}^0_t}{\mathcal{N}^0_t}\right) \\ +k_1(t)\left(\frac{\mathcal{N}^1_{t+\frac{1}{t}}-\mathcal{N}^1_t}{\mathcal{N}^1_t}\right) \\ +k_2(t)\left(\frac{\mathcal{N}^2_{t+\frac{1}{t}}-\mathcal{N}^2_t(t)}{\mathcal{N}^2_t(t)}\right) \end{pmatrix} \\ -\mathcal{B}X(t)\frac{1}{t}-0.25tc\aleph_{\frac{1}{t},e_0+t}+tc\frac{1}{t} \end{bmatrix} \quad (9)$$

From He and Liang, 2013), we have

$$\begin{cases} \mathcal{B}\frac{1}{t}=\mathcal{B}dt,\ tc\frac{1}{t}\to tcdt,\frac{\mathcal{N}^0_{t+\frac{1}{t}}-\mathcal{N}^0_t}{\mathcal{N}^0_t}\to\frac{d\mathcal{N}^0_t}{\mathcal{N}^0_t}, \\ \frac{\mathcal{N}^1_{t+\frac{1}{t}}-\mathcal{N}^1_t}{\mathcal{N}^1_t}\to\frac{d\mathcal{N}^1_t}{\mathcal{N}^1_t},\ \frac{\mathcal{N}^2_{t+\frac{1}{t}}-\mathcal{N}^2_t(t)}{\mathcal{N}^2_t(t)}\to\frac{d\mathcal{N}^2_t(t)}{\mathcal{N}^2_t(t)} \end{cases} \quad (10)$$

Substituting (1) and (10) into (9), we have

$$dX(t)=$$

$$\begin{bmatrix} X(t)\begin{pmatrix} k_1\frac{d\mathcal{N}^1_t}{\mathcal{N}^1_t}+k_2\frac{d\mathcal{N}^2_t}{\mathcal{N}^2_t}-\mathcal{B}dt \\ +(1-k_1-k_2) \\ \times\begin{pmatrix}(1-0.25\emptyset(e_0+t)dt \\ +\frac{d\mathcal{N}^0_t}{\mathcal{N}^0_t}(2-0.25\emptyset(e_0+t)dt)\end{pmatrix} \end{pmatrix} \\ +tcdt-0.25tc\emptyset(e_0+t)dt \end{bmatrix} \quad (11)$$

Substituting (4), (5), (6) and (3) into (11) and simplifying it, we have

$$dX(t)=\begin{bmatrix} \left(X(t)\begin{bmatrix} k_1\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right) \\ +k_2\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right) \\ +\left(2r-\frac{0.25}{e-e_0-t}\right) \end{bmatrix}\right)dt \\ -\mathcal{B}+tc\left(1-\frac{0.25}{e-e_0-t}\right) \\ +X(t)\begin{pmatrix}(k_1m_{11}+k_2m_{21})d\mathcal{W}_1 \\ +(k_1m_{12}+k_2m_{22})d\mathcal{W}_2\end{pmatrix} \end{bmatrix} \quad (12)$$

Hence, the wealth of the PSM with return of contribution for the purpose of mortgage is given in (12).

2.3 Legendre Transformation Method and Dual Theory

The Legendre transform is used to convert a non-linear optimization problem into a linear one. By using the LTM, complex equations can be reformulated and its solution can be determined by solving a system of linear equations. The dual theory, on the other hand, is based on the idea that the value of the system can be expressed as the maximum expected reward of a portfolio of non-anticipative assets.

Theorem 1: Let $f:R^n\to R$ be a convex function for $z>0$, defined the Legendre transform $K(z)=\max_x\{f(x)-zx\}$ where $K(z)$ is the Legendre dual of $f(x)$ (Jonson and Sircar, 2002).

Since $f(x)$ is convex, from the theorem 1, we can define the Legendre transform as follows

$$y(t,z)=\inf\left\{\begin{matrix} x \mid M(t,x)\geq \\ zx+\hat{M}(t,x)\end{matrix}\right\} 0<t<T \qquad (13)$$

Where $\hat{M}$ is the dual of M and $z>0$ is the dual variable of x .The value of x where this optimum is attained is denoted by $y(t,z)$, so that the functions y and $\hat{M}$ are closely related and can be refers to the dual of M . These functions are related as follows

$$\hat{M}(t,z)=M(t,y)-zy$$

where

$$y(t,z)=x, M_x=z, y=-\hat{M}_z \qquad (14)$$

At terminal time, we denote

$$\hat{U}(z)=\sup\{U(x)-zx \mid 0<x<\infty\}$$

and

$$G(z)=\sup\left\{x \mid U(x)\geq zx+\hat{U}(z)\right\}$$

As a result

$$G(z)=\left(U^1\right)^{-1}(\text{z}) \qquad (15)$$

Where G is the inverse of the marginal utility U and note that

At terminal time T , we can define

$$y(T,x)=\inf_{x>0}\left\{x \mid U(x)\geq zx+\hat{M}(t,z)\right\}$$

and

$$\hat{M}(t,z)=\sup_{x>0}\{U(x)-zx\}$$

So that

$$y(T,z)=\left(U^1\right)^{-1}(z) \qquad (16)$$

Next we differentiate (14) with respect to t and x

$$M_t=\hat{M}_t, M_x=z, M_{xx}=\frac{-1}{\hat{M}_{zz}} \qquad (17)$$

3.0 Main Result

3.1 Hamilton Jacobi Bellman Equation

Let $k(t)$ be the control variable known as the optimal investment strategy and we define the utility attained by the investor from a given state x at time t as

$$M_{k(t)}(t,x)=E_{k(t)}\left[U\big(X(T)\big) \mid X(t)=x\right] \qquad (18)$$

Where t is the time, x is the investor's wealth. The objective here is to determine the optimal investment strategy and the optimal value function of the investor given as

$$k(t)^* and M(t,x)=Sup M_{k(t)}(t,x), \qquad (19)$$

Respectively, such that

$$M_{k(t)^*}(t,x)=M(t,x). \qquad (20)$$

A careful observation of the problem described in (18) shows that it however turns out to be time-consistent since the Bellman optimality condition is satisfied. This law states that the optimal control is independent of the initial point. More precisely: if a control law is optimal on the full-time interval $[0,T]$, then it is also optimal for any subinterval $[t,T]$. see (Björk et al., 2012). A typical example of a utility function that satisfies (18) is the exponential utility function which is an example of time time-consistent problem.

From (Ihedioha et al, 2020) we apply Ito's lemma to derive the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (12).This is obtained by maximizing (18) subject to (12) as follows

$$dM=\frac{\partial M}{\partial t}dt+\frac{\partial M}{\partial X}dX+\frac{1}{2}\frac{\partial^2 M}{\partial^2 X}(dX)^2+\frac{1}{2}\frac{\partial^2 M}{\partial^2 t}(dt)^2+\frac{1}{2}\frac{\partial^2 M}{\partial X\partial t}(dX)(dt) \qquad (21)$$

Substituting (12) into (21), we have

$$\left\{\begin{matrix} \\ +Sup \\ k_1k_2 \\ \end{matrix}\left[\begin{matrix} M_t+\left[\begin{matrix}\left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right)x \\ +tc\left(1-\frac{0.25}{e-e_0-t}\right)\end{matrix}\right]M_x \\ \left(\begin{matrix}\frac{1}{2}x^2k_1^2\mathcal{P}+x^2k_1k_2\mathcal{R} \\ +\frac{1}{2}x^2k_2^2\mathcal{Q}\end{matrix}\right)M_{xx} \\ +\left(\begin{matrix}xk_1\left(\ell_1+\frac{0.25}{e-e_0-t}-t\right) \\ +xk_2\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)\end{matrix}\right)M_x\end{matrix}\right]\right\} \qquad (22)$$

Where $m_{11}^2+m_{12}^2=\mathcal{P}, m_{21}^2+m_{22}^2=\mathcal{Q} and m_{11}m_{21}+m_{12}m_{22}=\mathcal{R}$,

Next, we differentiate (22) with respect to k_1 and k_2, to obtain the first-order` maximizing condition for equation (22) as

$$k_1^*=-\frac{\mathcal{R}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{Q}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)}{x(\mathcal{PQ}-\mathcal{R}^2)}\frac{M_x}{M_{xx}} \qquad (23)$$

$$k_2^*=-\frac{\mathcal{R}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{P}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)}{x(\mathcal{PQ}-\mathcal{R}^2)}\frac{M_x}{M_{xx}} \qquad (24)$$

Substituting (23) and (24) into (22), we have

$$\left\{M_t+\frac{1}{2}(\mathcal{C}-\mathcal{D}-\mathcal{E})\frac{M_x^2}{M_{xx}}+\left[\left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right)x+tc\left(1-\frac{0.25}{e-e_0-t}\right)\right]M_x\right\}=0) \qquad (25)$$

Where

$$\begin{cases} \mathcal{C} = \frac{2\mathcal{R}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)}{(\mathcal{PQ}-\mathcal{R}^2)}, \\ \mathcal{D} = \frac{\mathcal{Q}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)^2}{(\mathcal{PQ}-\mathcal{R}^2)}, \mathcal{E} = \frac{\mathcal{P}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)^2}{(\mathcal{PQ}-\mathcal{R}^2)} \end{cases} \quad (26)$$

Next, we shall apply the Legendre transformation theorem to reduce (25) to a linear PDE

Substituting (17) into (23), (24) and (25), we have

$$\begin{pmatrix} \widehat{M_t} - \frac{1}{2}(\mathcal{C}-\mathcal{D}-\mathcal{E})z^2\widehat{M_{zz}} \\ + \begin{bmatrix} \left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right)y \\ +tc\left(1-\frac{0.25}{e-e_0-t}\right) \end{bmatrix} z = 0 \end{pmatrix} \quad (27)$$

$$k_1^* = -\left[\frac{\mathcal{R}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{Q}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)}{y(\mathcal{PQ}-\mathcal{R}^2)}\right] z\widehat{M_{zz}} \quad (28)$$

$$k_2^* = -\left[\frac{\mathcal{R}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{P}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)}{y(\mathcal{PQ}-\mathcal{R}^2)}\right] z\widehat{M_{zz}} \quad (29)$$

Differentiate with respect to z, we have

$$\begin{pmatrix} y_t + \frac{1}{2}(\mathcal{C}-\mathcal{D}-\mathcal{E})z^2 y_{zz} \\ + \begin{bmatrix} \left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right)y \\ +tc\left(1-\frac{0.25}{e-e_0-t}\right) \end{bmatrix} \\ -\begin{pmatrix} 2r-\mathcal{B}-\frac{0.25}{e-e_0-t} \\ -\frac{1}{2}(\mathcal{C}-\mathcal{D}-\mathcal{E}) \end{pmatrix} zy_z \end{pmatrix} = 0 \quad (30)$$

$$k_1^* = -\frac{\left[\mathcal{R}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{Q}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)\right]}{y(\mathcal{PQ}-\mathcal{R}^2)} zy_z \quad (31)$$

$$k_2^* = -\frac{\left[\mathcal{R}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{P}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)\right]}{y(\mathcal{PQ}-\mathcal{R}^2)} zy_z \quad (32)$$

3.2 Optimal Investment Strategies for a PSM Exhibiting Constant Absolute Risk Averse

We assume that the member takes an exponential utility Li*et al*, (2013), Osu*et al*, (2017)

$$\begin{cases} U(x) = -\frac{1}{\gamma}e^{-x\gamma} \\ \gamma > 0. \end{cases} \quad (33)$$

The absolute risk averse of a PPM with the utility in (33) is constant.

Next, we conjecture a solution to our problem under exponential utility similar to Li et al, (2013), Osu *et al* (2017) as follows

$$y(T,z) = (U^1)^{-1}(z) \quad (34)$$

From (33), we have

$$y(T,z) = \frac{-1}{\eta} lnz \quad (35)$$

$$\begin{cases} y(t,z) = \frac{1}{\eta}\left[\mathcal{U}(t)\left(lnz + v(t)\right)\right] + \mathcal{w}(t) \\ \mathcal{U}(T) = 1, \ v(T) = \mathcal{w}(T) = 0 \end{cases} \quad (36)$$

Differentiating (3.3.36), we have

$$y_t = \begin{cases} \frac{-1}{\eta}\left[\mathcal{U}_t\left(lnz+v(t)\right) + \mathcal{U}(t)v_t\right] + \mathcal{w}_t, \\ y_z = \frac{-1}{z\eta}, \mathcal{U}(t), y_{zz} = -\frac{1}{z^2\eta}\mathcal{U}(t) \end{cases} \quad (37)$$

$$\begin{cases} \left[\mathcal{U}_t - \left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right)\mathcal{U}(t)\right] lnz \\ -\begin{bmatrix} \mathcal{w}_t - \left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right)\mathcal{w}(t) \\ -tc\left(1-\frac{0.25}{e-e_0-t}\right) \end{bmatrix} \eta = 0 \\ \frac{-\mathcal{U}(t)}{\eta}\begin{bmatrix} v_t - \left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right) \\ +\frac{1}{2}(\mathcal{C}-\mathcal{D}-\mathcal{E}) \end{bmatrix} \end{cases} \quad (38)$$

Splitting (38), we have

$$\begin{cases} \mathcal{U}_t - \left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right)\mathcal{U}(t) = 0 \\ \mathcal{U}(T) = 1 \end{cases} \quad (39)$$

$$\begin{cases} \mathcal{w}(t) - \left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right)\mathcal{w}(t) \\ -tc\left(1-\frac{0.25}{e-e_0-t}\right) \\ \mathcal{w}(T) = 0 \end{cases} \quad (40)$$

$$\begin{cases} v_t - \left(2r-\mathcal{B}-\frac{0.25}{e-e_0-t}\right) \\ +\frac{1}{2}(\mathcal{C}-\mathcal{D}-\mathcal{E}) \quad \mathcal{U}_t = 0 \\ v(T) = 0 \end{cases} \quad (41)$$

Solving (39), we have

$$\mathcal{U}(t) = \left(\frac{e-e_0-t}{e-e_0-T}\right)^{0.25} e^{(2r-\mathcal{B})(t-T)} \quad (42)$$

Solving (40), we have

$$\mathcal{w}(t) = \begin{pmatrix} ce^{-(2r-\mathcal{B})t}(e-e_0-t)^{0.25} \\ \int_t^T \tau e^{-(2r-\mathcal{B})t}\left(\frac{1}{e-e_0-t}\right)^{0.25}\left(1-\frac{0.25}{e-e_0-t}\right)d\tau \end{pmatrix} \quad (43)$$

Solving (41), we have

$$v(t) = \begin{pmatrix} \left[2r-\frac{1}{2}(C-D-E)\right](t-T) \\ +In\left(\frac{1}{e-e_0-t}\right)^{0.25} \end{pmatrix} \quad (44)$$

Therefore, substituting (42), (43) and (44) into (36), we have

$$\begin{cases} y(t,z) = \frac{1}{\eta}\left[\left(\frac{e-e_0-t}{e-e_0-T}\right)^{0.25} e^{2r(t-T)} \times \left(\begin{matrix} lnz \\ +\left[\begin{matrix}(2r-\mathcal{B}) \\ -\frac{1}{2}\left(\begin{matrix}C-D \\ -E\end{matrix}\right)\end{matrix}\right](t-T) \\ +In\left(\frac{1}{e-e_0-t}\right)^{0.25}\end{matrix}\right)\right] \\ +ce^{-(2r-\mathcal{B})t}(e-e_0-t)^{0.25} \\ \int_t^T \tau e^{-(2r-\mathcal{B})t}\left(\frac{1}{e-e_0-t}\right)^{0.25}\left(1-\frac{0.25}{e-e_0-t}\right)d\tau \end{cases} \quad (45)$$

Proposition 1

The optimal investment strategies under exponential utility is given by

$$k_1^* = \left(\begin{matrix}\frac{\mathcal{R}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{Q}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)}{x\eta(\mathcal{P}\mathcal{Q}-\mathcal{R}^2)} \\ \times\left(\frac{e-e_0-t}{e-e_0-T}\right)^{\frac{1}{4}} e^{(2r-\mathcal{B})(t-T)}\end{matrix}\right) \quad (46)$$

$$k_2^* = \left(\begin{matrix}\frac{\mathcal{R}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{P}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)}{x\eta(\mathcal{P}\mathcal{Q}-\mathcal{R}^2)} \\ \times\left(\frac{e-e_0-t}{e-e_0-T}\right)^{\frac{1}{4}} e^{(2r-\mathcal{B})(t-T)}\end{matrix}\right) \quad (47)$$

$$k_0^* = 1 - k_1^* - k_2^* \quad (48)$$

proof

from (37),

$$y_z = \frac{-1}{z\eta}\mathcal{U}(t)$$

Substituting (42) into (37), we have

$$y_z = -\frac{1}{z\eta}\left(\frac{e-e_0-t}{e-e_0-T}\right)^{0.25} e^{(2r-\mathcal{B})(t-T)}$$

Hence

$$zy_z = -\frac{1}{\eta}\left(\frac{e-e_0-t}{e-e_0-T}\right)^{\frac{1}{4}} e^{(2r-\mathcal{B})(t-T)} \quad (49)$$

Substituting (49) into (31) and (32), we have

$$k^*{}_1 = -\frac{\mathcal{R}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{Q}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)}{y(\mathcal{P}\mathcal{Q}-\mathcal{R}^2)} \times$$

$$-\frac{1}{\eta}\left(\frac{e-e_0-t}{e-e_0-T}\right)^{\frac{1}{4}} e^{(2r-\mathcal{B})(t-T)}$$

$$k^*{}_1 = \frac{\mathcal{R}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{Q}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)}{x\eta(\mathcal{P}\mathcal{Q}-\mathcal{R}^2)}\left(\frac{e-e_0-t}{e-e_0-T}\right)^{\frac{1}{4}} e^{(2r-\mathcal{B})(t-T)}$$

$$k^*{}_2 = -\frac{\mathcal{R}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{P}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)}{y(\mathcal{P}\mathcal{Q}-\mathcal{R}^2)} \times$$

$$-\frac{1}{\eta}\left(\frac{e-e_0-t}{e-e_0-T}\right)^{\frac{1}{4}} e^{(2r-\mathcal{B})(t-T)}$$

$$k^*{}_2 = \frac{\mathcal{R}\left(\ell_1+\frac{0.25}{e-e_0-t}-1\right)-\mathcal{P}\left(\ell_2+\frac{0.25}{e-e_0-t}-1\right)}{x\eta(\mathcal{P}\mathcal{Q}-\mathcal{R}^2)}\left(\frac{e-e_0-t}{e-e_0-T}\right)^{\frac{1}{4}} e^{(2r-\mathcal{B})(t-T)}$$

4.0 Results and Discussion

In this section, some results of our work showing the relationship between the OID and some sensitive parameters will be presented and discussed. To achieve this, the following data are used similar to (He & Liang, 2013; Zhao & Rong, 2012) unless otherwise stated: $\ell_1 = 1.6$, $\ell_2 = 0.05$, $r = 0.1$, $\gamma = 0.01$, $x_0 = 1$, $m_{11} = 0.12$, $m_{12} = 0.10$, $m_{21} = 1.12$, $m_{22} = 1.1$, $t = 0{:}\,0.00005{:}\,20$, $e = 100$, $e_0 = 20$, $T = 40$.

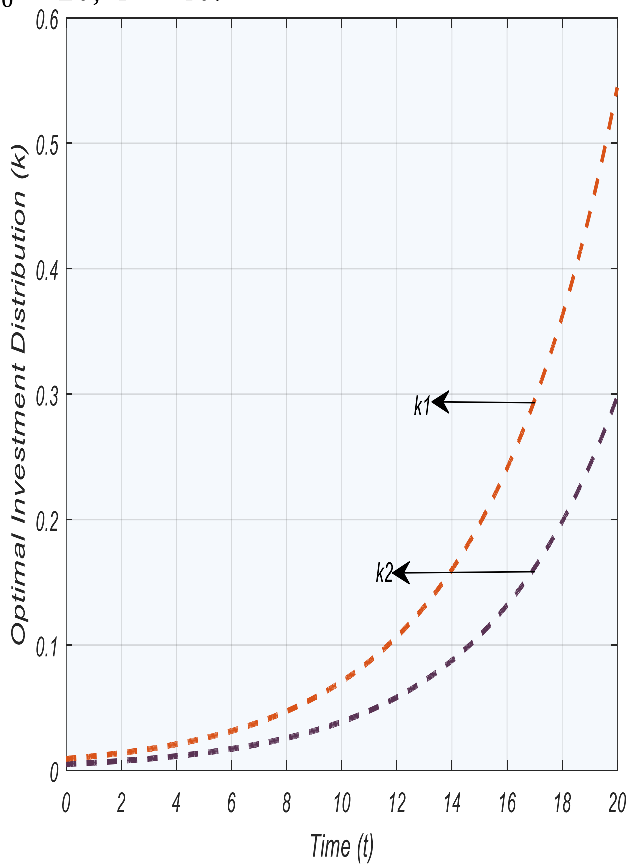

Fig. 1: Time evolution of the OID

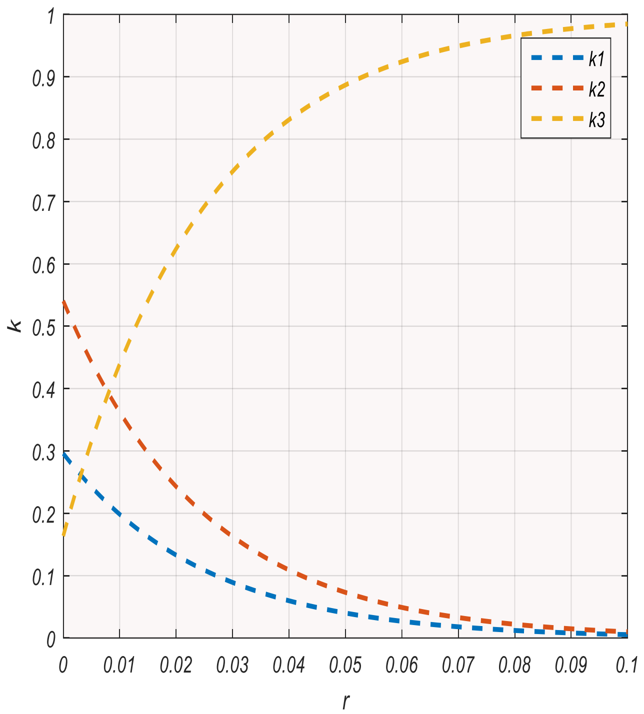

Fig. 2: Impact of risk free interest rate on the OID

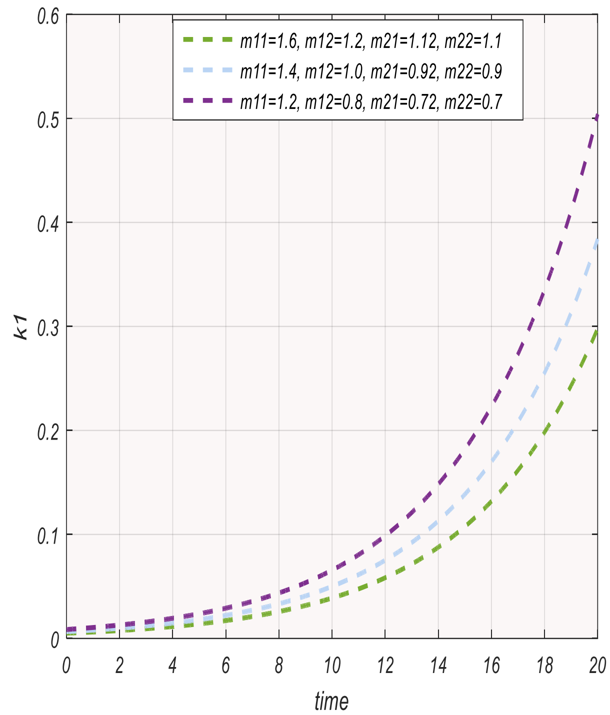

Fig. 4: Time evolution of stock 1 with different instantaneous volatilities

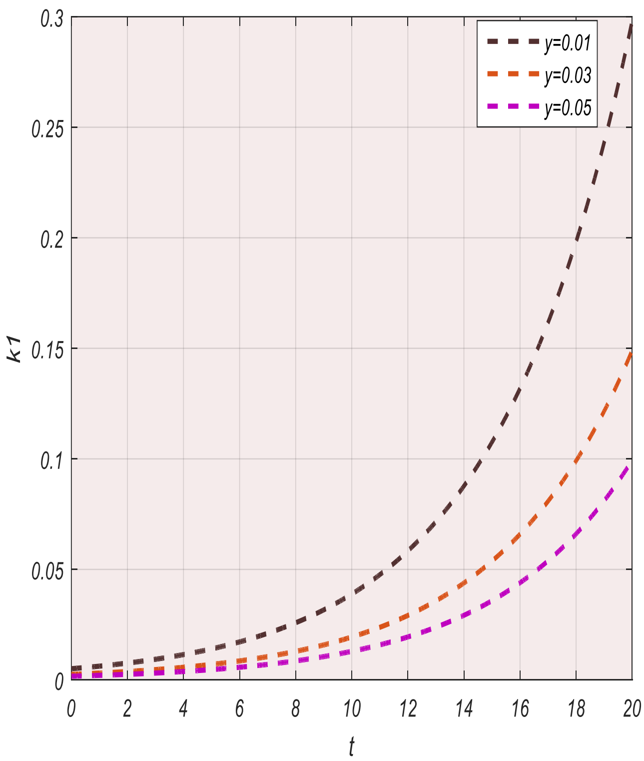

Fig. 3: Time evolution of stock 1 with different risk averse coefficient

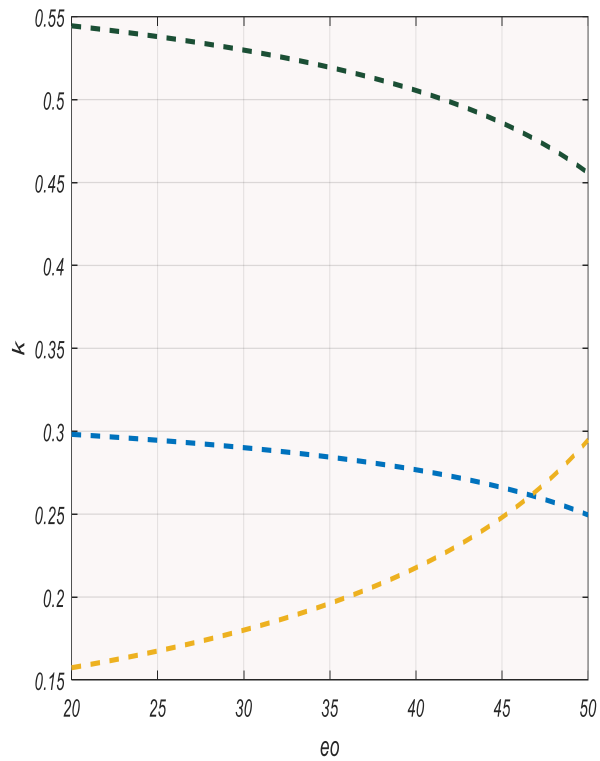

Fig. 5: Impact ofPSM's entry age on the OID

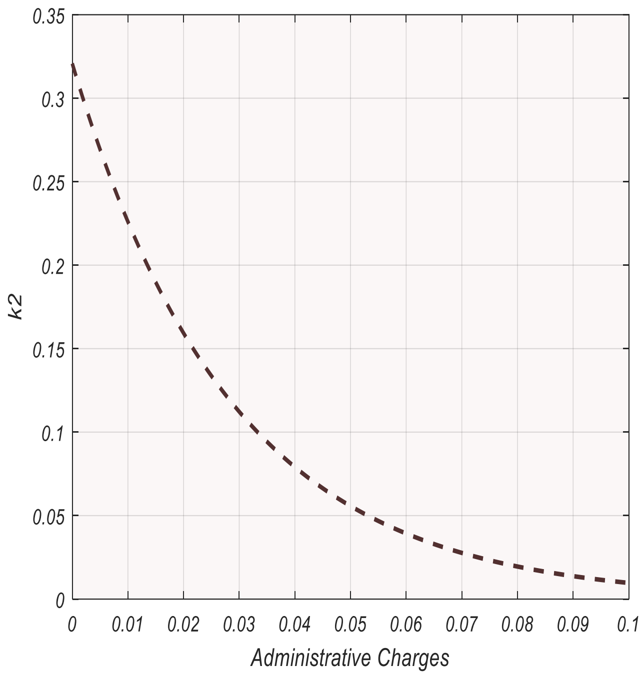

Fig. 6: impact of Administrative charges on OID of stock 2

In Fig. 1, the graph of OID for the two risky assets is presented as a function of time; the graph shows that the proportion of investment in the two risky assets increases with time and gets higher and higher as the PPM approaches retirement age while the proportion invested in the risk-free asset decreases with time and gets smaller and smaller as the PPM approaches retirement age. This is so because the initial fund size was used at the beginning of the investment instead of the optimal fund size.

In Fig. 2, the graph of the OID against the risk-free interest rate is presented; the graph shows that the optimal fund size is an increasing function of the risk-free interest rate. The graph implies that members with large funds prefer to invest where there is less risk since they may not want to lose what they have gathered already.

In Fig. 3, the graphs of OID against the risk averse coefficient are presented; the graphs show that the OID is inversely proportional to the risk averse coefficient. This implies that the more fearful the PPM is to investment in risky asset, the more likelihood that he will reduce the proportion of his wealth to be invested in risky asset. As a result, increases his investment in risk free asset.

In Fig. 4, the graph of OID against the instantaneous volatilities is presented; the graph shows that the OID is inversely proportional to the instantaneous volatility of the risky asset. This implies that the higher the instantaneous volatility of the risky asset, the higher the risk involved in the investment in such asset. Hence this may create more fears in the mind of the investor toward investing in the risky asset, furthermore, reduce the proportion of the investor's wealth in risky asset.

In Fig. 5, the graph illustrates the impact of PPM's entry age on the OID. It is observed that the OID decreases as the entry age of the PPM increases. The implication is that PPM with late entry in to the pension scheme, tends to decrease the proportion of their wealth to be invested in the risky assets.

In Fig. 6, the relationship between the OIDand the administrative charges is presented. It is observedthat the OID developed by the PFA is a decreasing function of the administrative charges. The implication in Fig. 6 is that, the higher the administrative charges on investment of the risky asset, the more likelihood that the PPM of the scheme will be discouraged from investing much proportion of their wealth in risky asset and vice versa.

5.0 Conclusion

This research work investigated the study of OID with the return of contribution for the mortgage housing scheme for a PSM exhibiting CARA. Abraham De Moivre's force function was used to determine the number of PSM interested in the mortgage housing scheme. Furthermore, the OID for the risky assets was obtained for time-consistent cases by using Legendre transformation, dual theory and variable change. It was observed that for a PFA to develop an efficient and robust investment plan with a return of 25% of contributions for a mortgage housing scheme, the risk-averse coefficient, instantaneous volatilities, administrative charges, initial wealth and entry age of the PSM must be carefully considered.

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Authors' Contributions

The first author developed the manuscript after discussion with the second and third authors. The first author formulate and solved the problem with the help of the second and third authors while the second author carried out numerical simulations for the results. The three authors also proof read the work.

Compliance with Ethical Standards Declarations

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable

Availability of data and materials

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