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Study of Symmetric Nuclear Matter Properties in Non-linear Walecka Model via Relativistic Mean-field approximation at zero-temperature

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Abstract: *Symmetric nuclear matter at zero temperature were analysed. The equations of state (EOS) of symmetric nuclear matter were studied in the non-linear Walecka models at different parameterizations. At normal nucleon density, strong correlations were observed among the different parameter sets, however the linear Walecka model gives values of nucleon effective mass* M_0^* *and nuclear incompressibility* (K) *at variance to the experimental values.The calculated values of saturation density ranges from 0.143* fm^{-3} *to 0.152* fm^{-3} *, nucleon effective mass 0.132 MeV to 0.157 MeV, binding energy per nucleon -16.01 MeV to -16.20 MeV, compression modulus 223.55 MeV to 271.36 MeV, and fermi-wavelength 1.30* fm^{-1} *to 1.31* fm^{-1} *for the non- linear Walecka model (NLWM). The results of the numerical computations were compared with the empirical analysis of the giant iso-scalar monopole resonance data. These quantities are important for understanding the structure of finite nuclei and neutron stars. The quantities have substantially described equation of state of other dense matter in astrophysical contexts.*

Keywords: *Symmetric nuclear matter, Lagrangian density, non-linear-Walecka model, relativistic mean field theory, equation of state, zero-temperature.*

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1.0 Introduction

The non-linear Walecka model (NLWM) is a relativistic quantum field theoretical framework used for describing nuclear matter properties (Bhattacharyya & Ghosh, 2012); (Aper *et al.*, 2019). At zero temperature, it incorporates interactions between nucleons mediated by scalar and vector mesons (Chung *et al.*, 2008). The lagrangian density for this model includes terms for nucleons (protons and neutrons), Scalar mesons (σ), vector mesons (ω), and self-interaction terms for the sigma meson field. This is aimed at addressing some limitations of the original linear Walecka model thereby providing a more accurate description of nuclear matter properties at high densities. Like the linear model, the equations of motion (EoM) for the various fields are derived from the Lagrangian density which involves contributions from the scalar and vector fields (Walecka, 2004); (Oppenheimer, 1939). Also, in the mean field approximation, the meson fields are replaced by their expectation values which provides the equations of state (EoS) for the symmetric nuclear matter. The equations of state (EOS) relates the energy density, pressure density to the baryon density which is crucial for understanding the properties of neutron stars and heavy-ion collision experiments (Sumiyoshi *et* al., 2019; Faise, 2011).This model will help to provide saturation properties of nuclear matter such as binding energy per nucleon, the nuclear matter incompressibility, Symmetry energy and the nucleon effective mass (Passamani, 2012). In an earlier attempt to study nuclear matter properties within the framework of quantum hydrodynamics (QHDI), Walecka and other co-workers were able to describe the saturation and other properties of nuclear matter using the well-studied linear σ-ω model (Patrigani, 2016); (Schmitt, 2010). However, the non-linear Walecka model yields nuclear incompressibility values (Ko) of around 550MeV. The value is unacceptably high and again the effective nucleon mass M* around 0.54M which seems too low (Chung *et al*., 2008); (Da-Silva, 2013); (Francesco, 2017), (Gambhir, 1989); (Parmer *et al*., 2023); and (Patrigani, 2016), hence the introduction of the non-linear model.

2.0 The Formalism of the Non-Linear Model

The non-linear Walecka model is otherwise known as the quantum hydrodynamic II (QHD II). It is a relativistic quantum field theory just like the linear Walecka model used for describing the main features of the nucleon-nucleon and nucleon-meson interactions. This model is characterized by the following lagrangian density (Parmer *et al*., 2023):

$$\mathcal{L} = \bar{\psi}\left[i\gamma_\mu\,(\partial^\mu + ig_\omega\omega^\mu) - (m - g_\sigma\sigma)\right]\psi + \frac{1}{2}\left[(\partial_\mu\sigma)\,(\partial^\mu\sigma) - m_\sigma^2\sigma^2\right]$$

$$-\frac{1}{4}\omega^{\mu\upsilon}\omega_{\mu\upsilon} + \frac{1}{2}m_\omega^2\omega^\mu\omega_\mu - \frac{1}{3}m_n b(g_\sigma\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4 \qquad (1)$$

where:

- ψ is nucleon field.
- σ is the sigma field with mass m_σ
- ω (the omega field), with mass m_ω
- g(σ) and g(ω) are the respective coupling constants for the nucleon-sigma and nucleon-omega interactions.
- b and c are coefficients of the non-linear sigma meson self-interaction terms.

The scalar self-interaction term is non-linear made up of cubic and quartic polynomials defined by the potential:

$$U(\sigma) = \frac{1}{3}m_n b(g_\sigma\sigma)^3 + \frac{1}{4}c(g_\sigma\sigma)^4 \qquad (2)$$

Where b and c are dimensionless constants and $m_n = 939\ MeV$ thought to be a mass equal to that of a neutron.

The equations of motion of the meson fields are obtained using the Euler-Lagrange equation (Diener, 2010); Francesco, 2017):

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0 \qquad (3)$$

Substituting equation (1) into equation (3), the meson field equations are obtain as follows:

$$\frac{\partial(\mathcal{L}_\sigma + \mathcal{L}_{int} - U(\sigma)}{\partial\sigma} = -m_\sigma^2\sigma(x) + g_\sigma(\bar{\psi}\psi - m_n b\big(g_\sigma\sigma(x)\big)^2 - c(\big(g_\sigma\sigma(x)\big)^3 \qquad (4)$$

So that after imposing mean-field procedures, equation (4), turns out to:

$$m_\sigma^2\langle\sigma\rangle = g_\sigma\left(\langle\bar{\psi}\psi\rangle\right) - m_n b\left(g_\sigma\langle\sigma\rangle\right)^2 - C\left(g_\sigma\langle\sigma\rangle\right)^3 \qquad (5)$$

Recalling the expression for the computed $\langle\bar{\psi}\psi\rangle$, in terms of $g_\sigma\langle\sigma\rangle$ which becomes:

$$g_\sigma\langle\sigma\rangle = \left(\frac{g_\sigma}{m_\sigma}\right)^2\left[\begin{array}{c} -m_n b\left(g_\sigma\langle\sigma\rangle\right)^2 - C\left(g_\sigma\langle\sigma\rangle\right)^3 + \\ \frac{2}{\pi^2}\int_0^{p_F} dp\ \frac{p^2\left(m - g_\sigma\langle\sigma\rangle\right)}{\sqrt{p^2 + \left(m - g_\sigma\langle\sigma\rangle\right)}} \end{array}\right] \qquad (6)$$

The expectation value of the Lagrangian also modified as:

$$\langle\mathcal{L}\rangle = -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 - \frac{1}{3}m_n b\left(g_\sigma\langle\sigma\rangle\right)^3 - \frac{1}{4}C\left(g_\sigma\langle\sigma\rangle\right)^4 \qquad (7)$$

3.0 Energy density and Pressure for the non-linear Walecka model

The energy(ε) and pressure (P) for the expectation values are in the rest frame are on the diagonal of the matrix form.

$$T_{\mu\upsilon} = T^{\mu\upsilon} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \qquad (8)$$

But by definition, the energy-momentum tensor is given by (Diener, 2010); Francesco, 2017) as:

$$T_{\mu\upsilon} = \eta_{\mu\upsilon}L - \frac{\partial L}{\partial(\partial^\mu\phi_i)}\partial_v\phi_i \qquad (9)$$

Where ϕ_i represents an arbitrary field example $\psi, \sigma, \omega -$ fields etc. with the Lagrangian for ψ nucleons in momentum space.

$$T_{\mu\upsilon} = \eta_{\mu\upsilon} L - \frac{\partial L}{\partial(\partial^{\mu}\psi)} \partial_{v}\psi \quad (10)$$

From the energy-momentum tensor, the energy and pressure densities are obtained respectively by (Parmer, 2019) as:

$$\varepsilon = -\langle \mathcal{L} \rangle + \langle \bar{\psi}\gamma_0 p_0 \psi \rangle \quad (11)$$

$$P = \langle \mathcal{L} \rangle + \frac{1}{3} \langle \bar{\psi}\gamma_i p_i \psi \rangle \quad (12)$$

Evaluating the above expectation values, we have

$$(\bar{\psi}\gamma_0 P_0 \psi) = \frac{2}{\pi^2} \int \partial P \; P^2 \sqrt{P^2 + (m - g_{\sigma}\langle\sigma\rangle)^2} \quad (13)$$

$$(\bar{\psi}\gamma_i P_i \psi) = \frac{1}{\pi^2} \int_0^{p_F} p^2 dp \frac{p^2}{\sqrt{P^2 + (m - g_{\sigma}\langle\sigma\rangle)^2}} \quad (14)$$

Substituting equations (7), (13) and (14) into equation (11) and (12) respectively, the equations of state (EoS) for the non-linear Walecka model with the self-interaction term are obtained as (Von-Maco, 2018):

$$\epsilon = \frac{1}{2} m_{\sigma}^2 \langle\sigma\rangle^2 + \frac{1}{2} m_{\omega}^2 \langle\omega_0\rangle^2 + \frac{1}{3} m_n b \left(g_{\sigma}\langle\sigma\rangle\right)^3 + \frac{1}{4} C \left(g_{\sigma}\langle\sigma\rangle\right)^4 + \frac{2}{\pi^2} \int_0^{p_F} dp \; p^2 \sqrt{p^2 + \left(m - g_{\sigma}\langle\sigma\rangle\right)^2} \quad (15)$$

Therefore

$$P = -\frac{1}{2} m_{\sigma}^2 \langle\sigma\rangle^2 + \frac{1}{2} m_{\omega}^2 \langle\omega_0\rangle^2 - \frac{1}{3} m_n b \left(g_{\sigma}\langle\sigma\rangle\right)^3 + \frac{1}{4} C \left(g_{\sigma}\langle\sigma\rangle\right)^4 + \frac{2}{3\pi^2} \int_0^{p_F} dp \frac{p^4}{\sqrt{p^2 + (m - g_{\sigma}\langle\sigma\rangle^2}} \quad (16)$$

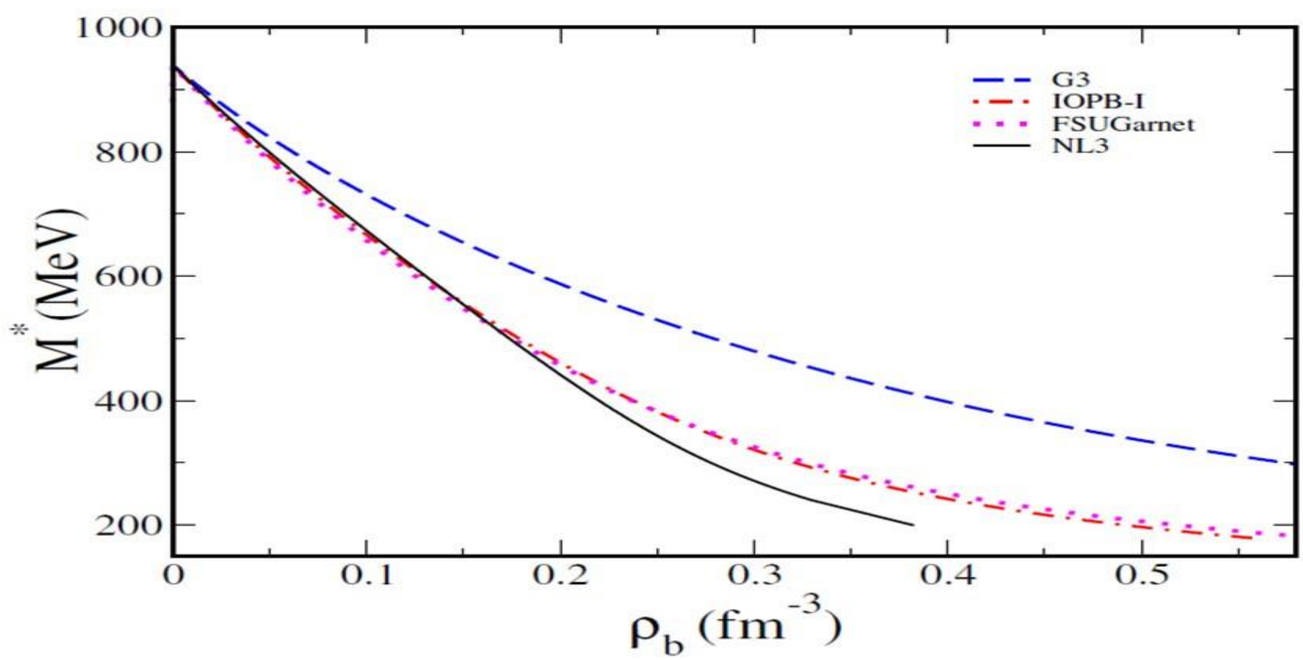
4.0 Numerical results and Discussions

In Fig. (1) we plotted the nuclear matter effective mass as a function of the baryon density for all the parameter sets. We noticed that the G3, FSUGarnet and IOPB-1 parameter set underestimate the EoS as shown by the NL3 set. These parameter sets showed similar behavior due to the fact that they share the same structure of couplings (Table1).The baryon effective mass decreases exponentially as density increases among the force parameters. This is because the solution of the self-consistent equation (Mpantis, 2020) will always yield a solutions of effective mass (M*) which is a decreasing function of the baryon density (Von-Maco, 2018; Mpantis, 2020). This pattern of monotonic decrease arises from the interaction of large condensed scalar field ($g_{\sigma}\sigma$) which is attractive and a large repulsive energy per baryon component coming from the vector field ($g_{\omega}\omega$).

The readiness for the NL3 set to over-estimate the EoS is well observed in the effective mass as a function of baryon density curve (Fig.1). This is because the effective masses determined the values of both the scalar and vector potentials through the self-consistent equation for scalar density. The plots of the binding energy of nuclear matter as a function of baryon density and Fermi-wavelength are also displayed in fig.(3).The binding energy per nucleon was estimated based on the force parameters to be about -16.41MeV at the saturation density of approximately 0.14 fm^{-3} and Fermi-wavelength (K_F^0=1.40 fm^{-1}). These results are within the range obtained in other literatures (Dhiman *et al*., 2007; Krane, 1988 and Sumiyoshi *et al*., 2019).The above results is an indication that nuclear matter is considered as a Fermi degenerate gas at super high density (Krane, 1988 and Sumiyoshi *et al*., 2019). Furthermore, it was noticed that fig. (3) depicted the softness of the G3 parameter set and the stiffness of NL3. Thus, the NL3 set is not a good tool for nuclear matter studies at super high density. It was observed that the symmetric nuclear matter is a dilute fermi system where the particles (nucleons) are interacting in a strongly repulsive potentials at short distances. The saturated values of the fermi momentum, density, binding energy, nand uclear incompressibility are in good agreement with accepted experimental values (Antic, 2015; Bhattacharyya & Ghosh, 2012). Values of the nuclear matter incompressibility among the various parameter sets are further enhanced and lowered as expected based on results obtained by other researchers due to the inclusion of non-linear scalar potential to the original linear langrangian density. Fig. (4) Depicted the binding energy versus baryon density for the various force parameters at different nuclear asymmetry parameter (α). For symmetric nuclear matter, α=0 (Mpantis, 2020; Parmer *et al*., 2023).

Here, nucleons are seriously bound at saturation point. It was noticed that boundness becomes weaker as the degree of asymmetry tends towards unity, while the energy per nucleon decreases as density increases. There is a substantial transition between symmetric nuclear matter (SNM) to pure neutron matter (PNM), at this points, the cusps or the pockets of the bound states begin to disappear. Increasing the asymmetry coefficient, the EoS become stiff which might become stiffer in high temperature studies. Here, it was noticed that at high densities, the system become unbound with the condition that $E/B > M$. Also at intermediate densities the attractive scalar interaction will dominate and the system will saturate, the relativistic nature of the scalar and vector fields that is responsible for this saturation. From the observed trends of behavior, these parameter sets can be used to explore the mass-radius profile of neutron stars with the aid of the well-known Tolman-Oppenheimer-Volkoff (TOV) equation for simulating the sites of gravitational waves strain, neutron star mergers, core-collapse supernovas and many more. (Chin, 1974; Dhiman *et al*., 2007; Gil *et al*., 2023; Ilona, 2007 and Oppenheimer, 1939). Thus, the force parameters IOPB-1, G3 and FSUGarnet can be used for estimating astrophysical properties of objects oscillating at supernormal densities (Sumiyoshi *et al*., 2019; Parmer *et al*., 2023).

Symmetric nuclear matter observables at zero temperature depicting nuclear matter incompressibility, nucleon effective mass, binding energy, and saturation density for these force parameters for the non-linear Walecka model is displayed in Table .2. The calculated values of saturation density ranges from (0.143-0.152) fm^{-3} nucleon effective mass (0.132-0.157) MeV, binding energy per nucleon (-16.01 to -16.20) MeV, compression modulus (223.55-271.36) MeV, and fermi-wavelength (1.30-1.31) fm^{-1} for the non- linear Walecka model (NLWM) in Table.2.



F 1: Self-consistent effective masses of nucleon as a function of baryon density for different parameter sets at T = 0 in the NLWM

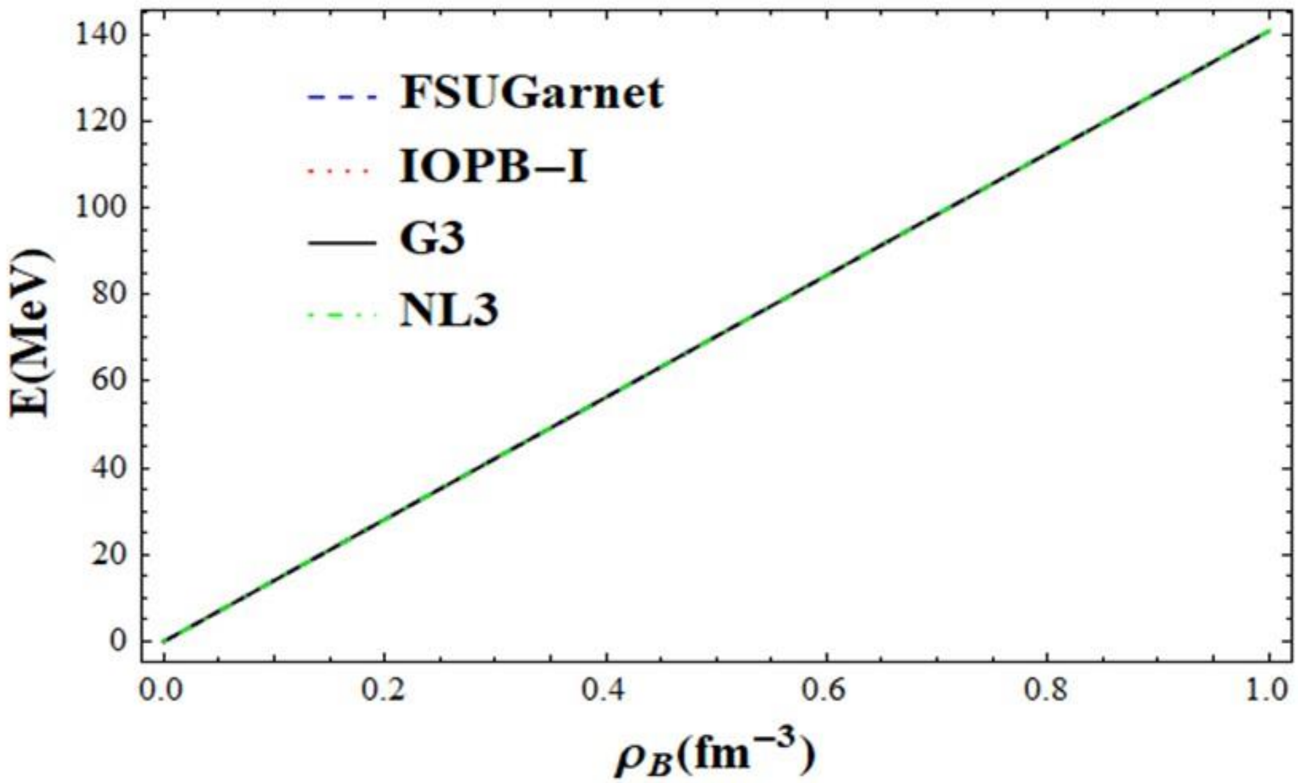


Fig. 2: Energy density against baryon density for NLWM at T= 0 for the parameter sets

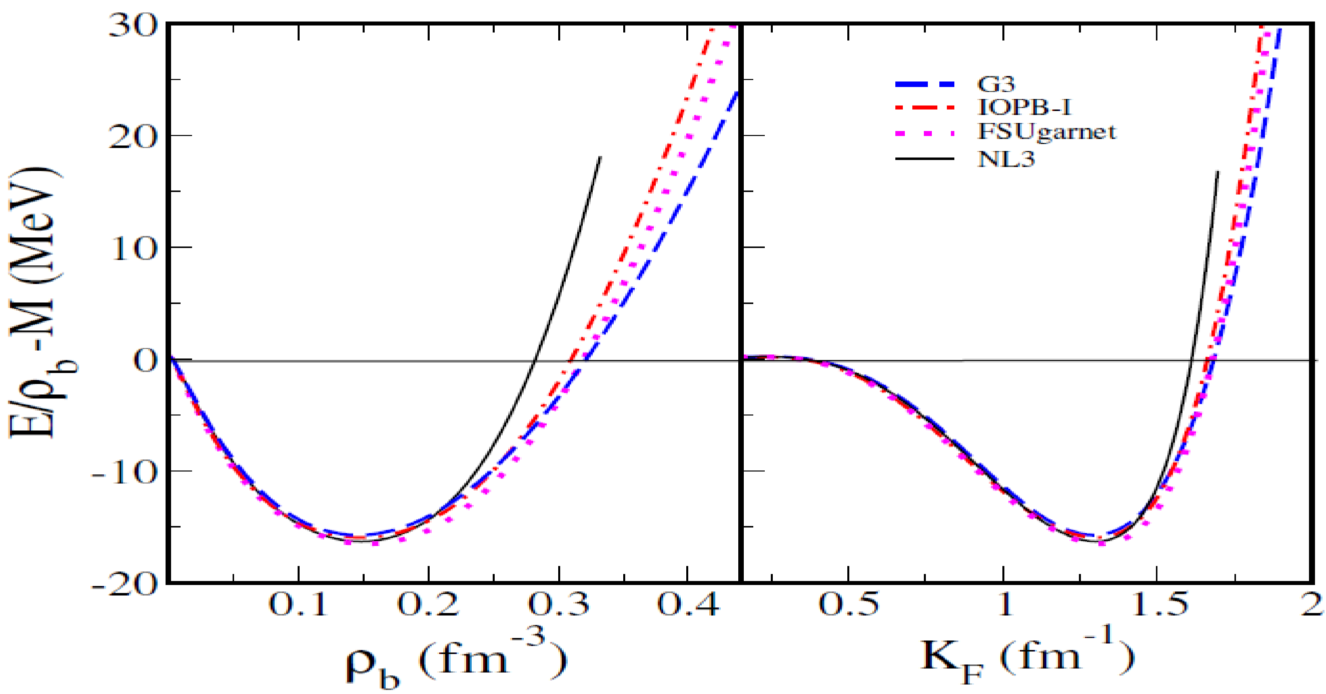


Fig. 3: Binding energy as a function of baryon density and Fermi-wavelength for the differentParameter sets using the NLWM

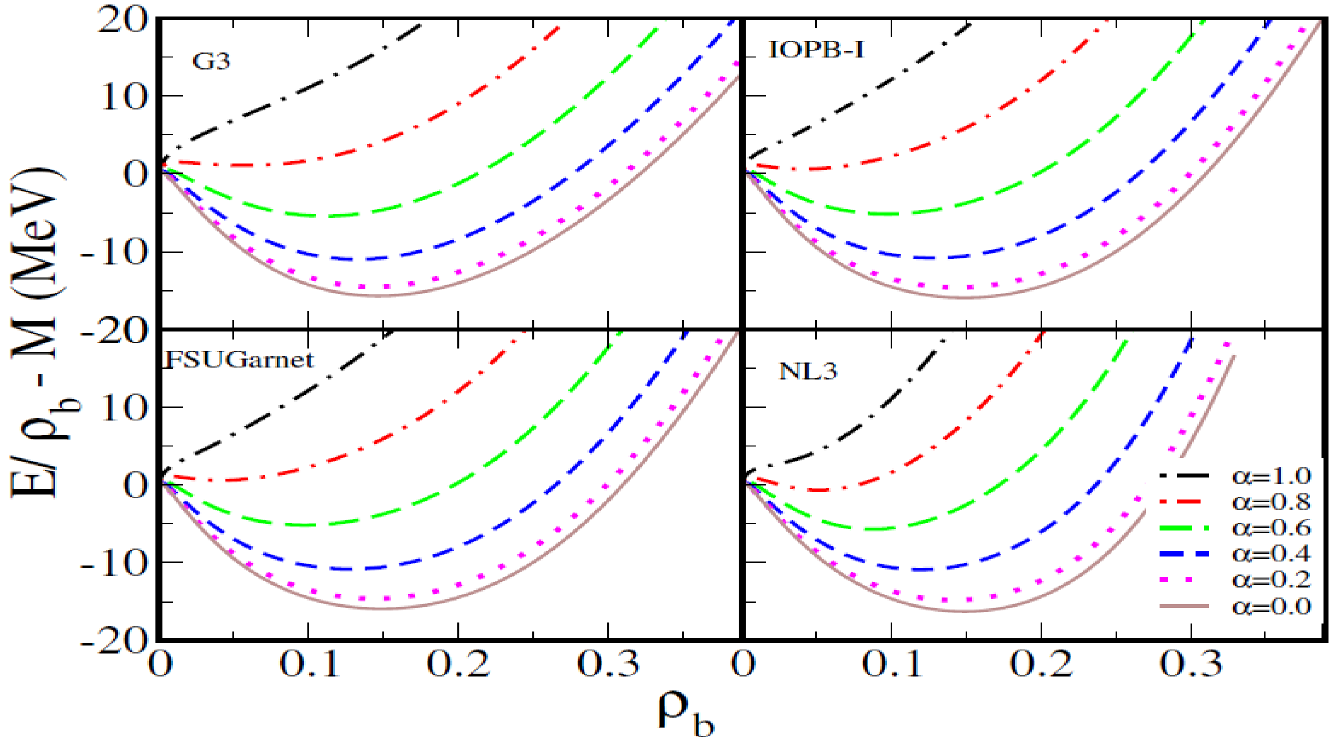


Fig. 4: Binding energy as a function of baryon density and Fermi-wavelength for the different Parameter sets for G3, IOPB-I, FSUGamet and NL3

Table 1: Parameter sets for the model. Nucleon mass is taken as 939MeV

	FSUGarnet	IOPB-1	G3	NL3
m_σ/M	0.529	0.533	0.559	0.541
m_ω/M	0.833	0.833	0.833	0.833
m_ρ/M	0.812	0.812	0.820	0.812
m_δ/M	0.000	0.000	1.043	0.000
$g_\sigma/4\pi$	0.837	0.827	0.782	0.813
$g_\omega/4\pi$	1.091	1..062	0.923	1.024
$g_\rho/4\pi$	1.105	0.885	0.962	0.712
$k_3(fm^{-1})$	1.368	1.496	2.606	1.465
k_4	-1.397	-2.932	1.694	-5.688
ς_0	4.410	3.103	1.010	0.000
η_1	0.000	0.000	0.424	0.000
η_2	0.000	0.000	0.114	0.000
	0,000	0.000	0.645	0.000
	0.043	0.024	0.038	0.000

Table 2: Calculated Nuclear Matter Observables for N LWM at zero temperature

	FSUGarnet	IOPB-1	G3	NL3
$\rho_{0(fm^{-3})}$	0.152	0.143	0.146	0.147
M^*/M	0.132	0.143	0.136	0.157
$\varepsilon_0(MeV)$	-16.01	-16.09	-16.03	-16.02
$p_F^0(fm^{-1}$	1.31	1.33	1.30	1.30
$K_\infty(MeV$	228.40	223.55	242.95	271.36

5.0 Conclusions\

The non-linear Walecka model introduces non-linear self-interaction terms of the sigma meson field into the Lagrangian. These self-interactions are necessary to reproduce the empirical properties of symmetric nuclear matter such as the binding energy per nucleon, the compressibility of nuclear matter and nucleon effective mass. From our results and calculations, we had that at the zero temperature limit, the non-linear model greatly enhances the compressional modulus to soften the EoS due to the inclusion of

The scalar meson field's cubic and quartic terms to the original lagrangian. On increasing the nuclear asymmetry parameter, symmetric nuclear matter (system) becomes unbound, EoS become stiffer and trends continue until SNM turns to pure neutron matter (PNM). The non-linear Walecka model (NLWM) significantly softens the nuclear matter equation of State (EoS) by reducing the incompressibility to an appreciable value at zero temperature. These quantities are important for understanding the structure of finite nuclei, neutron stars and equation of state of other dense matter in astrophysical contexts.

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Compliance with Ethical Standards Declaration

Ethical Approval

Not Applicable

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Authors' Contributions

YMA developed the non-linear Walecka model alongside some boundary conditions, RS, worked on the equations of motion of the meson fields which were obtained using the Euler-Lagrange equation. And MSI, interpreted the numerical results and provided a scholarly discussion.