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Exponentiated Power Ailamujia Distribution: Properties and Applications to Time Series

Kingsley Uchendu, Emmanuel Wilfred Okereke and Joy Chioma Nwabueze
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Abstract: *In this study, we propose a new three-parameter distribution, the* Exponentiated Power Ailamujia Distribution *(EPAD), as a flexible generalization of the Power Ailamujia Distribution (PAD). We derive key statistical properties of the EPAD, including its probability density function (PDF), cumulative distribution function (CDF), survival and hazard rate functions, raw and incomplete moments, quantile function, entropy, stochastic ordering, and order statistics. The PDF is shown to exhibit various shapes—unimodal, J-shaped, and bathtub-shaped—depending on the parameter configurations. A Monte Carlo simulation based on 1,000 replications and seven different sample sizes (n = 25 to 1000) confirms the consistency of the maximum likelihood estimators (MLEs), as root mean square errors (RMSEs) decrease from 0.2131 at n = 25 to 0.1031 at n = 1000. Two real-world datasets, including maximum flood levels and tax revenue data, are modeled using the EPAD and compared against PAD, the Exponentiated Power Lindley Distribution (EPLD), and the Exponentiated Weibull Distribution (EWD). Based on model selection criteria—Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (KS), Cramer-von Mises (CVM), and Anderson-Darling (AD)—EPAD yields superior fits, with the lowest AIC values of -26.52 and 382.81 for the respective datasets and p-values exceeding 0.90. These results demonstrate the potential of the EPAD in modeling time series data with complex distributional behavior.*

Keywords; *Exponentiated distributions, Ailamujia model, Statistical properties, Maximum likelihood, Time series modeling*

Kingsley Uchendu
Department of Statistics, Michael Okpara University of Agriculture, Umudike, Nigeria.
Email: **uchendu.kingsley@mouau.edu.ng**
Orcid id: 0000-0002-5986-5557

Emmanuel Wilfred Okereke
Department of Statistics, Michael Okpara University of Agriculture, Umudike, Nigeria.
Email: **okereke.emmanuel@mouau.edu.ng**
Orcid id: 0000-0002-6578-8324

Joy Chioma Nwabueze
Department of Statistics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria.

1.0 Introduction

Ailamujia distribution of Lv *et al*. (2002) is one of the existing one-parameter distributions that has witnessed several extensions in recent times. In particular, the area biased weighted Ailamujia distribution was proposed by Jayakumar and Elangovan (2019). Statistical properties and real-world applications of the power Lindley Ailmujia distribution were considered by Jan *et al*. (2020). Rather *et al*. (2022) studied the exponentiated Ailamujia distribution, demonstrating the superiority of the model to a number of comparable distributions when fitted to medical data. Alpha power Ailamujia distribution was developed by Gomaa *et al*. (2023). Additionally, the theoretical framework and applications of the power Ailamujia distribution were explored by Jamal *et al*. (2021). Ragab and Elgarhy. (2025) introduced the type II half-logistic Ailamujia, applying and comparing its fits to medical data with the fits of other notable distributions. Their empirical findings indicate that their proposed model can outperform some of the existing distributions.

The development of flexible probability distributions plays a central role in statistical modeling, especially in fields requiring accurate representation of skewness, kurtosis, and tail behavior. This is partly because of the ineffectiveness of classical distributions such as the exponential, normal and Weibull distributions to account for the complexity of some data observed in practice. In recent years, significant attention has been given to the construction of generalized families of distributions that can accommodate a wide range of data structures encountered in reliability analysis, survival analysis, hydrology, and financial modeling. The exponentiated-H family of distributions proposed by Gupta, Gupta and Gupta (1998) is useful in obtaining flexible generalizations of classical distributions. Members of this class of distributions, include the exponentiated Weibull distribution (Pal *et al*., 2006), exponentiated Gumbel distribution (Nadarajah, 2006), a generalized Lindley (Nadarajah *et al*., 2011), exponentiated Chen distribution (Dey *et al*.., 2017), exponentiated Akash (Okereke and Uwaeme, 2018), exponentiated transmuted Lindley distribution (Okereke, 2019) and exponentiated power half logistic distribution (Okereke *et al*., 2021), among others.

The essence of this study is to introduce and determine the statistical properties of a new member of the exponentiated-H family of distributions called the exponentiated power ailamujia distribution (EPAD), which doubles as a generalization of the power Ailamujia distribution.

Despite the growing number of generalizations of the Ailamujia distribution, many of the existing models still exhibit limitations in flexibility when coding heavy-tailed, or multimodal data structures. Furthermore, their performance in time series modeling remains underexplored, especially in comparison with newer exponentiated families. This necessitates the development of a more adaptable model that can offer improved fitting capacity across various real-world applications.

The significance of this study lies in its contribution to the ongoing development of flexible probability distributions for effective modeling of real-world data. By introducing the Exponentiated Power Ailamujia Distribution (EPAD), this research provides a versatile statistical tool that enhances the modeling of datasets characterized by skewness, heavy tails, and complex structures, which are often poorly captured by classical and some generalized distributions. The EPAD's ability to generalize the power Ailamujia distribution and its demonstrated superiority in fitting real-world data makes it a valuable addition to statistical modeling, particularly in fields such as reliability analysis, survival studies, hydrology, and time series analysis. Moreover, the application of EPAD to time series data opens up new possibilities for improved forecasting and pattern recognition, thereby expanding its relevance to financial modeling and other temporal data-driven domains.

2.0 Statistical Properties of the EPAD

In this section, the properties of the new distribution are determined. Since the PDF of the distribution is not in existence, we make effort to derive it first. To derive the PDF, it is necessary recall the PDF of the PAD. Statistically speaking, a continuous random variable X follows the PAD if its PDF is of the form (Jamal *et al*; 2021)

$$g(x)=\theta^2\beta x^{2\beta-1}e^{-\theta x^{\beta}},\ x>0,\beta,\theta>0. \quad (1)$$

The CDF of the PAD is

$$G(x)=1-(1+\theta x^{\beta})e^{-\theta x^{\beta}},\ x>0,\beta,\theta>0. \quad (2)$$

Applying (2) in (1), the CDF of the EPAD is obtained to be

$$f(x)=\alpha\beta\theta^2 x^{2\beta-1}[1-(1+\theta x^{\beta})e^{-\theta x^{\beta}}]^{\alpha-1}e^{-\theta x^{\beta}}, \alpha,\beta,\theta>0. \quad (3)$$

The differentiation with respect to x of (3) leads to the following PDF of the EPAD:

$$f(x)=\alpha\beta\,\theta^2 x^{2\beta-1}\left[1-(1+\theta x^{\beta})e^{-\theta x^{\beta}}\right]^{\alpha-1} \mathrm{X} e^{-\theta x^{\beta}},\ \alpha,\beta,\theta>0 \quad (4)$$

In Figure 1, we present the plots of the PDF of X for various values of its parameters. The figure revealed that the EPAD's PDF can be increasing, decreasing, left-skewed, right-skewed and unimodal.

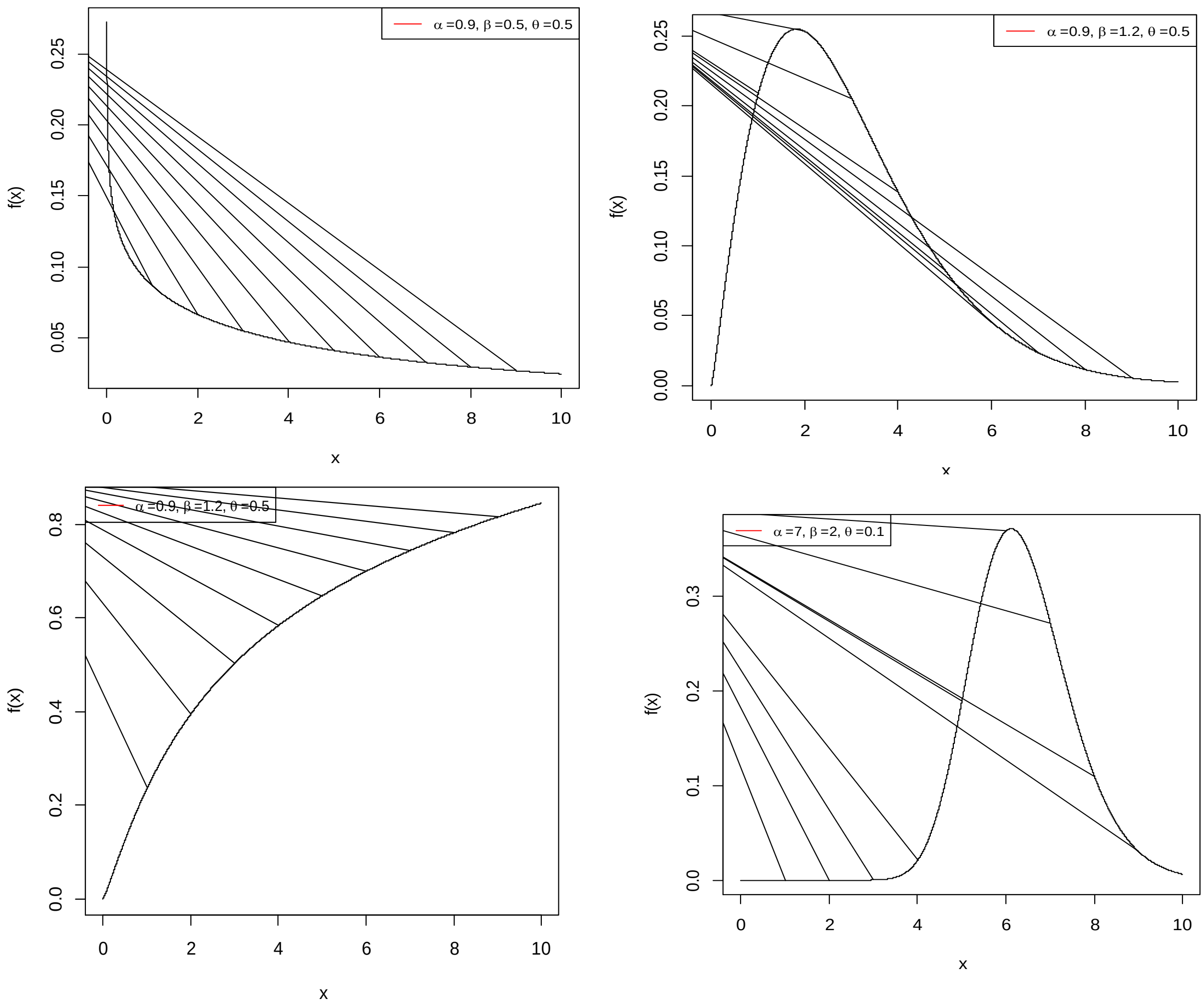

Fig.1; Graphical Representation of Different Shapes of the PDF of the EPAD

In accordance with (2.4), we have the following limit, when $\propto < 1$

$$\lim_{x\to 0} f(x)=\begin{cases}\infty, \beta\le\frac{1}{2}, \alpha<1\\ 0, \beta\le\frac{1}{2}, \alpha<1\end{cases} \quad (5)$$

If $\alpha>1$,

$$\lim_{x\to 0} f(x)=\alpha\beta\theta^2\lim_{x\to 0} x^{2\beta-1}[1-(1+\theta x^{\beta})e^{-\theta x^{\beta}}]^{\alpha-1}e^{-\theta x\beta}=0 \quad (6)$$

Furthermore,

$$\lim_{x\to 0} f(x)=\alpha\beta\theta^2 \lim_{x\to 0} x^{2\beta-1}[1-(1+\theta x^{\beta})e^{-\theta x^{\beta}}]^{\alpha-1}e^{-\theta x\beta}=0 \quad (7)$$

It can be deduced from equations 5 to 7 that the EPAD is unimodal if $\alpha>1$, and $\beta>\frac{1}{2}$, The unimodality condition also holds when $\alpha>1$.
For the EPAD, the survival function is $S(x)=1-F(x)$

$$=1-[1-(1+\theta x^{\beta})e^{-\theta x^{\beta}}]^{\alpha}, x>0, \alpha,\beta,\theta>0 \quad (8)$$

The related hazard rate function (HRF) is

$$h(x)=\frac{f(x)}{s(x)}$$

$$=\frac{\alpha\beta\theta^2 x^{2\beta-1}[1-(1+\theta x^{\beta})e^{-\theta x^{\beta}}]^{\alpha-1}e^{-\theta x^{\beta}}}{[1-(1+\theta x^{\beta})e^{\theta x^{\beta}}]^{\alpha}} \quad (9)$$

The graphical representation of the HRF of the EPAD is provided as Fig. 2. It is easily seen that the HRF can be increasing, decreasing, J shaped, bathtub shaped or unimodal, depending on the values of the parameters of the distribution.

2.1 Moments of EPAD

Moment of a continuous random variable are of paramount importance. They are fundamental to the capturing of critical features of the shape of distribution, including location, spread, skewness and kurtosis. Sometimes they can be used to obtain method of moments estimates of parameters of distribution.

Given that a continuous random variable X follows the EPAD with parameters $\propto, \beta\ and\ \theta,$ as indicated in the PDF (2.4), we write $x\sim$ EPAD $(\propto,\beta,\theta)$. The rth raw moment of X is

$$\mathrm{E}[X^{\Gamma}]=\int_0^{\infty} x^{\Gamma} f\left(x\right)dx$$

$$=\alpha\beta\theta^2\int_0^{\infty} x^{2\beta+r-1}[1-(1+\theta x^{\beta})e^{-\theta x^{\beta}}]^{dy}e^{\theta x^{\beta}}dx^{j}$$

$$=\alpha\beta\theta^2\sum_{1=0}^{j}(-1)^{j}\binom{\alpha-1}{j}\int_0^{\infty}(1+\theta x^{\beta})x^{2\beta+r-1}e^{-(j+i)\theta x^{\beta}dx}$$

$$=\alpha\beta\theta^2\sum_{1=0}^{j}(-1)^{j}\binom{\alpha-1}{j}\binom{j}{i}\theta^{i}\int_0^{\infty}(1+\theta x^{\beta})x^{2\beta+r-1}e^{-(j+i)\theta x^{\beta}dx}$$

$$Let\, U=\theta(j+1)x^{\beta}$$

$$\to x=\frac{U\frac{1}{\beta}}{(\theta(J+1))\frac{1}{\beta}}, dx=\frac{U\frac{1}{\beta}-1}{\beta(\theta(J+1))\frac{1}{\beta}}du$$

Therefore,

$$\mathrm{E}[X^{\Gamma}] = \alpha\beta\theta^2 \sum_{i=0}^{j}\sum_{j=0}^{\infty}(-1)^j \binom{\alpha-1}{j}\theta^i \int_0^{\infty}\left[\frac{U\,\frac{2\beta+\beta i+r-1}{\beta}e^{-u}}{(\theta(j+i)\frac{2\beta+\beta i+r-1}{\beta}}\right]\left(\frac{U^{\frac{1}{\beta-1}}}{\beta(\theta(j+1))\frac{1}{\beta}}\right)du$$

$$=\alpha\beta\theta^2\sum_{i=0}^{j}\sum_{j=0}^{\infty}\left[\frac{(-1)^j\binom{\alpha-1}{j}\binom{j}{i}\theta^i}{\beta[\theta(j+i)]\frac{2\beta+\beta i+r-1}{\beta}}\right]\int_0^{\infty}U\,1+i+\frac{r}{\beta}e^{-u}du$$

$$=\alpha\sum_{i=0}^{j}\sum_{j=0}^{\infty}\frac{(-1)^j}{\theta\frac{r}{\beta}(j+i)^{\frac{r}{\beta}+i+2}}\binom{\alpha-1}{j}\binom{j}{i}\Gamma\left(\frac{r}{\beta}+i+2\right)$$

Specifically,

$$\mathrm{E}(X)=\alpha\sum_{i=0}^{j}\sum_{j=0}^{\infty}\frac{(-1)^j}{\theta\frac{1}{\beta}(j+1)\frac{1}{\beta}+i+2}\binom{\alpha-1}{j}\binom{j}{i}\Gamma\left(\frac{1}{\beta}+i+2\right)$$

$$\mathrm{E}(X^2)=\alpha\sum_{i=0}^{j}\sum_{j=0}^{\infty}\frac{(-1)^j}{\theta\frac{2}{\beta}(j+1)\frac{2}{\beta}+i+2}\binom{\alpha-1}{j}\binom{j}{i}\Gamma\left(\frac{2}{\beta}+i+2\right)$$

$$\mathrm{E}(X^3)=\alpha\sum_{i=0}^{j}\sum_{j=0}^{\infty}\frac{(-1)^j}{\theta\frac{3}{\beta}(j+1)\frac{3}{\beta}+i+2}\binom{\alpha-1}{j}\binom{j}{i}\Gamma\left(\frac{3}{\beta}+i+2\right)$$

$$\mathrm{E}(X^4)=\alpha\sum_{i=0}^{j}\sum_{j=0}^{\infty}\frac{(-1)^j}{\theta\frac{4}{\beta}(j+1)\frac{4}{\beta}+i+2}\binom{\alpha-1}{j}\binom{j}{i}\Gamma\left(\frac{4}{\beta}+i+2\right)$$

$$\mathrm{var}(X)=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}\left(X\right)\right]^2$$

Additionally, the coefficient of skewness of the EPAD $(\propto,\beta,\theta)$ *is*

$$\beta_1=\frac{\mathrm{E}\left(X-\mu\right)^3}{\left[Var\left(x\right)\right]^{\frac{3}{2}}}=\frac{\mathrm{E}(X)^3-3\mu\mathrm{E}(x^2)+2\mu^2\mathrm{E}(x)}{\left[Var(x)\right]^{\frac{3}{2}}}$$

The corresponding coefficient of Kurtosis has the form

$$\beta_1=\frac{\mathrm{E}\left(X-\mu\right)^4}{\left[Var\left(x\right)\right]^2}=\frac{\mathrm{E}(X)^4-3\mu\mathrm{E}(x^3)+2\mu^2\mathrm{E}(x)}{\left[Var(x)\right]^2}$$

Mean (μ), variance (var), coefficient of skewness (β_1) and values of coefficient of kurtosis (β_2) of the EPAD values that corresponds to various values of $\propto,\beta$ *and* θ are contained in Table 1. This table shows that the EPAD can be positively skewed and either platykurtic or leptokurtic.

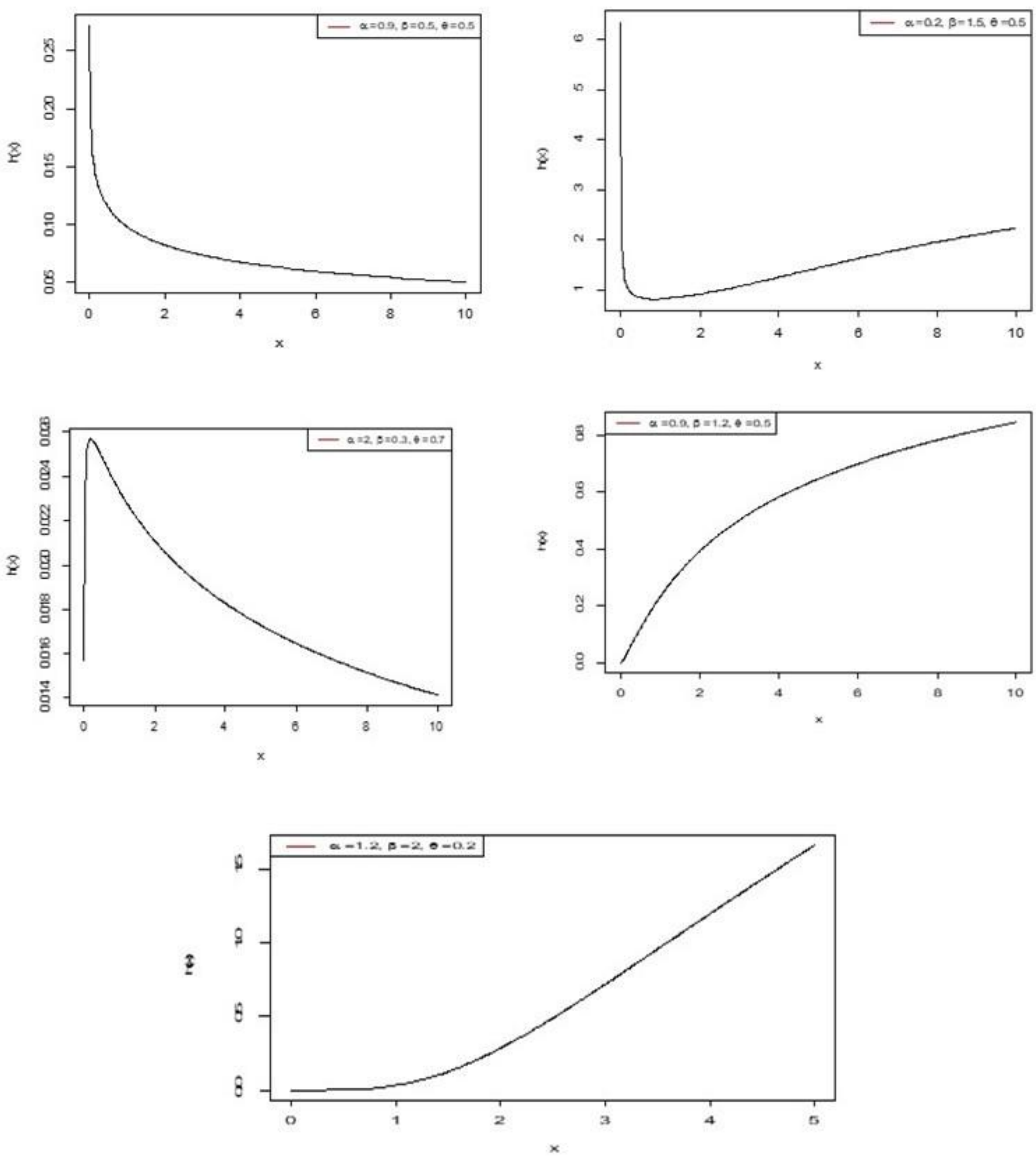

Fig. 2: Graphical Representation of Different Shapes of the HRF of the EPAD

Table 1: Descriptive Statistics for the EPAD

α	β	θ	μ	Var	β_1	β_2
0.5	0.5	0.5	13.8466	997.29	163023	46591472
0.5	1	1	1.3440	3.46	13	62
0.5	1.5	1.5	0.7307	0.67	1	1
0.5	2	2	24.0000	1920.00	322560	92897280
1	0.5	0.5	2.0000	6.00	24	120

1	1	1	0.9399	1.00	1	2
1	1.5	1.5	39.0000	3600.00	632520	184705920
1	2	2	2.7500	9.75	43	225
1.5	0.5	0.5	1.1322	1.38	2	2
1.5	1	1	13.8466	997.29	163023	46591472
1.5	1.5	1.5	1.3440	3.46	13	62
1.5	2	2	0.7307	0.67	1	1
2	0.5	0.5	24.0000	1920.00	322560	92897280
2	1	1	2.0000	6.00	24	120
2	1.5	1.5	0.9399	1.00	1	2
2	2	2	39.0000	3600.00	632520	184705920

2.2 Incomplete moments

Incomplete moments are useful in deriving important properties of statistical distributions, which include mean deviations, mean residual life function, Bonferroni and Lorenz curves.
Let $J_{r\,(x)}$ denote the rth incomplete moment of a random variable $X,$ where $x\sim EPAD\ (\propto, \beta,\ \theta).$
Then:

$$j_r(x)=\int_0^x m^r f(m)dm$$

$$=\alpha\beta\theta^2\sum_{i=0}^{j}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\binom{j}{i}\int_0^x m^{2\beta+\beta i+r-1}e^{-\theta(j+1)e^{\beta}}dm$$

$$=\theta(j+1)m^{\beta}\Rightarrow m=\frac{w^{\frac{1}{\beta}}}{(\theta(j+1)^{\frac{1}{\beta}}}$$

$$dm=\frac{w^{\frac{1}{\beta-1}}}{\beta[\theta(j+1)]^{\frac{1}{\beta}}}$$

$$J_r(x)=\alpha\,\theta^2\sum_{i=0}^{j}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\binom{j}{i}\theta^i$$

$$X\int_0^{\theta}(j+1)x^{\beta}\,\frac{w\dfrac{2\beta+\beta_i+r-1}{\beta}}{[\theta(j+1)]\dfrac{2\beta+\beta_i+r-1}{\beta}}\left[\frac{w^{\frac{1}{\beta-1}}}{(\theta(j+1))^{\frac{1}{\beta}}}\right]e^{-w}dw$$

$$=\alpha\theta^2\sum_{i=0}^{j}\sum_{j=0}^{\infty}\left[\frac{(-1)\binom{\alpha-1}{j}\binom{j}{i}\theta^i}{(\theta^{\frac{r}{\beta}}(j+1))^{2+i+\frac{r}{\beta}}}\right]\gamma\left(\frac{r}{\beta}+i+2,\theta(j+1)x^{\beta}\right)$$

$$=\alpha\sum_{i=0}^{j}\sum_{j=0}^{\infty}\left[\frac{(-1)^j\binom{\alpha-1}{j}\binom{j}{i}\theta^i}{(\theta(j+1))^{2+i+\frac{r}{\beta}}}\right]\int_0^{\theta\,(j+1)x^{\beta}}W^{\frac{r}{\beta+i+1}}e^{-w}dw$$

where $\gamma\ (.)$ is the lower incomplete gamma function.

1.3 Bonferroni and Lorenz curves

Bonferroni and Lorenz curves are useful in studying income distribution. For a probability value P, the Bonferroni curve for the EPAD is

$$B(p) = \int_0^q \frac{Ef(t)d_t}{p\mu} = \frac{J_{1(q)}}{p\mu}$$

The related Lorenz curve has the form

$$L\ (p) = \int_0^q \frac{Ef(t)}{\mu} = \frac{J_{1(q)}}{p\mu},$$

where $q = F^{-1}(p)$.

2.4 Stochastic Ordering

Here, we derive and present results pertaining to stochastic dominance of the EPAD over the exponentiated Weibull distribution (EWD) as well as likelihood ratio ordering based on EPAD. These results are presented in the proofs of Propositions 2.1 and 2.2 respectively.

Proposition 2.I: Let $F_1(x;\alpha,\beta,\theta)$ and $G_1(x;\alpha,\beta,\theta)$ be the cdfs of the EPAD and EWD, each with parameters $\propto, \beta\ and\ \theta$. Then $F_1(x;\alpha,\beta,\theta) \leq G_1(x;\alpha,\beta,\theta)$. That is the EPAD (α,β,θ) exhibits first-order stochastic dominance over the EWD (α,β,θ).

Proof: Certainly, if $x < 0$, $F_1(x,\propto,\beta,\theta) = 0$ and $G_1(x,\propto,\beta,\theta) = 0$. Thus, the inequality holds when $x < 0$. If $x \geq 0$.

$1 + \theta x^{\beta} \geq 0$ indicating that

$$F_1(x;\ \propto,\beta,\theta) = \left[1 - \left(1 + \theta x^{\beta}\right)e^{-\theta x^{\beta}}\right]^{\propto}$$

$$\leq G_1(x;\propto,\beta,\theta) = \left[1 - e^{-\theta x^{\beta}}\right]^{\propto}$$

This completes the proof.

The significance of Proposition 2.1 is that the EPAD can be applied where the EWD is empirically deficient. Proposition 2 contains an important stochastic ordering results on EPAD.

Proposition 2.2: let $X_1\ and\ X_2$ refers to two random variables having the EPAD such that the distribution of X_1 has the parameters $\propto_1, \beta_1, and\ \theta_1$ and the distribution of X_2 has the parameters $\propto_2, \beta_2, and\ \theta_2$.

If $\beta_1 = \beta_2\ and\ \theta_1 = \theta_2$, X_2 is larger than based on the likelihood ratio ordering if $\propto_1 < \propto_2$.

Proof

To prove proportion 2.2, it becomes important to determine the condition under which the ratio $\frac{f_{1(x)}}{f_{2(x)}}$ decreases in x,

Where $f_{1\,(x)}$ and $f_{2\,(x)}$ are the pdfs of $X_1\ and\ X_2$ respectively

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NOW

$$\frac{f_{1(x)}}{f_{2(x)}} = \frac{\alpha_1\beta_1\theta^{2\beta_{1-1}}[1-\left(1+\theta_1 x\beta^1\right)e^{-\theta_1 x\beta^1}]^{\alpha_1-1}e^{-\theta_1^{x^{\beta_1}}}}{\alpha_2\beta_2\theta_2^2 x^{2\beta_{2-1}}[1-\left(1+\theta_2 x\beta^2\right)e^{-\theta_2 x\beta^2}]^{\alpha_2-1}e^{-\theta_2^{x^{\beta_1}}}}$$

Taking natural log on both sides of the last equation results in

$$\ln\left[\frac{f_{1(x)}}{f_{2(x)}}\right] = \ln(x_1) + \ln(\beta_1) + 2\ln(\theta_1) + (2\beta_1 - 1)\ln x$$

$$+(\alpha_1 - 1)\ln\left[1 - \left(1 + \theta_1 x^{\beta_1}\right)e^{-\theta_1 x\beta_1}\right]$$

$$-\theta_1 x^{\beta_1} - \ln\alpha_2 - \ln\beta_2 - 2\ln\theta_2$$

$$-(2\beta_2 - 1)\ln x - (\alpha_2 - 1)\ln[1 - (1 + \theta_2 x^{\beta_2})e^{-\theta_2 x^{\beta_2}}] + \theta_2 x^{\beta_2}$$

Differentiating with respect to x, we have

$$\frac{d}{dx}\ln\left[\frac{f_{1(x)}}{f_{2(x)}}\right] = \frac{2\beta_1 - 1}{x} + \frac{(\alpha_1 - 1)\beta_1\theta_1^2 x^{2\beta_{1-1}}e^{-\theta_1 x\beta_1}}{1 - (1 + \theta_1 x^{\beta_1})e^{-\theta_1 x\beta_1}}$$

$$-\beta_1\theta_1 x^{\beta 1-1} - \frac{(2\beta_{2-1})}{x}$$

$$-\frac{\beta_2\theta_2^2(\alpha_2-1)(x^{2\beta 2-1}e^{-\theta_2 x^{\beta_2}})}{1-(1+\theta_2 x^{\beta_2})e^{-\theta_2 x\beta_2}} + \beta_2\theta_2 x^{\beta_{2-1}}$$

If $\beta_1 = \beta_2 = \beta$ and $\theta_1 = \theta_2 = \theta$, then

$$\frac{d}{dx}\ln\left[\frac{f_{1(x)}}{f_{2(x)}}\right] = \frac{\beta\theta^2 x^{2\beta-1}e^{-\theta_1 x^{\beta 1}}}{1-(1+\theta+x^\beta)e^{-\theta x^\beta}}(\alpha_1-\alpha_2)$$

In this case, $\frac{d}{dx}\ln\left[\frac{f_{1(x)}}{f_{2(x)}}\right] < 0 \; if \; \propto_1, <\propto_2$.

That is $\frac{f_1(x)}{f_2(x)}$ decreases in x whenever

$\propto_1, < \propto_2, \beta_1 = \beta_2$ and $\theta_1, = \theta_2$. This ends the proof.

2.5 Rényi Entropy

Entropy is a measure of uncertainty inherent in a random variable. For the EPAD $(\propto, \beta, \theta)$, Rényi entropy is given by

$$E_R = \frac{1}{1-\lambda}(\int_0^\infty f^\lambda(x)dx), \lambda \neq 0, \lambda \neq 1$$

$$= \frac{1}{1-\lambda}\ln[\alpha\beta\theta^2]^\lambda \int_0^\infty (x^{\lambda(2\beta-1)}[1-(1+\theta x^\beta)e^{-\theta x^\beta}]^{\lambda(\alpha-1)} Xe^{-\lambda\theta x^\beta}dx$$

$$= \frac{1}{1-\lambda}\ln[\alpha\beta\theta^2]^\lambda \sum_{j=0}^{\infty}(-1)\binom{\lambda(\alpha-1)}{j}\int_0^\infty x^{\lambda(2\beta-1)}(1+\theta x^\beta)^j e^{(\lambda-j)}\theta x^\beta dx$$

$$= \frac{1}{1-\lambda}\ln\sum_{i=0}^{j}\sum_{j=0}^{\infty}(-1)^j\theta^i\binom{\lambda(\alpha-1)}{j}\binom{j}{i}\int_0^\infty x^{\lambda(2\beta-1)+\beta_1}e^{\theta x^\beta(\lambda-j)}dx + \frac{1}{1-\lambda}\ln(\alpha\beta\theta^2)$$

Let Z = $\theta x^\beta(\lambda+j)$

$$\Rightarrow x\frac{Z^{\frac{1}{\beta}}}{(\theta(\lambda+j))^{\frac{1}{\beta}}} \Rightarrow dx = \frac{Z^{\frac{1}{\beta}-1}}{\beta(\theta(\lambda+j))^{\frac{1}{\beta}}}dz$$

$$\therefore ER = \frac{1}{1-\lambda}\left[\ln(\alpha\beta\theta^2)^\lambda\sum_{j=0}^{\infty}(-1)^j\theta^i\binom{\lambda(\alpha-1)}{j}\binom{j}{i}\right]\left(\frac{1}{[\theta(\lambda+1)]\lambda\frac{(2\beta-1)+\beta_1}{\beta}}\right)$$

$$\text{X}\frac{1}{\beta(\theta(\lambda+j)^{\frac{1}{\beta}}}\int_0^\infty Z\frac{(\lambda(2\beta-1)+\beta_1)}{\beta}+\frac{1}{\beta}-1e^{-z}dz$$

$$= \frac{1}{1-\lambda}\left[\ln(\alpha\beta\theta^2)^\lambda\sum_{i=0}^{j}\sum_{j=0}^{\infty}\frac{(-1)^j\theta^i}{\beta(\theta(\lambda+j)\frac{\lambda(2\beta-1)+\beta_i+1}{\beta}}\binom{\lambda(\alpha-1)}{j}\binom{j}{i}\right]\Gamma\left(\frac{\lambda(2\beta-1)+\beta_i+1}{\beta}\right)$$

2.6 Order statistics

Order statistics and their distributions have found applications in statistical inference, especially nonparametric inference. Let $X_{(1)}, X_{(2)}, , \ldots, X_{(n)}$, denote order statistics associated with EPAD $(\propto_1, \beta, \theta)$. The pdf of the kth order statistic has the form.

$$f_{x(k)}(x) = \frac{n!}{(k-1)!(n-k)!} f_{(x)} F^{k-1}(x)\left[1-F(x)\right]^{n-k}$$

Specifically, the smallest order statistic has the pdf

$$f_{x(k)}(x) = n\alpha\beta\theta^2 x^{2\beta-1}[1-(1+\theta x^{\beta})e^{-\theta x\beta}]^{\alpha-1}e^{-\theta x\beta}$$

$$X[1-(1-(1+\theta x^{\beta})e^{-\theta x^{\beta}})^{\alpha-1}]^{n-1}$$

Similarly, the pdf of the nth order statistic is

$$f_{x(n)}(x) = n\alpha\beta\theta^2 x^{2\beta-1}[1-(1+\theta x^{\beta})e^{-\theta x\beta}]^{\alpha n-1}e^{-\theta x\beta}$$

2.7 Quantile function

Suppose that $q \in (0,1)$. The quantile function for the EPAD $(\propto_1, \beta, \theta)$, denoted by xq, is obtained by solving for xq in the equation

$$F_{(xq)=q}$$

$$\Rightarrow [1-(1+\theta x_q^b)e^{\theta x_q^{\beta}}]^{\alpha} = q$$

Consequently,

$$(1+\theta x_q^b)e^{\theta x_q^{\beta}} = 1-q^{\frac{1}{\alpha}}$$

$$-(1+\theta x_q^b)e^{-(1-\theta x_q^{\beta})} = -(1-q^{\frac{1}{\alpha}})e^{-1}$$

It follows that

$$xq = \left[-\frac{1}{\theta}(W_{-1}\left(q^{\frac{1}{\alpha}}-1\right)e^{-1})+1\right]^{\frac{1}{\beta}},$$

where $W_{-1}(.)$ is the negative branch of the Lambert W function,

3.0 Maximum likelihood Estimation of parameters of EPAD

Consider a random sample $X_1, X_2, , \ldots, X_n$, of size n from the EPAD $(\propto, \beta, \theta)$. The likelihood function is

$$L = \prod_{i=1}^{n} f(x_i)$$

$$= (\alpha\beta\theta^2)^2 \prod_{i=1}^{n}\left[1-(1+\theta x_i^{\beta})e^{-\theta x_i^{\beta}}\right]^{\alpha-1} e^{-\theta x_i^{\beta}}$$

Therefore, the log-likelihood function is

$$\ln L = n \ln \propto + \ln\beta + 2n\ln\theta + \propto -1 \sum_{i=1}^{n} \left[\ln\left(1-\left(1+\theta x_i^{\beta}\right)e^{-\theta x_i^{\beta}}\right)\right] - \theta\sum_{i=1}^{n} x_i^{\beta}$$

Partial derivatives of $\ln L$ with respect to $\propto, \beta$ and θ are as follows

$$\frac{\partial}{\partial\propto}\ln L = \frac{n}{\propto} + \sum_{i=1}^{n} \ln\left[1-(1+\theta x_i^{\beta})\, e^{-\theta x_i^{\beta}}\right]$$

$$\frac{\partial}{\partial\beta}\ln L = \frac{n}{\beta} + (\propto -1)\sum_{i=1}^{n} \frac{\theta^2 x_i^{2\beta}\ln x_i\, e^{-\theta x_i^{\beta}}}{1-(1+\theta x_i^{\beta})e^{-\theta x_i^{\beta}}} - \theta\sum_{i=1}^{n} x_i^{\beta}\ln x_i$$

$$\frac{\partial}{\partial\theta}\ln L = \frac{2n}{\theta} + (\propto -1)\theta\sum_{i=1}^{n} \frac{x_i^{2\beta} e^{-\theta x_i^{\beta}}}{1-(1+\theta x_i^{\beta})e^{-\theta x_i^{\beta}}} - \sum_{i=1}^{n} x_i^{\beta}$$

Solving $\frac{\partial}{\partial\propto}\ln L = 0, \frac{\partial}{\partial\beta}\ln L = 0$ and $\frac{\partial}{\partial\theta}\ln L$ =0 simultaneously enables us to obtain the maximum likelihood estimates of the parameters $\propto, \beta\ and\ \theta$. Consequently, only numerical solutions can be obtained due to the non-linearity of the likelihood equations.

4.0 Simulation Results Based EPAD $(\propto, \beta, \theta)$

The consistency property of the maximum likelihood estimators of $\propto, \beta\ and\ \theta$ in EPAD $(\propto, \beta\ and\ \theta)$ is explored via the Monte Carlo simulation procedure. The simulation process is predicted on 1000 samples for each of the sample sizes n = 25. 50, 75, 100, 200, 500 and 1000 obtained from the EPAD $(\propto, \beta\ and\ \theta)$ under different sets of values of $\propto, \beta\ and\ \theta$. The different sets of values of $\propto, \beta\ and\ \theta$ considered in the Monte Carlo experiment include $(\propto, \beta\ and\ \theta) = (0.5, 0.5, 0.5)$. For each sample size and sets of parameter values, the average estimate (AE) of each parameter as well as its associated average (AB) and mean squared error are computed. The requisite numerical results are presented in Table 2. Table 2 show that the MSEs tend to zero as n increases indicating consistency of the estimators. Average bais for each parameters estimators decrease as n increases.

Table 1: Simulation Results Based on EPAD

	PARAMETERS			AVERAGE BIASES			RMSES		
n	α	β	θ	AB ($\hat{\alpha}$)	AB ($\hat{b}$)	AB ($\hat{\theta}$)	RMSE($\hat{\alpha}$)	RMSE($\hat{b}$)	RMSE($\hat{\theta}$)
25	0.5	0.5	0.5	-0.0118	-0.0747	-0.0564	0.1544	0.1372	0.2131
50	0.5	0.5	0.5	-0.0178	0.0557	-0.0401	0.1441	0.1236	0.2073
100	0.5	0.5	0.5	-0.0122	0.0437	-0.032	0.1374	0.1127	0.1988
200	0.5	0.5	0.5	0.0043	0.0291	-0.0073	0.1331	0.1025	0.1891
500	0.5	0.5	0.5	-0.0026	0.0211	-0.0091	0.1161	0.038	0.1662
1000	0.5	0.5	0.5	0.0062	0.0031	0.0070	0.0702	0.0435	0.1031
25	105	0.4	2.9	0.1732	0.0337	-0.2037	0.1947	0.0498	0.2909
50	105	0.4	2.9	0.1449	0.0211	-0.1261	0.1918	0.0389	0.2311
100	105	0.4	2.9	0.1248	0.0082	-0.0481	0.1926	0.0269	0.1624
200	105	0.4	2.9	0.0898	0.0039	-0.0224	0.0199	0.0199	0.1221
500	105	0.4	2.9	0.0527	0.0011	-0.0065	0.1933	0.01295	0.0823
1000	105	0.4	2.9	0.0428	0.0003	-0.0002	0.1947	0.0088	0.0557

5.0 Applications

Two real-world time series data are used to demonstrate the applicability of the EPAD in this section. The first data (Data I) are Maximum flood levels data (Dumonceaux and Antle, 2012):

0.654,0.613,0.315,0.449,0.297,0.402,0.379,0.423,0.379,0.3235,0.269,0.740,0.418,0.412,0.494,0.416,0.338,0.392,0.484,0.265.

The second data set (Data II) which refers to the actual taxes revenue data (in 1000 million Egyptian pounds) is as follows (Nassar and Nada, (2011).):

5.9, 20.4, 14.9, 16.2, 17.2,
7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1,6.7,17,8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1,
36, 8.5, 8, 9.2, 26.2, 21.9,16.7,21.3,35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7,
8.6,12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

The fits of the EPAD to each of the data stated above is compared with the fits of the PAD, exponentiated power Lindley distribution (EPLD) (Ashour and Eltehiwy, 2014) and EWD. The CDFs of the last two distributions, denoted by $f_3(x)$ and $f_4(x)$ are defined below:

$$f_3(x) = \left[1 - \left(1 + \frac{\theta x^{\beta}}{\theta + 1}\right)e^{-\theta x^{\beta}}\right]^{\alpha}, x > 0, \alpha, \beta, \theta > 0.$$

$$f_4(x) = \left[1 - e^{-(\theta x)^{\beta}}\right]^{\alpha}, x > 0, \alpha, \beta, \theta > 0.$$

The criteria used to compare fits of these distributions are the Akaike information criterion (AIC) and Bayesian information criterion (BIC). Additionally, we employ the Kolmogorov-Smirnov (KS), Cramer-von Mises (CVM) and Anderson-Darling (AD) statistics in comparing the goodness of fits of the distributions. Accordingly, any of the distributions that corresponds to the minimum values of the criteria and statistics is adjudged the best model for the concerned. The maximum likelihood estimates of the parameters of the distributions in conjunction with the calculated values of the model evaluation criteria and the goodness of fit statistics are presented in Table 3 for the two data sets. The numerical results in the table indicate that the EPAD gives the best fit to the data compared to the other three data.

Table 3: Maximum likelihood estimates and the associated results based on the distributions fitted to Data I and Data II.

DATA	Distribution	Estimate	(L) Loglikelihood	AIC	BIC	KS	CVM	AD	P-value
	PAD	$\hat{\beta}=2.4924$ $\hat{\theta}=14.6858$	14.1896	-24.3791	-22.3877	0.1870	0.1085	0.6503	0.4865
I	EPAD	$\hat{\alpha}$ =98.6435 $\hat{\beta}$ =0.7111 $\hat{\theta}$ =13.5306	16.2604	-26.5208	-23.5336	0.1188	0.0426	0.2696	0.9403
	EPLD	$\hat{\alpha}$ =42.4215							

		$\hat{\beta}$ =1.0406 $\hat{\theta}$ =11.4077	16.1217	-26.2434	-23.2562	0.1266	0.0447	0.2907	0.9056
	EWD	$\hat{\alpha}$ =79.1864 $\hat{\beta}$ =0.9234 $\hat{\theta}$ =13.4452	16.2047	-26.4094	-23.4222	0.1213	0.0432	0.2798	0.9302
II	PAD	$\hat{\beta}$ =1.3027 $\hat{\theta}$ =0.0636	-194.557	393.1094	397.2645	0.1348	0.2168	1.3939	0.2338
	EPAD	$\hat{\alpha}$ =105.0266 $\hat{\beta}$ =0.3680 $\hat{\theta}$ =2.9186	-188.4055	382.8111	389.0437	0.0674	0.0580	0.3357	0.9513
	EPLD	$\hat{\alpha}$ =58.5503 $\hat{\beta}$ =0.4680 $\hat{\theta}$ =1.7742	-188.6991	383.3981	389.6307	0.0755	0.0697	0.04058	0.8894
	EWD	$\hat{\alpha}$ =108.0223 $\hat{\beta}$ =0.4542 $\hat{\theta}$ =3.1248	-188.5102	383.0203	389.2529	0.0707	0.0629	0.3656	0.9296

6.0 Conclusion

This study introduced a novel and flexible three-parameter distribution known as the *Exponentiated Power Ailamujia Distribution* (EPAD), developed as a generalization of the Power Ailamujia Distribution. We systematically derived and analyzed the statistical properties of the EPAD, including its probability density and hazard rate functions, moments, entropy, quantile function, stochastic ordering, and order statistics. The shape flexibility of the distribution was demonstrated, with the PDF and hazard rate function accommodating various forms such as increasing, decreasing, unimodal, J-shaped, and bathtub-shaped patterns. Through Monte Carlo simulation, the maximum likelihood estimators of the parameters were shown to be consistent, with decreasing bias and RMSE as sample size increased.

Furthermore, the applicability of the EPAD was validated using two real-world time series datasets—maximum flood levels and tax revenue data. In comparison with other well-known models such as PAD, EPLD, and EWD, the EPAD exhibited superior performance based on goodness-of-fit criteria including AIC, BIC, KS, CVM, and AD statistics. These findings underscore the robustness and practical relevance of the EPAD in modeling complex time series data in hydrological and economic domains. The distribution holds promise for wider applications in fields where capturing diverse data behaviors is essential.

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Authors’ Contribution

Kingsley Uchendu conceptualized the study, developed the mathematical formulation of the Exponentiated Power Ailamujia Distribution (EPAD), and derived its statistical properties. Emmanuel Wilfred Okereke contributed to the methodological design, carried out the Monte Carlo simulations, and performed the comparative model fitting using real-world datasets. Joy Chioma Nwabueze assisted in data collection, literature review, computational analysis, and interpretation of results, as well as the preparation and proofreading of the manuscript.