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Development and Applications of the Type II Half-Logistic Exponentiated Inverse Weibull Distribution

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Abstract: *A variety of distribution classes have emerged by expanding or generalizing well-known continuous distributions to enhance their flexibility and adaptability across various fields. One such distribution is the Inverse Weibull (IW) distribution, introduced by Keller and Kanath in 1982, which has proven effective in modelling failure characteristics. Over the years, several extensions of the IW distribution have been developed, including the Beta Inverse Weibull, Kumaraswamy-Inverse Weibull, and many others. This paper introduces a novel extension called the Type II Half-Logistic Inverse Weibull (TIIHLEtIW) distribution, derived from the Type II Half-Logistic Exponentiated-G (TIIHLEt-G) family proposed by Bello et al. in 2021. The TIIHLEtIW distribution incorporates two additional shape parameters, enhancing its flexibility. We provide the cumulative distribution function (cdf), probability density function (pdf), and key statistical properties, including moments, moment-generating function, reliability function, hazard function, and quantile function. Maximum likelihood estimation (MLE) is employed for parameter estimation, and a simulation study evaluates the performance of the MLEs. Finally, the applicability and superiority of the TIIHLEtIW distribution are demonstrated through a comparative study using two real datasets, showcasing its improved fit over several established distributions.*

Keywords: *Type II Half-Logistic Exponentiated-G, Inverse Weibull distribution, Hazard function, Reliability function, Maximum likelihood, Order Statistics*

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1.0 Introduction

A variety of distribution classes have been developed by expanding or generalizing well-known continuous distributions. These new families of continuous distributions enhance and extend classic distributions, offering increased flexibility for applications across various fields. The newly created distributions have been extensively researched, demonstrating their broad utility and improved adaptability. The inverse Weibull (IW) distribution is an adaptation of the Weibull distribution using transformed variables. Its appeal lies in its flexibility and simplicity, making it suitable for the modelling of various failure characteristics. Introduced by Keller and Kanath (1982), the Inverse Weibull distribution was initially designed for analyzing the degradation of mechanical components in survival and reliability studies.

In recent years, the Inverse Weibull distribution has attracted considerable attention, resulting in the creation of numerous extensions. These include the Beta Inverse Weibull by Khan (2010), the Kumaraswamy-Inverse Weibull by Shahbaz *et al.,* (2012), the Reflected Generalized Beta Inverse Weibull by Elbatal *et al.,* (2016), the Topp-Leone Inverse Weibull by Abbas *et al.*, (2017), the Marshall-Olkin Extended Inverse Weibull by Pakungwati *et al.,* (2018), the Odd Frechet Inverse Weibull by Fayomi (2019), the Gamma-Inverse Weibull by Abbas *et al.,* (2020), the Extended Inverse Weibull by Alkarni *et al.,* (2020), and the modified Burr XII Inverse Weibull by Bhatti *et al.,* (2020).

Bello *et al.,* (2021) proposed a new distribution family known as the Type II Half-Logistic Exponentiated-G (TIIHLEt-G), which includes two additional shape parameters. For any arbitrary cumulative distribution function (cdf) $H(x,\Theta)$ as a baseline, the TIIHLEt-G family with two positive shape parameters l and a has the cumulative distribution function (cdf) and probability density function (pdf) given by:

$$F_{TIIHLEt\_G}(x;\lambda,\alpha,\Theta)=\frac{2H^{\alpha\lambda}(x;\Theta)}{\left[1+H^{\alpha\lambda}(x;\Theta)\right]} \quad , \quad x>0,\ \lambda,\alpha>0 \qquad (1)$$

and

$$f_{TIIHLEt-G}(x;\lambda,\alpha,\Theta)=\frac{2\lambda\alpha h(x;\Theta)H^{\alpha-1}(x;\Theta)\left[H^{\alpha(\lambda-1)}(x;\Theta)\right]}{\left[1+H^{\alpha\lambda}(x;\Theta)\right]^{2}},\ x>0,\ \lambda,\alpha>0 \qquad (2)$$

The cdf and pdf of the Inverse Weibull distribution are given as

$$H(x;q,b)= e^{-qx^{-b}},\ x>0,q,b>0 \qquad (3)$$

$$h(x;q,b)= qb\,x^{-b-1}e^{-qx^{-b}},\ x>0,q,b>0 \qquad (4)$$

This paper aims at developing a more flexible model by extending the two-parameter Inverse Weibull distribution. The new model is named the Type II Half-Logistic Inverse Weibull (TIIHLEtIW) distribution. We derive the TIIHLEtIW distribution from the framework proposed by Bello *et al.,* (2021) and present several key statistical properties.

2.0 The Type II Half-Logistic Exponentiated Inverse Weibull (TIIHLEtIW) Distribution

We define a new model called the TIIHLEtIW model. A random variable X is said to follow the TIIHLEtIW distribution if its cumulative distribution function (cdf) is obtained by substituting equation (3) into equation (1) as follows:

$$F_{TIIHLEtIW}(x;\lambda,\alpha,\theta,\beta)=\frac{2\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda}}{1+\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda}}\ ,x>0,\lambda,\alpha,\theta,\beta>0 \qquad (5)$$

and its corresponding pdf is

$$f_{TIIHLEtIW}(x;\lambda,\alpha,\theta,\beta)=\frac{2\lambda\alpha\theta\beta x^{-\beta-1}e^{-\theta x^{-\beta}}\left[e^{-\theta x^{-\beta}}\right]^{\alpha-1}\left[e^{-\theta x^{-\beta}}\right]^{\alpha(\lambda-1)}}{\left[1+\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda}\right]^{2}},x>0,\lambda,\alpha,\theta,\beta>0 \qquad (6)$$

where β is a scale parameter and λ,α,θ are shape parameters.

3.0 Important Representation

We have provided a useful representation for the pdf and cdf of the TIIHLEtIW distribution. This representation utilizes the generalized binomial series, which is known for

$$(1+Z)^{-\nu}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\nu+i-1}{i}z^{i} \qquad (7)$$

For $|z|<1$ and ν is a positive real non-integer. The density function of the TIIHLEtIW distribution is then obtained by using the binomial theorem (7) to (6).

$$f_{TIIHLEtIW}(x;\lambda,\alpha,\theta,\beta)=2\lambda\alpha\theta\beta x^{-\beta-1}\sum_{i=0}^{\infty}(-1)^{i}\binom{1+i}{i}\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(i-1)} \qquad (8)$$

Furthermore, an expansion for the $\left[F(x,\lambda,\alpha,\theta,\beta)\right]^{h}$ is derived, where h is an integer, and the binomial expansion is computed once more.

$$\left[F(x;\lambda,\alpha,\theta,\beta)\right]^{h}=2^{h}\sum_{t=0}^{h}(-1)^{t}\binom{h+t-1}{t}\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(h+t)} \qquad (9)$$

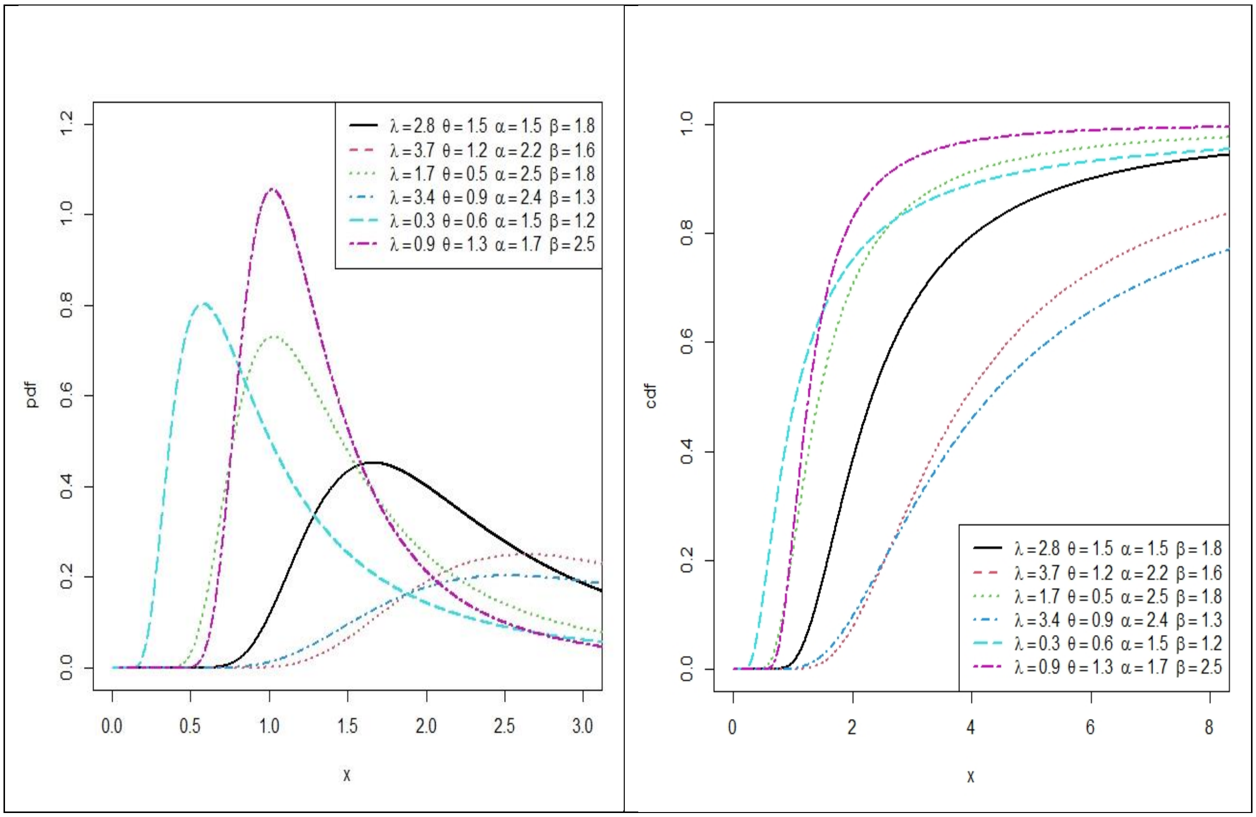

Fig. 1: Plots of Pdf and Cdf of TIIHLEtIW distribution for different values of parameters.

4.0 Statistical Properties

4.1 Probability weighted moments

Probability weighted moments (PWMs) were introduced by Greenwood *et al.*, (1979). They are used to derive inverse form estimators for the parameters and quantiles of a distribution. The PWMs, denoted by $\kappa_{r,s}$, can be derived for a random variable X using the following relationships.

$$\kappa_{r,s}=E\left[X^{r}F(X)^{s}\right]=\int_{-\infty}^{\infty}x^{r}f(x)(F(x))^{s}dx \qquad (10)$$

The PWMs of the TIIHLEtIW distribution are developed by substituting equations (8) and (9) into equation (10), and replacing h with s in the process.

$$\kappa_{r,s} = 2^{s+1}\lambda\alpha\theta\beta x^{-\beta-1}\sum_{i=0}^{\infty}\sum_{t=0}^{s}(-1)^{i+t}\binom{1+i}{i}\binom{s+t-1}{t}\int_0^\infty x^r\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(i+s+t-1)} \tag{11}$$

Consider the integral

$$\int_0^\infty x^r\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(i+s+t-1)}$$

Let $y = \alpha\lambda(i+s+t-1)\theta x^{-\beta} \Rightarrow x = \left[\frac{y}{\alpha\lambda(i+s+t-1)\theta}\right]^{\frac{-1}{\beta}}; dx = \frac{dy}{\alpha\lambda(i+s+t-1)\theta\beta x^{-\beta-1}}$

Then

$$\int_0^\infty\left[\frac{y}{\alpha\lambda(i+s+t-1)\theta}\right]^{\frac{-r}{\beta}} e^{-y}\frac{dy}{\alpha\lambda(i+s+t-1)\theta\beta x^{-\beta-1}} = \frac{1}{[i+s+t-1]^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}[\alpha\lambda]^{\frac{r}{\beta}}}\int_0^\infty y^{\frac{-r}{\beta}}e^{-y}dy$$

$$\int_0^\infty y^{\frac{-r}{\beta}}e^{-y}dy = \Gamma\left(1-\frac{r}{\beta}\right)$$

The PWMs of TIIHLEtW can be written as proceed

$$\kappa_{r,s} = \frac{2^{s+1}\sum_{i=0}^{\infty}\sum_{t=0}^{s}(-1)^{i+t}\binom{1+i}{i}\binom{s+t-1}{t}}{[i+s+t-1]^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}[\alpha\lambda]^{\frac{r}{\beta}}}\Gamma\left(1-\frac{r}{\beta}\right)$$

4.2 Moments

Since moments are crucial in any statistical analysis, particularly in practical applications, we derive the r^th moment for the new distribution.

$$\mu_r' = E(x^r) = \int_0^\infty x^r f(x)dx \tag{13}$$

By using the important representation of the pdf in equation (8), we have

$$E(X^r) = 2\lambda\alpha\theta\beta x^{-\beta-1}\sum_{i=0}^{\infty}(-1)^i\binom{1+i}{i}\int_0^\infty x^r\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(i-1)} \tag{14}$$

Consider the integral

$$\int_0^\infty x^r\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(i-1)}$$

Let $w = \alpha\lambda(i-1)\theta x^{-\beta} \Rightarrow x = \left[\frac{w}{\alpha\lambda(i-1)\theta}\right]^{\frac{-1}{\beta}}; dx = \frac{dw}{\alpha\lambda(i-1)\theta\beta x^{-\beta-1}}$

Then

$$\int_0^\infty\left[\frac{w}{\alpha\lambda(i-1)\theta}\right]^{\frac{-r}{\beta}} e^{-w}\frac{dw}{\alpha\lambda(i-1)\theta\beta x^{-\beta-1}} = \frac{1}{[i-1]^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}[\alpha\lambda]^{\frac{r}{\beta}}}\int_0^\infty w^{\frac{-r}{\beta}}e^{-w}dw = \int_0^\infty w^{\frac{r}{\beta}}e^{-w}dw = \Gamma\left(1-\frac{r}{\beta}\right)$$

The r[th] moment for TIIHLEtIW distribution can be written as a proceed

$$E(X^r)=\frac{2\sum_{i=0}^{\infty}\left(-1\right)^i\binom{1+i}{i}}{\left[i-1\right]^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}\left[\alpha\lambda\right]^{\frac{r}{\beta}}}\Gamma\left(1-\frac{r}{\beta}\right)\tag{15}$$

The mean and variance of TIIHLEtIW distribution are as follows

$$E(X)=\frac{2\sum_{i=0}^{\infty}\left(-1\right)^i\binom{1+i}{i}}{\left[i-1\right]^{\frac{1}{\beta}+1}\theta^{\frac{1}{\beta}}\left[\alpha\lambda\right]^{\frac{1}{\beta}}}\Gamma\left(\frac{\beta-1}{\beta}\right)\tag{16}$$

and

$$\operatorname{var}\left(X\right)=\frac{2\sum_{i=0}^{\infty}\left(-1\right)^i\binom{1+i}{i}}{\left[i-1\right]^{\frac{2}{\beta}+1}\theta^{\frac{2}{\beta}}\left[\alpha\lambda\right]^{\frac{2}{\beta}}}\Gamma\left(\frac{\beta-2}{\beta}\right)-\left[\frac{2\sum_{i=0}^{\infty}\left(-1\right)^i\binom{1+i}{i}}{\left[i-1\right]^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}\left[\alpha\lambda\right]^{\frac{r}{\beta}}}\Gamma\left(1-\frac{r}{\beta}\right)\right]^2\tag{17}$$

4.3 Moment-generating function (mgf)

The Moment Generating Function of x is given as:

$$M_x\left(t\right)=E\left(e^{tx}\right)=\int_0^{\infty}e^{tx}f(x)dx\tag{18}$$

where the expansion of $e^{tx}=\sum_{m=0}^{\infty}\frac{t^m x^m}{m!}$

The moment-generating function of TIIHLEtIW distribution is given by

$$M_x(t)=\frac{2\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}t^m\left(-1\right)^i\binom{1+i}{i}}{m!\left[i-1\right]^{\frac{m}{\beta}+1}\theta^{\frac{m}{\beta}}\left[\alpha\lambda\right]^{\frac{m}{\beta}}}\Gamma\left(1-\frac{m}{\beta}\right)\tag{19}$$

4.4 Reliability function

The reliability function represents the probability that a patient will survive beyond a specified period. It is defined as follows:

$$R(x;\lambda,\alpha,\theta,\beta)=\frac{1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda}}{1+\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda}}\tag{20}$$

4.5 Hazard function

The hazard function represents the probability of an event of interest occurring within a short time interval and is defined as follows:

$$T(x;\lambda,\alpha,\theta,\beta)=\frac{2\lambda\alpha\theta\beta x^{-\beta-1}e^{-\theta x^{-\beta}}\left[e^{-\theta x^{-\beta}}\right]^{\alpha-1}\left[e^{-\theta x^{-\beta}}\right]^{\alpha(\lambda-1)}}{1-\left[e^{-\theta x^{-\beta}}\right]^{2\alpha\lambda}}\tag{21}$$

4.6 Quantile Function

The quantile function is a crucial tool for generating random variables from any continuous probability distribution, making it significant in probability theory. For a given x, the quantile function is F(x) = u, where u is distributed as U(0,1). The TIIHLEtIW distribution can be easily simulated by inverting equation (5), resulting in the quantile function Q(u), defined as follows:

$$x = Q(u) = \left[\frac{-1}{\theta} \log \left[\frac{u}{2-u} \right]^{\frac{1}{\alpha\lambda}} \right]^{\frac{-1}{\beta}} \tag{22}$$

The first quartile, the median and the third quartile of TIIHLEtIW distribution are obtained by putting u = 0.25, 0.5 and 0.75, respectively in equation (22).

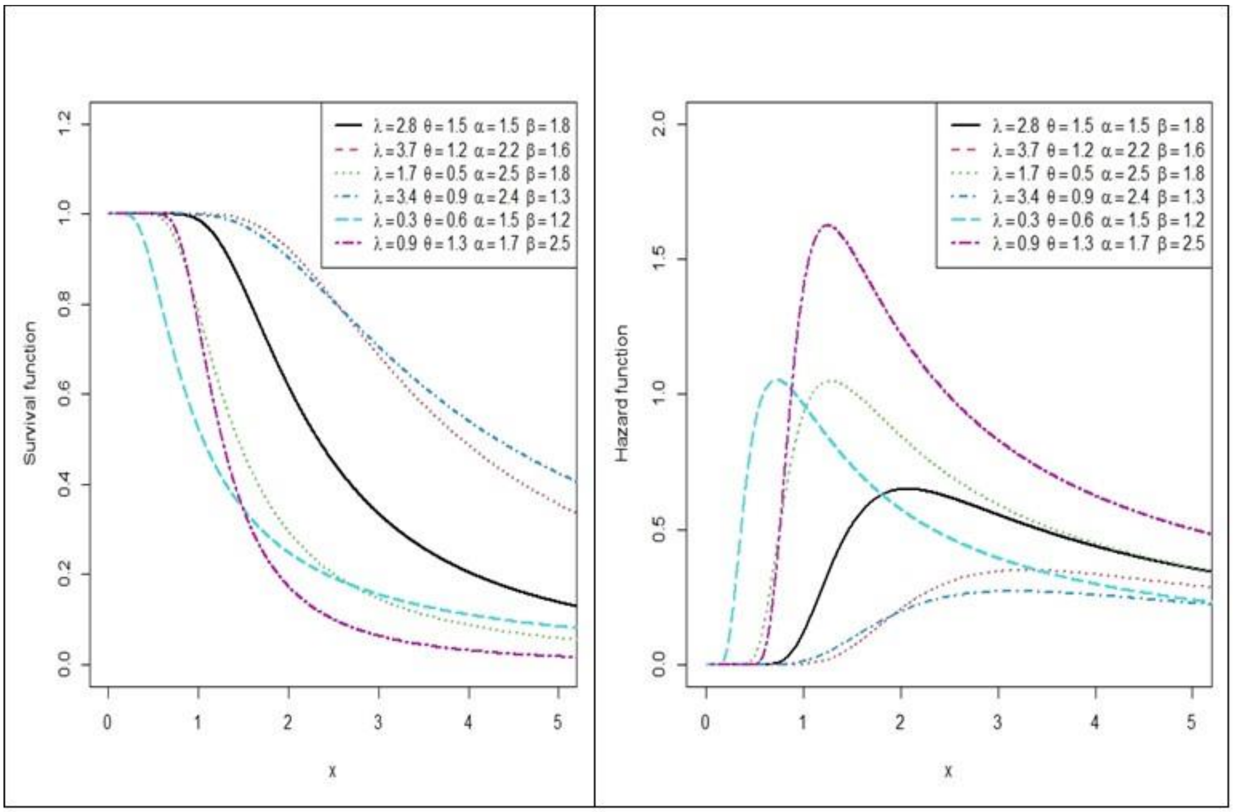

Fig. 2: Plots of reliability and hazard of the TIIHLEtIW distribution for different valves of parameters

5.0 Order Statistics

Order statistics are widely used in various statistical fields, including reliability and life testing. Consider $X_1, X_2, \ldots, X_n$ as independent and identically distributed random variables with a continuous distribution function F(x). Let $X_1, X_2, \ldots, X_n$ be n independently distributed and continuous random variables from the TIIHLEtIW distribution. Let $F_{\{r:n\}}(x)$ and $f_{\{r:n\}}(x)$, where r = 1, 2, 3,…, n, denote the cdf and pdf of the r^{th} order statistic $X_{\{r:n\}}$, respectively. According to David (1970), the probability density function of $X_{\{r:n\}}$ is given by:

$$f_{r:n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \tag{23}$$

Substituting equation (8) and equation (9) into equation (23), also replacing h with v+r-1 in equation (9). We have

$$f_{r:n}(x;\lambda,\alpha,\theta,\beta)=\frac{2^{v+r}\lambda\alpha\theta\beta x^{-\beta-1}}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i=0}^{\infty}\sum_{t=0}^{v+r-1}(-1)^{v+i+t}\binom{n-r}{v}\binom{1+i}{i}\binom{v+r+t-2}{t}\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(i+v+r+t-2)} \quad (24)$$

The equation above is called the rth order statistics for the TIIHLEtIW distribution.
Let r = n, then the probability density function of the maximum order statistics of TIIHLEtW distribution is

$$f_{n:n}(x;\lambda,\alpha,\theta,\beta)=2^{v+n}n\lambda\alpha\theta\beta x^{-\beta-1}\sum_{i=0}^{\infty}\sum_{t=0}^{v+n-1}(-1)^{v+i+t}\binom{1+i}{i}\binom{v+n+t-2}{t}\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(i+v+n+t-2)} \quad (25)$$

Also, let r = 1, then the probability density function of the minimum order statistics of TIIHLEtW distribution is

$$f_{1:n}(x;\lambda,\alpha,\theta,\beta)=2^{v+1}n\lambda\alpha\theta\beta x^{-\beta-1}\sum_{v=0}^{n-1}\sum_{i=0}^{\infty}\sum_{t=0}^{v}(-1)^{v+i+t}\binom{n-1}{v}\binom{1+i}{i}\binom{v+t-1}{t}\left[e^{-\theta x^{-\beta}}\right]^{\alpha\lambda(i+v+t-1)} \quad (26)$$

6.0 Parameter Estimation

We examine the maximum likelihood method for estimating the unknown parameters of the TIIHLEtIW distribution using complete data. Maximum likelihood estimates (MLEs) are appealing because they can generate confidence intervals and provide straightforward approximations that perform well with finite samples. The approximation for MLEs is easy to manage in distribution theory, both analytically and numerically. Let $x_1, x_2, x_3,...,x_n$ be a random sample of size n from the TIIHLEtIW distribution. The likelihood function based on the observed sample for the vector of parameters $(\lambda, \alpha, \theta, \beta)^T$ is given by:

$$\begin{aligned}\log L &= n\log(2)+n\log(\lambda)+n\log(\alpha)+n\log(\theta)+n\log(\beta)-(\beta-1)\sum_{i=1}^{n}\log(x_i)\\ &\quad -\theta\alpha\sum_{i=1}^{n}x_i^{-\beta}-\theta\alpha(\lambda-1)\sum_{i=1}^{n}x_i^{-\beta}-2\sum_{i=1}^{n}\log\left[1+\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha\lambda}\right]\end{aligned} \quad (27)$$

The components of score vector $\Delta L(\phi)=\left(\frac{\partial L(\phi)}{\partial\lambda},\frac{\partial L(\phi)}{\partial\alpha},\frac{\partial L(\phi)}{\partial\theta},\frac{\partial L(\phi)}{\partial\beta}\right)^T$ are given as

$$\frac{\partial\log L}{\partial\lambda}=\frac{n}{\lambda}-\theta\alpha\sum_{i=1}^{n}x_i^{-\beta}-2\sum_{i=1}^{n}\frac{\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha\lambda}\log\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}}{\left[1+\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha\lambda}\right]} \quad (28)$$

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha}-\theta\sum_{i=1}^{n}x_i^{-\beta}-\theta(\lambda-1)\sum_{i=1}^{n}x_i^{-\beta}-2\sum_{i=1}^{n}\frac{\lambda\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha(\lambda-1)}\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\log\left[e^{-\theta x_i^{-\beta}}\right]}{\left[1+\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha\lambda}\right]} \quad (29)$$

$$\frac{\partial\log L}{\partial\theta}=\frac{n}{\theta}-\alpha\sum_{i=1}^{n}x_i^{-\beta}-\alpha(\lambda-1)\sum_{i=1}^{n}x_i^{-\beta}-2\sum_{i=1}^{n}\frac{\lambda\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha(\lambda-1)}\alpha\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha-1}e^{-\theta x_i^{-\beta}}x_i^{-\beta}}{\left[1+\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha\lambda}\right]} \quad (30)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log(x_i) - \theta\alpha \sum_{i=1}^{n} x_i^{-\beta} \log(x_i) - \theta\alpha(\lambda - 1) \sum_{i=}^{n} x_i^{-\beta} \log(x_i)$$
$$-2\sum_{i=1}^{n} \frac{\lambda\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha(\lambda-1)} \alpha\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha-1} e^{-\theta x_i^{-\beta}} \theta x_i^{-\beta} \log(x_i)}{\left[1 + \left[e^{-\theta x_i^{-\beta}}\right]^{\alpha\lambda}\right]} \tag{31}$$

The MLEs are determined by setting $\frac{\partial L(\phi)}{\partial \lambda}, \frac{\partial L(\phi)}{\partial \alpha}, \frac{\partial L(\phi)}{\partial \theta}$ and $\frac{\partial L(\phi)}{\partial \beta}$ to zero and solving them simultaneously. Since these equations cannot be solved analytically, numerical methods must be employed.

6.0 Simulation Study

In this section, a numerical analysis will be performed to assess the performance of MLE for the TIIHLEtIW distribution. Table 1 demonstrates that as the sample size increases, the values of biases and RMSEs approach zero, and the estimates converge to the true values. This indicates that the estimates are efficient and consistent.

Table 1: MLEs, biases and RMSE for some values of parameters

		(1.5,1.5,3,3)			(2,0.5,1.5,1.5)		
n	Parameters	Estimated Values	Bais	RMSE	Estimated Values	Bais	RMSE
	λ	1.6002	0.1002	0.4997	2.0160	0.0160	0.1818
20	α	1.7487	0.2487	0.6059	0.5089	0.0089	0.1188
	θ	3.0310	0.0310	0.4781	1.5017	0.0017	0.1504
	β	3.2805	0.2805	0.7360	1.6413	0.1413	0.3692
	λ	1.5489	0.0489	0.2960	2.0606	0.0606	0.1179
50	α	1.6186	0.1186	0.3361	0.5074	0.0074	0.0617
	θ	3.0533	0.0533	0.3549	1.5093	0.0093	0.0904
	β	3.1103	0.1103	0.4007	1.5558	0.0558	0.2018
100	λ	1.5358	0.0358	0.2062	2.0726	0.0726	0.0951
	α	1.5572	0.0572	0.2207	0.5039	0.0039	0.0434
	θ	3.0678	0.0678	0.2588	1.5161	0.0161	0.0646
	β	3.0545	0.0545	0.2570	1.5275	0.0275	0.1291
250	λ	1.5239	0.0239	0.1280	2.0746	0.0746	0.0860
	α	1.5101	0.0101	0.1324	0.5017	0.0017	0.0324
	θ	3.0839	0.0839	0.1812	1.5201	0.0201	0.0522
	β	3.0158	0.0158	0.1575	1.5079	0.0079	0.0791
500	λ	1.5014	0.0014	0.0858	2.0705	0.0705	0.0773
	α	1.5058	0.0058	0.0906	0.5024	0.0024	0.0277
	θ	3.0775	0.0775	0.1306	1.5188	0.0188	0.0421
	β	3.0064	0.0064	0.1110	1.5032	0.0032	0.0556

1000	λ	1.5020	0.0020	0.0677	2.0672	0.0672	0.0736
	α	1.5109	0.0109	0.0712	0.5012	0.0012	0.0256
	θ	3.0501	0.0501	0.0885	1.5164	0.0164	0.0368
	β	3.0044	0.0044	0.0753	1.5022	0.0022	0.0377

6.1 Applications to Real Data

We fit the TIIHLEtIW distribution to two real datasets and conducted a comparative study with several other distributions: the Extended Inverse Weibull (TIHLIW) Distribution by Alkarni *et al.,* (2020), the Marshall-Olkin Extended Inverse Weibull (MOIW) Distribution by Pakungwati *et al.,* (2018), the Generalized Inverse Weibull (GIW) Distribution by De Gusmao *et al.,* (2011), the Kumaraswamy–Inverse Weibull (KIW) Distribution by Shahbaz *et al.,* (2012), and the Inverse Weibull (IW) Distribution by Keller and Kanath (1982). This comparison serves to demonstrate the performance of the TIIHLEtIW distribution.

The TIHLIW distribution developed by Alkarni *et al.,* (2020) has pdf defined as:

$$f(x;\alpha,\lambda,\beta)=\frac{2\lambda\alpha\beta x^{-\beta-1}\exp\left(-\alpha x^{-\beta}\right)\left(1-\exp\left(-\alpha x^{-\beta}\right)\right)^{\lambda-1}}{\left[1+\left(1-\exp\left(-\alpha x^{-\beta}\right)\right)^{\lambda}\right]^{2}} \tag{32}$$

The MOIW distribution developed by Pakungwati *et al.,* (2018) has pdf defined as:

$$f(x;\beta,\alpha,\theta)=\frac{\alpha\beta\theta^{-\beta}x^{-\beta-1}\exp\left(-(x\theta)^{-\beta}\right)}{\left(\alpha-(\alpha-1)\exp\left(-(x\theta)^{-\beta}\right)\right)^{2}} \tag{33}$$

The GIW distribution proposed by De Gusmao *et al.,* (2011) has pdf given as:

$$f(x;\alpha,\lambda,\beta)=\lambda\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right) \tag{34}$$

The KIW Distribution proposed by Shahbaz *et al.,* (2012) has pdf given as:

$$f(x;\lambda,\alpha,\theta,\beta)=\frac{\lambda\theta\alpha\beta}{x^{\beta+1}}\exp\left(-\frac{\lambda\alpha}{x^{\beta}}\right)\left(1-\exp\left(-\frac{\lambda\alpha}{x^{\beta}}\right)\right)^{\theta-1} \tag{35}$$

The IW distribution developed by Keller and Kanath (1982) has pdf defined as:

$$f(x;\alpha,\lambda)=\alpha\lambda x^{-\lambda-1}\exp\left(-\alpha x^{-\lambda}\right) \tag{36}$$

The two datasets used as illustrations in the application demonstrate the new proposed distribution's flexibility, applicability, and superior fit when modelling the datasets empirically compared to the above comparator distributions. All calculations are executed using the R programming language.

Data set 1

The first data set below represents the tensile strength of carbon fibers, as previously utilized by Akanji *et al.,* (2023):

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

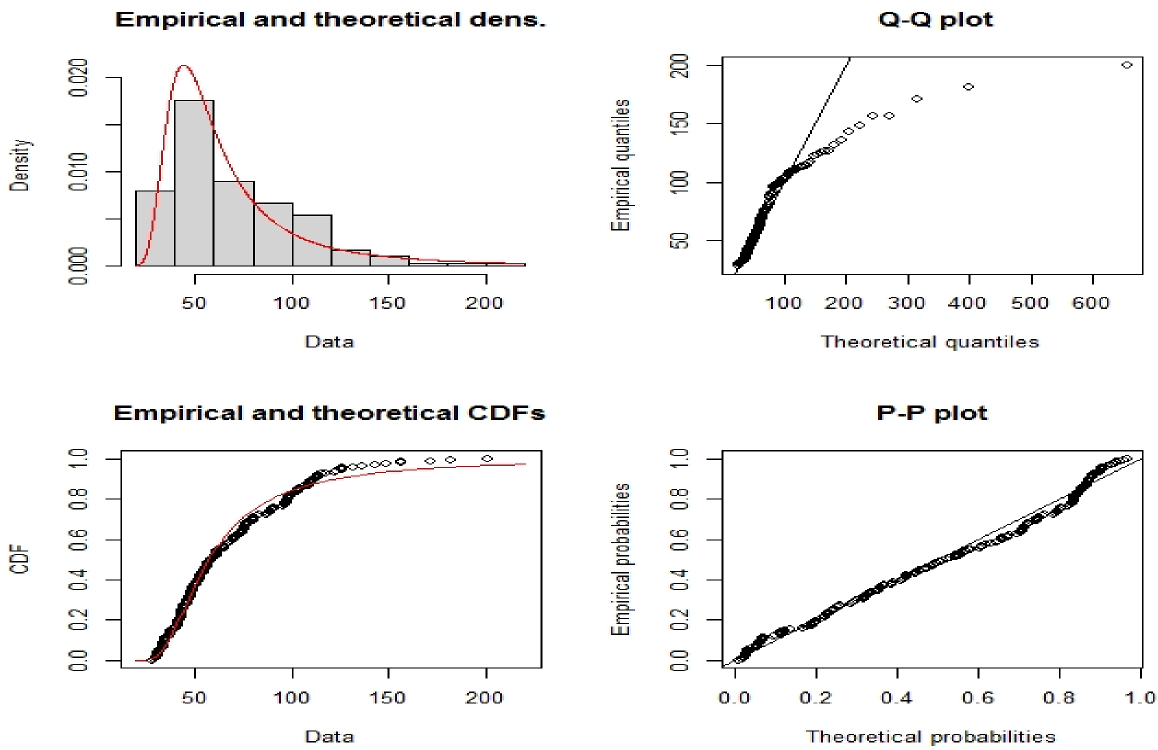

Fig. 3: Empirical and theoretical pdfs and cdfs, Q-Q and P-P plots for data set 1

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	α	λ	θ	β	LL	AIC
TIIHLEtIW	3.0022	1.6079	0.8392	1.5099	- 172.9524	353.9048
TIHLIW	4.2502	4.5785	-	1.1657	-177.6114	361.2228
KIW	2.1516	6.1424	2.5170	1.0249	-174.8953	357.7906
GIW	1.3121	1.7737	-	1.9059	-178.4966	362.9932
MOIW	4.7410	-	0.8693	2.3922	-179.2674	364.5348
IW	3.0856	1.7737	-	-	-182.4966	368.9932

The parameters of the new proposed distribution and five comparator distributions were estimated using maximum likelihood, with the results shown in Table 2. Based on the AIC goodness-of-fit measure, the new proposed distribution achieved the lowest value, although the KIW distribution was a close second. The superiority of the proposed distribution is further supported by empirical and theoretical pdfs and cdfs, as well as the Q-Q and P-P plots visual examination shown in Fig. 3. Therefore, the newly proposed distribution is considered the most suitable fit for the carbon fibers data set among the distributions evaluated.

Data set 2

The second data set presented below pertains to civil engineering data involving hailing times, as previously studied by Akanji *et al.*, (2023):

4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80.

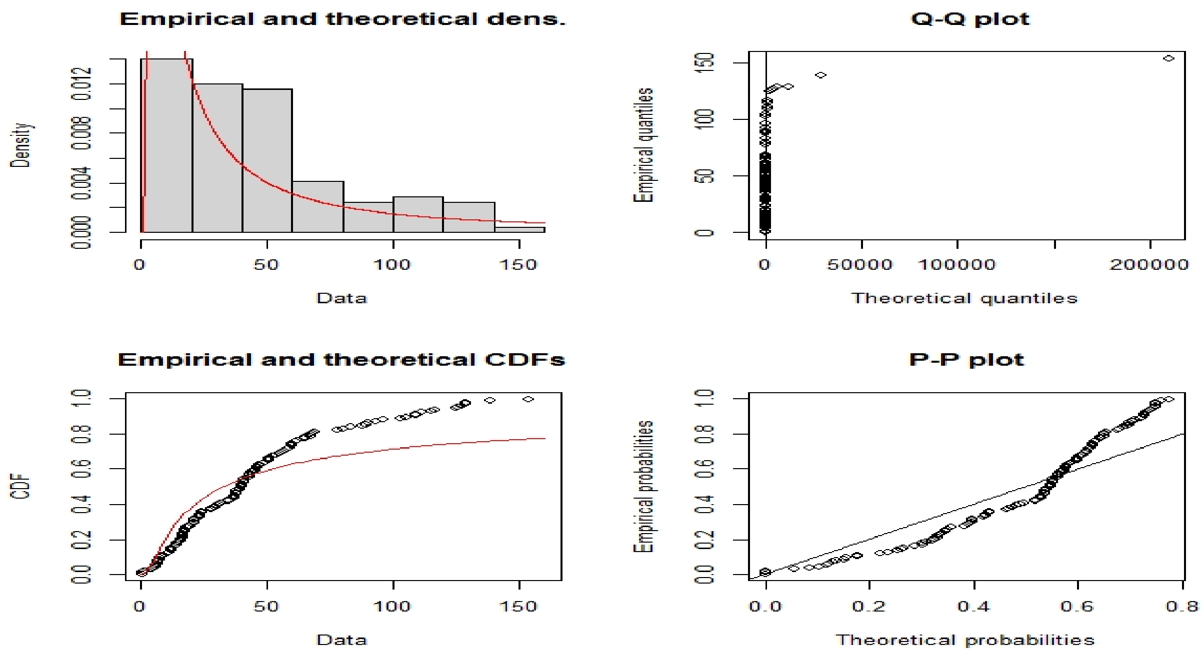

Fig. 4: Empirical and theoretical pdfs and cdfs, Q-Q and P-P plots for data set 2

Table 3: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	α	λ	θ	β	LL	AIC
TIIHLEtIW	2.0782	6.2267	0.0033	0.3960	-123.8043	255.6085
TIHLIW	24.7313	12.8858	-	1.3409	-139.265	284.5299
KIW	11.6924	14.5415	5.1887	1.7337	-134.0294	276.0588
GIW	3.6026	4.2042		4.4080	-146.009	298.018
MOIW	2.4732	-	0.2318	3.8414	-151.3612	308.7225
IW	4.2340	4.4077	-	-	-146.009	296.018

The parameters of the TIIHLEtIW distribution and five comparator distributions were estimated using maximum likelihood, as shown in Table 3. According to the goodness-of-fit measure AIC, the new distribution achieved the lowest value, indicating it is the best fit for the hailing times data set. The new distribution's superiority is further confirmed by the visual examination of the empirical and theoretical pdfs, cdfs, Q-Q plots, and P-P plots, as depicted in Fig. 4.

7.0 Conclusion

The study introduces the Type II Half-Logistic Inverse Weibull (TIIHLEtIW) distribution, an extension of the Inverse Weibull (IW) distribution that incorporates two additional shape parameters to enhance flexibility and adaptability in modeling data. This novel distribution is derived from the Type II Half-Logistic Exponentiated-G (TIIHLEt-G) family. Key statistical properties of the TIIHLEtIW distribution, such as the cumulative distribution function (cdf), probability density function (pdf), moments, moment-generating function, reliability function, hazard function, and quantile function, are thoroughly discussed. Parameter estimation is carried out using maximum likelihood estimation (MLE), and a simulation study is conducted to assess the performance of the MLEs. The practical applicability and superiority of the TIIHLEtIW distribution are demonstrated through a comparative analysis using two real datasets, revealing a better fit compared to several established distributions.

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Compliance with Ethical Standards Declaration

Ethical Approval

Not Applicable

Competing interests

The authors declare that they have no known competing financial interests

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Availability of data and materials

Data would be made available on request.

Authors Contribution

This research was a collaborative effort among all authors. Author Yakubu Isa was responsible for the study design, statistical analysis, protocol development, and drafting the initial manuscript. Authors Radiya Muhammad Said, Juliet Wallen Piapna, and Abdulhaq Bashir oversaw the data analysis and conducted the literature review. All authors reviewed and approved the final version of the manuscript.