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Nonlinear Dynamic Buckling Behaviour of Viscously Damped Columns on Elastic Foundations Under Step Loading

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***Abstract**: This study investigates the influence of viscous damping on the load-carrying capacity of an imperfect finite column. The analysis is based on an established governing equation for the lateral displacement of a viscously damped finite column, which is solved using the regular perturbation method. To facilitate the solution, the governing equation is first non-dimensionalized through the introduction of appropriate dimensionless parameters. This transformation naturally introduces a slow time scale into the formulation, and nonlinear terms of order higher than cubic are neglected to simplify the analysis. The resulting expression for the lateral displacement, as a function of both time and spatial coordinates, is derived analytically. An asymptotic series expansion is then employed to determine the column's maximum lateral displacement, from which the dynamic buckling load is obtained. The actual buckling load values are computed using MATLAB, and the results are subsequently used to assess the quantitative effect of viscous damping on the dynamic buckling capacity.*

***Keywords**:Dynamic buckling; column; nonlinear foundation; viscously damped; step-load.*

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1. 0 Introduction

In this study, we examine the influence of light viscous damping on the dynamic buckling load of a finite column supported by a quadratic-cubic nonlinear elastic foundation. The column is subjected to a step load applied immediately after the initial time. The configuration of the column was previously investigated by Hansen and Roorda(1973, 1974), where they analyzed the sensitivity of such a structure to imperfections, assuming that the inherent imperfections were spatially stochastic. Elishakoff (1979) later explored the reliability approach to imperfection sensitivity for a similar column. Similarly, Boyce (1961) studied the buckling of a stochastically imperfect column on a nonlinear elastic foundation, while Amazigo (1971) focused on the buckling behavior of a column with randomly distributed initial displacement. Other relevant investigations in this area include Dumir et al. (1987) and Kevorkian (2003), among others. This buckling is a form of deformation which are caused by stresses as the papers by Egbuhuzor and Erumaka(2020) and Egbuhuzor and Udoh (2023). The papers solved for displacement distributions and stresses acting under internal pressure.

However, while the aforementioned studies primarily focused on the stability of these structures under static loading, there were also investigations that delved into the dynamic behavior of these structures. Amazigo and Frank (1971), Ette (1992), and others examined the dynamic stability of these structures when subjected to a step load. Additional research on dynamic buckling in relation to columns has been conducted by Ette et al. (2018; 2019), among others. Some of the problems addressed in this study involve two small non-dimensional parameters, and the analysis employs two-timing multi-scaling perturbation techniques, similar to the approaches used in Ozoigbo and Ette (2019) and Ette et al. (2019), among other references.

2. 0 Formulation of the Differential Equation Governing Lateral Displacement

The dimensional differential equation satisfied by displacement W(X, T) of a finite imperfect damped column lying on a quadratic-cubic nonlinear elastic foundation as given in Ette(1992) is

$$m_0 W_{,TT} + C_0 W_{,T} + EIW_{,XXXX} + P(T)W_{,XX} + k_1 W - \beta_1 W^2 - \beta k_3 W^3 = -P(T)\frac{d^2\underline{W}}{dX^2}, \quad T > 0, \quad 0 < X < \Pi \tag{1}$$

$$W = W_{,XX} = 0 \quad at \;\; X = 0, \Pi \tag{2}$$

$$W(X,0) = W_{,T}(X,0) = 0 \quad 0 < X < \Pi \tag{3}$$

where m_0 is the mass per unit length of the column, X and T are the spatial and time variables respectively, C_0 is the damping constant where the damping is taken proportional to the first degree of the velocity $W_{,T}$. Here EI is the bending stiffness where E is the Young modulus and I is the moment of inertia and k_1, β_1, k_3 are spring constants, considered positive, while β is the imperfection-sensitivity parameter which is such that if the column were lying on a strictly nonlinear cubic elastic foundation $(ie\ if\ \beta_1 = 0)$ then, β would take the values $\beta = -1\ or\ \beta = 1$. In this case, the resultant column is said to be on a "hardening" foundation if $\beta = -1$ and on a "softening" foundation if $\beta = 1$. However, for the case of a column on a quadratic-cubic foundation, we still maintain β without substituting the above values for it. We note that the elastic foundation exerts a force per unit length given by $k_1 W - \beta_1 W^2 - \beta k_3 W^3$. Equally of note is the term $\underline{W}(X)$ which is a time independent twice-differentiable stress-free imperfection function while P(T) is the load history. We have neglected axial inertia as well as nonlinearities higher than cubic. We have also neglected nonlinear derivatives of $\underline{W}(X)$ and a subscript following a comma indicates partial differentiation.

3. 0 Non-dimensionalization

As in Elishakoff (1980), we shall nondimensionalize the equations (1) – (3) by introducing the following quantities

$$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X, \qquad w = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} W, \qquad \lambda f(t) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}}, \qquad t = \left(\frac{k_1}{m_0}\right)^{\frac{1}{2}} T \tag{4}$$

$$\underline{\xi w} = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} \underline{W}, \qquad 2\alpha\underline{\xi} = \frac{C_0}{(m_0 k_1)^{\frac{1}{2}}}, \tag{5}$$

$$0 < \alpha < 1, 0 < \lambda < 1, 0 < \underline{\xi} < 1$$

On substituting (4) and (5) into (1) – (3) and simplifying, we get

$$w_{,tt} + 2\alpha\underline{\xi} w_{,t} + w_{,xxxx} + 2\lambda f(t) w_{,xx} + w - k_2 w^2 - \beta w^3 = -2\lambda f(t)\xi\underline{w}_{,xx}, \quad t > 0, \quad 0 < x < \pi \tag{6}$$

$$w(x,t) = w_{,xx}(x,t) = 0 \quad at \;\; x = 0, \pi \tag{7}$$

$$w(x,0) = w_{,t}(x,0) = 0, \; 0 < x < \pi \tag{8}$$

where

$$k_2 = \left(\frac{\beta_1}{k_1 k_3}\right)$$

We note that $\underline{\xi}$ is the amplitude of the imperfection, deemed small relative to unity, while the damping is taken in the order of the amplitude of the imperfection $\underline{\xi}$. Here, f(t) is the time dependent loading history which in our case is the step load characterized by

$$f(t) = 1,\ t > 0 \qquad (9)$$

and λ is the amplitude of the loading history whose specific value at buckling, namely λ_D, is what we are to evaluate. Normally, as a result of our non-dimensionalization, the inequality $0 < \lambda < 1$ holds. We call λ_D, the dynamic buckling load and it is defined as the largest value of λ for the solution of (2.6) – (2.8) to be bounded for all time t > 0. As in Amazigo and Frank (1971), we determine λ_D from the maximization

$$\frac{d\lambda}{dW_a} = 0 \qquad (10)$$

where $\boldsymbol{W_a}$, as the maximum lateral (normal) displacement, is a function of the load function $\boldsymbol{\lambda}$. In order to utilize this maximization, we shall first determine the displacement w(x, t) subsequent upon which we determine the maximum lateral displacement $\boldsymbol{W_a}$.

4. 0 Perturbation and Asymptotics Analysis

In solving Eq.3,we introduce a slow time scale τ as in Egbuhuzor and Mezie (2014)and is meant to take care of the effects of imperfection and damping, such that

$$\tau = \underline{\xi} t \qquad (11)$$

Thus, our displacement $w(x,t)$, will now be expressed as

$$w(x,t) \equiv U\left(x,t,\tau,\underline{\xi}\right)$$

$$w_{,t}(x,t) = U_{,t} + U_{,\tau} \qquad (12)$$

$$w_{,tt}(x,t) = U_{,tt} + 2\underline{\xi} U_{,t\tau} + \underline{\xi}^2 U_{,\tau\tau} \qquad (13)$$

By regular perturbation method, we let

$$U\left(x,t,\tau,\underline{\xi}\right) = \sum_{i=1}^{\infty} U^i(x,t,\tau)\underline{\xi}^i \qquad (14)$$

On substituting (12)-(14) into (4)-(6), and assuming term-wise differentiation we get

$$O\left(\underline{\xi}\right):\ U_{,tt}^{(1)} + U_{,xxxx}^{(1)} + 2\lambda U_{,xx}^{(1)} + U^{(1)} = -2\lambda \underline{w}_{,xx} \quad (15)$$

$$O\left(\underline{\xi}^2\right):\ U_{,tt}^{(2)} + U_{,xxxx}^{(2)} + 2\lambda U_{,xx}^{(2)} + U^{(2)} = -2U_{,t\tau}^{(1)} - 2\alpha U_{,t}^{(1)} + k_2 (U^{(1)})^2 \quad (16)$$

$$O\left(\underline{\xi}^3\right):\ U_{,tt}^{(3)} + U_{,xxxx}^{(3)} + 2\lambda U_{,xx}^{(3)} + U^{(3)} = -2U_{,t\tau}^{(2)} - 2\alpha U_{,t}^{(2)} - 2\alpha U_{,\tau}^{(1)} - U_{,\tau\tau}^{(1)}$$
$$+2k_2 U^{(1)} U^{(2)} + \beta (U^{(1)})^3 \qquad (17)$$

with initial conditions:

$$U^i(x,0,0) = 0 \qquad for\ i = 1,2,3,\ldots \tag{18}$$

$$U_{,t}^{(1)}(x,0,0) = 0 \tag{19}$$

$$U_{,t}^{(2)}(x,0,0) + U_{,\tau}^{(1)}(x,0,0) = 0 \tag{20}$$

$$U_{,t}^{(3)}(x,0,0) + U_{,\tau}^{(2)}(x,0,0) = 0 \tag{21}$$

The associated boundary conditions are

$$U^{(i)}(x,t,\tau) = U_{,xx}^{(i)}(x,t,\tau) = 0 \quad at\ \ x = 0, \pi\ , i = 1,2,3,\ldots \tag{22}$$

Based on the boundary conditions (2.2), we let

$$\underline{w}(x) = \underline{a}_m\ sin\ sinmx, |\underline{a}_m| < 1, m\ is\ a\ fixed\ integer \tag{23}$$

$$U^i(x,t,\tau) = \sum_{n=1}^{\infty} \quad U_n^i(t,\tau) sin\ nx \tag{24}$$

We now substitute (23), (24) into (17) and obtain for i =1, multiply the resultant equation through $sinmx$ and integrate over the entire column from 0 to 1, and note that the only non vanishing value of $U_n^{(1)}$ is when n=m. Thus, for n=m, we have that

$$U_{m,tt}^{(1)} + \varphi_m^2 U_m^{(1)} = 2m^2 \lambda \underline{a}_m \tag{25}$$

$$U_m^{(1)}(0,0) = 0, U_{m,t}^{(1)}(0,0) = 0 \tag{26}$$

On solving (25, 26), we obtain

$$U_m^{(1)}(t,\tau) = \alpha_m^{(1)}(\tau) cos\varphi_m t + \beta_m^{(1)}(\tau) sin\varphi_m t + B_m \tag{27}$$

$$B_m = \frac{2m^2 \lambda \underline{a}_m}{\varphi_m^2} \tag{28}$$

$$\alpha_m^{(1)}(0) = -B_m, \qquad \beta_m^{(1)}(0) = 0 \tag{29}$$

Where (29) is obtained by using (28)

Thus from (24), we obtain the first component of the lateral displacement as

$$U^{(1)}(x,t,\tau) = U_m^{(1)}(t,\tau) sin\ mx \qquad (29)$$

Where $\alpha_m^{(1)}(\tau)\ and\ \beta_m^{(1)}(\tau)$ are yet to be determined in full

Next, we substitute Eq.(24), for $i = 2$ into Eq(16) using Eq.(23), multiply the resultant equation by $sinmx$ and integrate over the entire column from 0 to π

$$U_{m,tt}^{(2)} + \varphi_m^2 U_m^{(2)} = -2U_{m,t\tau}^{(1)} - 2\alpha U_{m,t}^{(1)} + \frac{8k_2}{3m\pi}(U_m^{(1)})^2 \tag{30}$$

$$U_m^{(2)}(0,0) = 0, \ \ U_{m,t}^{(2)}(0,0) + U_{m,\tau}^{(1)}(0,0) = 0 \tag{31}$$

Meanwhile we note the following, using (25, 26)

$$U_{m,t\tau}^{(1)} = -\varphi_m \dot{\alpha}_m^{(1)} sin\ \varphi_m t + \varphi_m \dot{\beta}_m^{(1)} cos\ \varphi_m t \tag{32}$$

$$where \qquad (\ \dot{}\ ) = \frac{d(\,)}{d\tau}$$

$$U_{m,t}^{(1)} = -\varphi_m \alpha_m^{(1)} sin\ \varphi_m t + \varphi_m \beta_m^{(1)} scos\ \varphi_m t \tag{33}$$

$$(U_m^{(1)})^2 = S_m + \ \theta_m cos\ 2\varphi_m t + \alpha_m^{(1)} \beta_m^{(1)} sin2\ \varphi_m t + 2\alpha_m^{(1)} B_m cos\ \varphi_m t + \ 2\beta_m^{(1)} B_m sin\ \varphi_m t \tag{34}$$

$$where \qquad S_m(\tau) = \frac{1}{2}\alpha_m^{(1)^2} + \frac{1}{2}\beta_m^{(1)^2} + B_m^2 \tag{35}$$

$$\theta_m(\tau) = \frac{1}{2}\alpha_m^{(1)^2} - \frac{1}{2}\beta_m^{(1)^2} \tag{36}$$

Using (33), we obtain from (35, 36) $S_m(0) = \frac{3}{2}B_m^2$, $\theta_m(0) = \frac{1}{2}B_m^2$

We now rearrange (35) in terms of $sin\ \varphi_m t$,$cos\ \varphi_m t$, $sin\ 2\varphi_m t$, and $cos\ 2\varphi_m t$, to obtain

$$U_{m,tt}^{(2)} + \varphi_m^2 U_m^{(2)} = R_0 cos\ \varphi_m t + R_1 sin\ \varphi_m t$$

$$+\frac{8k_2}{3m\pi}\left\{S_m + \theta_m cos\ 2\varphi_m t + \alpha_m^{(1)}\beta_m^{(1)} sin2\ \varphi_m t\right\} \quad (42) where \quad R_0$$

$$= -2\varphi_m \dot{\beta}_m^{(1)} - 2\alpha\varphi_m\beta_m^{(1)} + \frac{16k_2\alpha_m^{(1)}B_m}{3m\pi} \tag{37}$$

$$R_1 = 2\varphi_m\dot{\alpha}_m^{(1)} + 2\alpha\varphi_m\alpha_m^{(1)} + \frac{16k_2\beta_m^{(1)}B_m}{3m\pi} \tag{38}$$

Now, since we are dealing with a real life situation, for us to obtain a uniformly valid solution of (38) in terms of t, we must equate the coefficients of $sin\ \varphi_m t$ and $cos\ \varphi_m t$ to zero. This will result to

$$\dot{\beta}_m^{(1)} + \alpha\beta_m^{(1)} = w_m\alpha_m^{(1)} \tag{39}$$

$$\dot{\alpha}_m^{(1)} + \alpha\alpha_m^{(1)} = -w_m\beta_m^{(1)} \tag{40}$$

$$w_m = \frac{8k_2B_m}{3m\varphi_m\pi} \tag{41}$$

Solving (39, 36) simultaneously, we obtain that

$$\beta_m^{(1)}(\tau) = -B_m e^{-\alpha\tau} sin\ w_m\tau \tag{42}$$

$$\alpha_m^{(1)}(\tau) = -B_m e^{-\alpha\tau} cos\ w_m\tau \tag{43}$$

Thus with (48, 49), we have determined $U_m^{(1)}(t,\tau)$ in full

From (40, 41), we also obtain that

$$\dot{\beta}_m^{(1)}(0) = -w_mB_m\ , \qquad \dot{\alpha}_m^{(1)}(0) = \alpha B_m \tag{44}$$

Now, the remaining equation in (42) is

$$U_{m,tt}^{(2)} + \varphi_m^2 U_m^{(2)} = \frac{8k_2}{3m\pi}\left\{S_m + \theta_m cos\ 2\varphi_m t + \alpha_m^{(1)}\beta_m^{(1)} sin2\ \varphi_m t\right\} \tag{45}$$

we obtain that

$$U_m^{(2)}(t,\tau) = \alpha_m^{(2)}(\tau)cos\varphi_m t + \beta_m^{(2)}(\tau)sin\varphi_m t$$

$$+\frac{8k_2}{3m\pi}\left\{\frac{S_m}{\varphi_m^2} - \frac{\theta_m cos\ 2\varphi_m t}{3\varphi_m^2} - \frac{-\alpha_m^{(1)}\beta_m^{(1)} sin\ 2\varphi_m t}{3\varphi_m^2}\right\} \tag{46}$$

$$\alpha_m^{(2)}(0) = -\frac{32k_2B_m^2}{9m\varphi_m^2\pi} \tag{47}$$

$$\beta_m^{(2)}(0) = -\frac{\alpha B_m}{\varphi_m} \tag{48}$$

$$U^{(2)}(x,t,\tau) = U_m^{(2)}(t,\tau) sin\ mx \tag{49}$$

We obtain, for i=3,

$$\sum_{n=1}^{\infty} \left\{U_{n,tt}^{(3)} + \varphi_n^2 U_n^{(3)}\right\} sin\, nx = -2U_{,t\tau}^{(2)} - 2\alpha U_{,t}^{(2)} - 2\alpha U_{,\tau}^{(1)} - U_{,\tau\tau}^{(1)}$$
$$+2k_2 U^{(1)} U^{(2)} + \beta (U^{(1)})^3 \qquad (50)$$

Also, we substitute for (34) and (55) on the right hand side of (56) obtaining

$$\sum_{n=1}^{\infty} \left\{U_{n,tt}^{(3)} + \varphi_n^2 U_n^{(3)}\right\} sin\, nx = -2U_{m,t\tau}^{(2)} sin\, mx - 2\alpha U_{m,t}^{(2)} sin\, mx$$
$$-2\alpha U_{m,\tau}^{(1)} sin\, mx - U_{m,\tau\tau}^{(1)} sin\, mx \; + 2k_2 U^{(1)} U^{(2)} sin^2 mx + \beta (U^{(1)})^3 sin^3 mx (51)$$

This in linear form gives

$$\sum_{n=1}^{\infty} \left\{U_{n,tt}^{(3)} + \varphi_n^2 U_n^{(3)}\right\} sin\, nx = -2U_{m,t\tau}^{(2)} sin\, mx - 2\alpha U_{m,t}^{(2)} sin\, mx$$

$$-2\alpha U_{m,\tau}^{(1)} sin\, mx - U_{m,\tau\tau}^{(1)} sin\, mx \; + k_2 U^{(1)} U^{(2)} (1 - cos\, 2mx)$$
$$+ \frac{\beta (U^{(1)})^3}{4} (3 sin\, mx - sin\, 3mx) \qquad (52)$$

We multiply (52) in turn by Sin mx and Sin $3mx$ respectively obtaining

$$\sum_{n=1}^{\infty} \left\{U_{n,tt}^{(3)} + \varphi_n^2 U_n^{(3)}\right\} sin\, nx sin\, mx = -2U_{m,t\tau}^{(2)} sin^2 mx - 2\alpha U_{m,t}^{(2)} sin^2 mx$$
$$-2\alpha U_{m,\tau}^{(1)} sin^2 mx - U_{m,\tau\tau}^{(1)} sin^2 mx + k_2 U^{(1)} U^{(2)} (1 - cos\, 2mx) sin\, mx$$
$$+ \frac{\beta (U^{(1)})^3}{4} (3 sin\, mx - sin\, 3mx) sin\, mx \qquad (53)$$

and

$$\sum_{n=1}^{\infty} \left\{U_{n,tt}^{(3)} + \varphi_n^2 U_n^{(3)}\right\} sin\, nx sin\, 3mx = -2U_{m,t\tau}^{(2)} sin\, mx sin\, 3mx$$
$$-2\alpha U_{m,t}^{(2)} sin\, mx sin\, 3mx - 2\alpha U_{m,\tau}^{(1)} sin\, mx sin\, 3mx - U_{m,\tau\tau}^{(1)} sin\, mx sin\, 3mx$$
$$k_2 U^{(1)} U^{(2)} (1 - cos\, 2mx) sin\, 3mx + \frac{\beta (U^{(1)})^3}{4} (3 sin\, mx - sin\, 3mx) sin\, 3mx (54)$$

We integrate (53) over the entire column and notice, as in (24), that the only non- vanishing value of $U_n^{(3)}$ is when n=m. Thus for n=m, where values of m are odd, we have that

$$U_{m,tt}^{(3)} + \varphi_m^2 U_m^{(3)} = -2U_{m,t\tau}^{(2)} - U_{m,\tau\tau}^{(1)} - 2\alpha U_{m,t}^{(2)} - 2\alpha U_{m,\tau}^{(1)}$$
$$+ \frac{16 k_2}{3m\pi} U_m^{(1)} U_m^{(2)} - \frac{3\beta}{4} (U_m^{(1)})^3 \qquad (55)$$
$$U_m^{(3)}(0,0) = 0, U_{m,t}^{(3)}(0,0) + U_{m,\tau}^{(2)}(0,0) = 0 \qquad (56)$$

We note the following

$$U_{m,\tau}^{(1)} = \dot{\alpha}_m^{(1)} cos\, \varphi_m t + \dot{\beta}_m^{(1)} sin\, \varphi_m t \qquad (57)$$
$$U_{m,\tau\tau}^{(1)} = \ddot{\alpha}_m^{(1)} cos\, \varphi_m t + \ddot{\beta}_m^{(1)} sin\, \varphi_m t \qquad (58)$$

$$U_{m,t}^{(2)} = -\varphi_m\left(\alpha_m^{(2)}\sin\varphi_m t - \beta_m^{(2)}\cos\varphi_m t\right) + \frac{16\varphi_m k_2}{3m\pi}\left\{\frac{\theta_m \sin 2\varphi_m t}{3\varphi_m^2} - \alpha_m^{(1)}\beta_m^{(1)}\frac{\cos 2\varphi_m t}{3\varphi_m^2}\right\} \quad (59)$$

$$U_{m,t\tau}^{(2)} = -\varphi_m\left(\dot{\alpha}_m^{(2)}\sin\varphi_m t - \dot{\beta}_m^{(2)}\cos\varphi_m t\right) + \frac{16\varphi_m k_2}{3m\pi}\left\{\frac{\dot{\theta}_m \sin 2\varphi_m t}{3\varphi_m^2} - \frac{\left(\alpha_m^{(1)}\beta_m^{(1)}\right)^{\cdot}\cos 2\varphi_m t}{3\varphi_m^2}\right\} \quad (60)$$

$$\begin{aligned} U_m^{(1)}U_m^{(2)} = {} & \frac{\alpha_m^{(1)}\alpha_m^{(2)}}{2} + \frac{\beta_m^{(1)}\beta_m^{(2)}}{2} + \frac{8k_2 B_m S_m}{3m\pi\varphi_m^2} \\ & + \left\{\frac{8k_2\alpha_m^{(1)}}{3m\pi\varphi_m^2}\left(S_m - \frac{\theta_m}{6} - \frac{{\beta_m^{(1)}}^2}{6}\right) + B_m\alpha_m^{(2)}\right\}\cos\varphi_m t \\ & + \left\{\frac{8k_2\beta_m^{(1)}}{3m\pi\varphi_m^2}\left(S_m + \frac{\theta_m}{6} - \frac{{\alpha_m^{(1)}}^2}{6}\right) + B_m\beta_m^{(2)}\right\}\sin\varphi_m t \\ & + \left\{\frac{\alpha_m^{(1)}\alpha_m^{(2)}}{2} - \frac{\beta_m^{(1)}\beta_m^{(2)}}{2} - \frac{8k_2 B_m\theta_m}{9m\pi\varphi_m^2}\right\}\cos 2\varphi_m t \\ & + \left\{\frac{\alpha_m^{(1)}\beta_m^{(2)}}{2} + \frac{\beta_m^{(1)}\alpha_m^{(2)}}{2} - \frac{8k_2 B_m\alpha_m^{(1)}\beta_m^{(1)}}{9m\pi\varphi_m^2}\right\}\sin 2\varphi_m t \\ & + \left\{\frac{8k_2\alpha_m^{(1)}}{3m\pi\varphi_m^2}\left(-\frac{\theta_m}{6} + \frac{{\beta_m^{(1)}}^2}{6}\right)\right\}\cos 3\varphi_m t \\ & + \left\{\frac{8k_2\beta_m^{(1)}}{3m\pi\varphi_m^2}\left(-\frac{\theta_m}{6} - \frac{{\alpha_m^{(1)}}^2}{6}\right)\right\}\sin 3\varphi_m t \end{aligned} \quad (61)$$

$$\begin{aligned} {U_m^{(1)}}^3 = {} & \frac{B_m}{2}\left(3{\alpha_m^{(1)}}^2 + 3{\beta_m^{(1)}}^2 + 2B_m^2\right) + \left\{\frac{3\alpha_m^{(1)}}{4}\left({\alpha_m^{(1)}}^2 + {\beta_m^{(1)}}^2 + 4B_m^2\right)\right\}\cos\varphi_m t \\ & + \frac{3\beta_m^{(1)}}{4}\left({\alpha_m^{(1)}}^2 + {\beta_m^{(1)}}^2 + 4B_m^2\right)\sin\varphi_m t + \frac{3B_m}{2}\left({\alpha_m^{(1)}}^2 - {\beta_m^{(1)}}^2\right)\cos 2\varphi_m t \\ & + 3B_m\alpha_m^{(1)}\beta_m^{(1)}\sin 2\varphi_m t + \frac{\alpha_m^{(1)}}{4}\left({\alpha_m^{(1)}}^2 - 3{\beta_m^{(1)}}^2\right)\cos 3\varphi_m t \\ & + \frac{\beta_m^{(1)}}{4}\left(3{\alpha_m^{(1)}}^2 - {\beta_m^{(1)}}^2\right)\sin 3\varphi_m t \end{aligned} \quad (62)$$

We also integrate (3.40) over the entire column and notice that, the only non- vanishing value of $U_n^{(3)}$ is when n=3m. Thus, for n=3m, where values of m are odd, we have that

$$U_{3m,tt}^{(3)} + \varphi_{3m}^2 U_{3m}^{(3)} = -\frac{16k_2}{15m\pi}U_m^{(1)}U_m^{(2)} - \frac{\beta}{4}(U_m^{(1)})^3 \quad (63)$$

$$U_{3m}^{(3)}(0,0) = 0, \qquad U_{3m,t}^{(3)}(0,0) = 0 \quad (64)$$

$$where \quad \varphi_{3m}^2 = 81m^4 - 18m^2\lambda + 1 \quad (65)$$

If we substitute for $U_m^{(1)} and\ U_m^{(2)}$ in (69) and simplify, we get

$$U_{3m,tt}^{(3)} + \varphi_{3m}^2 U_{3m}^{(3)} = T_0 + T_1 cos\ \varphi_m t + T_2 sin\ \varphi_m t + T_3 cos\ 2\varphi_m t + T_4 sin 2\varphi_m t + T_5 cos\ 3\varphi_m t + T_6 sin\ 3\varphi_m t \quad (66)$$

where

$$T_0(\tau) = -\frac{16k_2}{15m\pi}\left\{\frac{\alpha_m^{(1)}\alpha_m^{(2)}}{2} + \frac{\beta_m^{(1)}\beta_m^{(2)}}{2} + \frac{8k_2 B_m S_m}{3m\pi\varphi_m^2}\right\} - \frac{\beta B_m}{8}\left\{3{\alpha_m^{(1)}}^2 + 3{\beta_m^{(1)}}^2 + 2B_m^2\right\} \quad (67)$$

$$T_0(0) = -\frac{832k_2^2 B_m^3}{135(m\pi\varphi_m)^2} - \frac{5\beta B_m^3}{8} \quad (68)$$

$$T_1(\tau) = -\frac{16k_2}{15m\pi}\left\{\frac{8k_2\alpha_m^{(1)}}{3m\pi\varphi_m^2}\left(S_m - \frac{\theta_m}{6} - \frac{{\beta_m^{(1)}}^2}{6}\right) + B_m\alpha_m^{(2)}\right\} - \frac{3\beta\alpha_m^{(1)}}{16}\left\{{\alpha_m^{(1)}}^2 + {\beta_m^{(1)}}^2 + 4B_m^2\right\} \quad (69)$$

$$T_1(0) = \frac{352k_2^2 B_m^3}{45(m\pi\varphi_m)^2} + \frac{15\beta B_m^3}{16} \quad (70)$$

$$T_2(\tau) = -\frac{16k_2}{15m\pi}\left\{\frac{8k_2\beta_m^{(1)}}{3m\pi\varphi_m^2}\left(S_m + \frac{\theta_m}{6} - \frac{{\alpha_m^{(1)}}^2}{6}\right) + B_m\beta_m^{(2)}\right\} - \frac{3\beta\beta_m^{(1)}}{16}\left\{{\alpha_m^{(1)}}^2 + {\beta_m^{(1)}}^2 + 4B_m^2\right\} \quad (71)$$

$$T_2(0) = \frac{16\alpha k_2 B_m^2}{15m\pi\varphi_m} \quad (72)$$

$$T_3(\tau) = -\frac{16k_2}{15m\pi}\left(\frac{\alpha_m^{(1)}\alpha_m^{(2)}}{2} - \frac{\beta_m^{(1)}\beta_m^{(2)}}{2} - \frac{8k_2 B_m\theta_m}{9m\pi\varphi_m^2}\right) - \frac{3\beta B_m\theta_m}{4} \quad (73)$$

$$T_3(0) = -\frac{64k_2^2 B_m^3}{45(m\pi\varphi_m)^2} - \frac{3\beta B_m^3}{8} \quad (74)$$

$$T_4(\tau) = -\frac{16k_2}{15m\pi}\left(\frac{\alpha_m^{(1)}\beta_m^{(2)}}{2} - \frac{\beta_m^{(1)}\alpha_m^{(2)}}{2} - \frac{8k_2 B_m\alpha_m^{(1)}\beta_m^{(1)}}{9m\pi\varphi_m^2}\right) - \frac{3\beta B_m\alpha_m^{(1)}\beta_m^{(1)}}{4} \quad (75)$$

$$T_4(0) = -\frac{8\alpha k_2 B_m^2}{15m\pi\varphi_m} \quad (76)$$

$$T_5(\tau) = -\frac{16k_2}{15m\pi}\left\{\frac{8k_2\alpha_m^{(1)}}{3m\pi\varphi_m^2}\left(-\frac{\theta_m}{6} + \frac{{\beta_m^{(1)}}^2}{6}\right)\right\} - \frac{\beta\alpha_m^{(1)}}{16}\left({\alpha_m^{(1)}}^2 - 3{\beta_m^{(1)}}^2\right) \quad (77)$$

$$T_5(0) = -\frac{32k_2^2 B_m^3}{135(m\pi\varphi_m)^2} + \frac{\beta B_m^3}{16} \quad (78)$$

$$T_6(\tau) = -\frac{16k_2}{15m\pi}\left\{\frac{8k_2\beta_m^{(1)}}{3m\pi\varphi_m^2}\left(-\frac{\theta_m}{6} - \frac{{\alpha_m^{(1)}}^2}{6}\right)\right\} - \frac{\beta\beta_m^{(1)}}{16}\left(3{\alpha_m^{(1)}}^2 - {\beta_m^{(1)}}^2\right) \quad (79)$$

$$T_6(0) = 0 \quad (80)$$

We now arrange (61) in terms of $cos\ \varphi_m t, sin\ \varphi_m t, cos\ 2\varphi_m t$, sin$2\varphi_m t, cos\ 3\varphi_m t$ and sin $3\varphi_m t$

and obtain

$$U_{m,tt}^{(3)} + \varphi_m^2 U_m^{(3)} = S_0 + S_1 cos\ \varphi_m t + S_2 sin\ \varphi_m t + S_3 cos\ 2\varphi_m t$$
$$+ S_4 sin 2\varphi_m t + S_5 cos\ 3\varphi_m t + S_6 sin\ 3\varphi_m t \quad (81)$$

$$S_1(\tau) = -2\varphi_m\left(\dot{\beta}_m^{(2)} + 2\alpha\beta_m^{(2)}\right) - \ddot{\alpha}_m^{(1)} - 2\alpha\dot{\alpha}_m^{(1)} + \frac{16k_2\alpha_m^{(2)}B_m}{3m\pi}$$
$$+ \alpha_m^{(1)}\left\{\frac{128k_2^2}{9(m\pi\varphi_m)^2}\left(S_m - \frac{\theta_m}{6} - \frac{{\beta_m^{(1)}}^2}{6}\right) + \frac{9\beta}{16}\left({\alpha_m^{(1)}}^2 + {\beta_m^{(1)}}^2 + 4B_m^2\right)\right\} \quad (82)$$

$$S_2(\tau) = 2\varphi_m\left(\dot{\alpha}_m^{(2)} + 2\alpha\alpha_m^{(2)}\right) - \ddot{\beta}_m^{(1)} - 2\alpha\dot{\beta}_m^{(1)} + \frac{16k_2\beta_m^{(2)}B_m}{3m\pi}$$
$$+ \beta_m^{(1)}\left\{\frac{128k_2^2}{9(m\pi\varphi_m)^2}\left(S_m + \frac{\theta_m}{6} - \frac{{\alpha_m^{(1)}}^2}{6}\right)\right.$$
$$\left. + \frac{9\beta}{16}\left({\alpha_m^{(1)}}^2 + {\beta_m^{(1)}}^2 + 4B_m^2\right)\right\} \quad (83)$$

$$S_0(\tau) = \frac{16k_2}{3m\pi}\left(\frac{\alpha_m^{(1)}\alpha_m^{(2)}}{2} + \frac{\beta_m^{(1)}\beta_m^{(2)}}{2} + \frac{8k_2B_mS_m}{3m\pi\varphi_m^2}\right) + \frac{3\beta B_m}{8}\left(3{\alpha_m^{(1)}}^2 + 3{\beta_m^{(1)}}^2 + 2B_m^2\right) \quad (84)$$

$$S_0(0) = \frac{832k_2^2B_m^3}{27(m\pi\varphi_m)^2} + \frac{15\beta B_m^3}{8} \quad (85)$$

$$S_3(\tau) = \frac{32k_2}{9m\pi\varphi_m}\left\{\left(\alpha_m^{(1)}\beta_m^{(1)}\right)^{\cdot} + \alpha\alpha_m^{(1)}\beta_m^{(1)}\right\}$$
$$+ \frac{16k_2}{3m\pi}\left(\frac{\alpha_m^{(1)}\alpha_m^{(2)}}{2} - \frac{\beta_m^{(1)}\beta_m^{(2)}}{2} - \frac{8k_2B_m\theta_m}{9m\pi\varphi_m^2}\right) + \frac{9\beta\theta_mB_m}{4} \quad (86)$$

$$S_3(0) = \frac{448k_2^2B_m^3}{27(m\pi\varphi_m)^2} + \frac{9\beta B_m^3}{8} \quad (87)$$

$$S_4(\tau) = -\frac{32k_2}{9m\pi\varphi_m}\left(\dot{\theta}_m - \alpha\theta_m\right) + \frac{16k_2}{3m\pi}\left(\frac{\alpha_m^{(1)}\beta_m^{(2)}}{2} + \frac{\beta_m^{(1)}\alpha_m^{(2)}}{2} - \frac{8k_2B_m\alpha_m^{(1)}\beta_m^{(1)}}{9m\pi\varphi_m^2}\right)$$
$$+ \frac{9\beta\alpha_m^{(1)}\beta_m^{(1)}B_m}{4} \quad (94)$$

$$S_4(0) = \frac{40\alpha k_2B_m^2}{9m\pi\varphi_m} \quad (88)$$

$$S_5(\tau) = \frac{16k_2}{3m\pi}\left\{\frac{8k_2\alpha_m^{(1)}}{3m\pi\varphi_m^2}\left(-\frac{\theta_m}{6} + \frac{{\beta_m^{(1)}}^2}{6}\right)\right\} + \frac{3\beta\alpha_m^{(1)}}{16}\left({\alpha_m^{(1)}}^2 - 3{\beta_m^{(1)}}^2\right) \quad (89)$$

$$S_5(0) = \frac{32k_2^2B_m^3}{27(m\pi\varphi_m)^2} - \frac{3\beta B_m^3}{16} \quad (90)$$

$$S_6(\tau)=\frac{16k_2}{3m\pi}\left\{\frac{8k_2\beta_m^{(1)}}{3m\pi\varphi_m^2}\left(-\frac{\theta_m}{6}-\frac{{\alpha_m^{(1)}}^2}{6}\right)\right\}+\frac{3\beta\beta_m^{(1)}}{16}\left(3{\alpha_m^{(1)}}^2-{\beta_m^{(1)}}^2\right) \quad (91)$$

$$S_6(0)=0 \quad (92)$$

Using $S_i(\tau)$, i=0,1,…,6 at $\tau=0$ above.

Now, to ensure a uniformly valid solution in terms of t, we equate the coefficient s of cos $\varphi_m t,$ and sin $\varphi_m t,$ to zero and obtain the respective equations

$$\dot{\beta}_m^{(2)}+\alpha\beta_m^{(2)}=\frac{1}{2\varphi_m}\left\{\frac{16k_2\alpha_m^{(2)}B_m}{3m\pi}+\alpha_m^{(1)}\left(\frac{128k_2^2}{9(m\pi\varphi_m)^2}\left(S_m-\frac{\theta_m}{6}+\frac{{\beta_m^{(1)}}^2}{6}\right)+\frac{9\beta}{16}\left({\alpha_m^{(1)}}^2+{\beta_m^{(1)}}^2+4B_m^2\right)\right)-\ddot{\alpha}_m^{(1)}-2\alpha\dot{\alpha}_m^{(1)}\right\} \quad (93)$$

$$\dot{\alpha}_m^{(2)}+\alpha\alpha_m^{(2)}=-\frac{1}{2\varphi_m}\left\{\frac{16k_2\beta_m^{(2)}B_m}{3m\pi}+\beta_m^{(1)}\left(\frac{128k_2^2}{9(m\pi\varphi_m)^2}\left(S_m+\frac{\theta_m}{6}-\frac{{\alpha_m^{(1)}}^2}{6}\right)+\frac{9\beta}{16}\left({\alpha_m^{(1)}}^2+{\beta_m^{(1)}}^2+4B_m^2\right)\right)-\ddot{\beta}_m^{(1)}-2\alpha\dot{\beta}_m^{(1)}\right\} \quad (94)$$

Since explicit values of $\beta_m^{(2)}$ $and\alpha_m^{(2)}$ are not needed in this analysis, we shall omit determining them. However, values of $\dot{\beta}_m^{(2)}(0)$ and $\dot{\alpha}_m^{(2)}(0)$ will be needed and so from (93, 94), we get at $\tau=0$

$$\dot{\beta}_m^{(2)}(0)=-\frac{368k_2^2B_m^3}{27\varphi_m(m\pi\varphi_m)^2}-\frac{45\beta B_m^3}{32\varphi_m} \quad (95)$$

$$\dot{\alpha}_m^{(2)}(0)=\frac{56\alpha k_2B_m^2}{9m\pi\varphi_m^2} \quad (96)$$

The remaining equation is

$$U_{m,tt}^{(3)}+\varphi_m^2U_m^{(3)}=S_0+S_3cos\,2\varphi_m t+S_4sin2\varphi_m t+S_5cos\,3\varphi_m t+S_6sin\,3\varphi_m t \quad (97)$$

On solving (95), we obtain the following

$$U_m^{(3)}(t,\tau)=\alpha_m^{(3)}(\tau)cos\varphi_m t+\beta_m^{(3)}(\tau)sin\varphi_m t+\frac{S_0}{\varphi_m^2}-\frac{S_3cos\,2\varphi_m t}{3\varphi_m^2}-\frac{S_4sin2\varphi_m t}{3\varphi_m^2}-\frac{S_5cos\,3\varphi_m t}{8\varphi_m^2}-\frac{S_6sin\,3\varphi_m t}{8\varphi_m^2} \quad (98)$$

$$\alpha_m^{(3)}(0) = -\frac{2036k_2^2B_m^3}{81\varphi_m^2(m\pi\varphi_m)^2} - \frac{195\beta B_m^3}{128\varphi_m^2} \qquad (99)$$

$$\beta_m^{(3)}(0) = -\frac{40\alpha k_2 B_m^2}{27(m\pi\varphi_m^2)\varphi_m} \qquad (100)$$

We obtain the following

$$U_{3m}^{(3)}(t,\tau) = \alpha_{3m}^{(3)}(\tau)cos\varphi_{3m}t + \beta_{3m}^{(3)}(\tau)sin\varphi_{3m}t + \frac{T_0}{\varphi_{3m}^2} + \frac{T_1 cos\ \varphi_m t}{\varphi_{3m}^2 - \varphi_m^2} + \frac{T_2 sin\varphi_m t}{\varphi_{3m}^2 - \varphi_m^2} + \frac{T_3 cos\ 2\varphi_m t}{\varphi_{3m}^2 - 4\varphi_m^2} + \frac{T_4 sin2\varphi_m t}{\varphi_{3m}^2 - \varphi_m^2} + \frac{T_5 cos\ 3\varphi_m t}{\varphi_{3m}^2 - 9\varphi_m^2} + \frac{T_6 sin\ 3\varphi_m t}{\varphi_{3m}^2 - 9\varphi_m^2} \qquad (101)$$

$$\alpha_{3m}^{(3)}(0) = \frac{32k_2^2B_m^3}{135(m\pi\varphi_m)^2}\left\{\frac{26}{\varphi_{3m}^2} - \frac{33}{\varphi_{3m}^2-\varphi_m^2} - \frac{6}{\varphi_{3m}^2-4\varphi_m^2} + \frac{1}{\varphi_{3m}^2-9\varphi_m^2}\right\} - \frac{\beta B_m^3}{16}\left\{\frac{10}{\varphi_{3m}^2} - \frac{15}{\varphi_{3m}^2-\varphi_m^2} + \frac{6}{\varphi_{3m}^2-4\varphi_m^2} - \frac{1}{\varphi_{3m}^2-\varphi_m^2}\right\} \qquad (102)$$

$$\beta_{3m}^{(3)}(0) = \frac{16\alpha k_2 B_m^2}{15m\pi\varphi_{3m}}\left\{-\frac{1}{\varphi_{3m}^2-\varphi_m^2} + \frac{1}{\varphi_{3m}^2-4\varphi_m^2}\right\} \qquad (103)$$

Thus, the expression for the third component of the lateral displacement is

$$U^{(3)}(x,t,\tau) = U_m^{(3)}(t,\tau)sin\ mx + U_{3m}^{(3)}(t,\tau)sin\ 3mx \qquad (104)$$

we obtain the expression for the lateral displacement as $U\left(x,t,\tau,\underline{\xi}\right) = U^{(1)}(x,t,\tau)\underline{\xi} + U^{(2)}(x,t,\tau)\underline{\xi}^2 + U^{(3)}(x,t,\tau)\underline{\xi}^3 + \cdots$

$$= \left(U_m^{(1)}(t,\tau)\underline{\xi} + U_m^{(2)}(t,\tau)\underline{\xi}^2 + U_m^{(3)}(t,\tau)\underline{\xi}^3\right) sin\ m + U_{3m}^{(3)}(t,\tau)\underline{\xi}^3 sin\ 3mx + \cdots \qquad (105)$$

5. 0 The Dynamic Buckling Load

The maximum load, sometimes called the critical load causes a column to be in a state of unstable equilibrium, that is, any increase in the loads or the introduction of the slightest lateral force will cause the column to fail by buckling.

We can determine the dynamic buckling load $\lambda_D$of the damped imperfect column by maximizing the load parameter λ with respect to the maximum displacement $U_a(\lambda)$as in (13). This can be achieved as in Elishakoff (1979), by reversing (121) and obtaining $\underline{\xi}$as an asymptotic series in powers of $U_a(\lambda)$, that is, setting

$$\underline{\xi} = e_1U_a(\lambda) + e_2U_a^2(\lambda) + e_3U_a^3(\lambda) + \ ... \qquad (106)$$

and determining the$e_i's$, i=1,2,3,…, which are functions of λ. Substituting and higher powers we obtain

$$\underline{\xi} = e_1C_1\underline{\xi} + \left(e_1C_2 + e_2{C_1}^2\right)\underline{\xi}^2 + \left(e_1C_3 + 2e_2C_1C_2 + e_3{C_1}^3\right)\underline{\xi}^3 + \cdots \qquad (107)$$

Equating the coefficients in order of $\underline{\xi}$, we obtain the following

$$e_1 = \frac{1}{C_1}, \qquad e_2 = \frac{-C_2}{{C_1}^3}, \qquad e_3 = \frac{2{C_2}^2 - C_1C_3}{{C_1}^5} \qquad (108)$$

The dynamic buckling load λ_D, is obtained from the following maximization using the modified form as

$$\frac{d\lambda}{dU_a} = 0 \qquad (109)$$

On differentiating (109) with respect to $U_a(\lambda)$, we obtain

$$e_1 + 2e_2U_a(\lambda_D) + 3e_3U_a^2(\lambda_D) = 0 \qquad (110)$$

$$where\ \ \lambda = \lambda_D\ \ at\ buckling$$

Evaluating (110) we obtain

$$U_a(\lambda_D) = \frac{-e_2 \pm ({e_2}^2 - 3e_1e_3)^{\frac{1}{2}}}{3e_3} \qquad (111)$$

$$e_3 = \frac{1}{32B_m^5}\left\{2\left(\frac{64k_2B_m^2F_1}{9m\pi\varphi_m^2}\right)^2 - 2B_m\left(\frac{3\beta B_m^3F_2}{\varphi_m^2}\right)\right\} = \frac{-1}{16B_m}\left\{\frac{3\beta F_2}{\varphi_m^2} - \frac{(64k_2F_1)^2}{81m^2\varphi_m^4\pi^2}\right\}$$

$$= \frac{-3\beta F_2 Y}{16\varphi_m^2 B_m} \qquad (112)$$

where

$$Y = 1 - \frac{(64k_2F_1)^2}{243\beta F_2 m^2\varphi_m^2\pi^2} \qquad (113)$$

We consider only the negative square root since the positive part is of no real physical significance and obtain

$$-e_2 - ({e_2}^2 - 3e_1e_3)^{\frac{1}{2}} = -\frac{1}{8B_m}\left\{\left(\frac{18\beta F_2}{\varphi_m^2} - 5\left(\frac{64k_2F_1}{9m\pi\varphi_m^2}\right)^2\right)^{\frac{1}{2}} - \frac{64k_2F_1}{9m\pi\varphi_m^2}\right\}$$

$$= -\frac{3\sqrt{\beta F_2}R}{4\sqrt{2}} \qquad (114)$$

where

$$R = \left\{\left(1 - \frac{5(64k_2F_1)^2}{1458\beta F_2(m\varphi_m\pi)^2}\right)^{\frac{1}{2}} - \frac{64k_2F_1}{27m\pi\varphi_m\sqrt{2\beta F_2}}\right\} \qquad (115)$$

By substitute we obtain

$$U_a(\lambda_D) = \frac{4\varphi_m RY^{-1}}{3\sqrt{2\beta F_2}} \qquad (116)$$

We however evaluate at $\lambda = \lambda_D$ to obtain

$$\varphi_m = (m^4 - 2\lambda_D m^2 + 1)^{\frac{1}{2}} \qquad (117)$$

$$e_3U_a^3(\lambda_D) = -\frac{e_1U_a(\lambda_D)}{3} - \frac{2e_2U_a^2(\lambda_D)}{3} \qquad (118)$$

$$3\underline{\xi} = \frac{2U_a}{C_1}\left(1 - \frac{U_aC_2}{2{C_1}^2}\right) \qquad (119)$$

$$3\underline{\xi} = \frac{2\varphi_m^3 RY^{-1}}{3\lambda_D m^2\underline{a}_m\sqrt{2\beta F_2}}\left(1 - \frac{32k_2F_1RY^{-1}}{27m\pi\varphi_m\sqrt{2\beta F_2}}\right)$$

Leading to the expression of dynamic buckling load as

$$\varphi_m^3 = \frac{9YR^{-1}\lambda_D m^2\underline{a}_m\sqrt{2\beta F_2}\underline{\xi}}{2}\left(1 - \frac{32k_2F_1RY^{-1}}{27m\pi\varphi_m\sqrt{2\beta F_2}}\right)^{-1} \qquad (120)$$

On substituting for φ_m, we get

$$(m^4 - 2\lambda_D m^2 + 1)^{\frac{3}{2}} = \frac{9YR^{-1}\lambda_D m^2\underline{a}_m\sqrt{2\beta F_2}\underline{\xi}}{2}\left(1 - \frac{32k_2F_1RY^{-1}}{27m\pi\varphi_m\sqrt{2\beta F_2}}\right)^{-1} \qquad (121)$$

Where we have retained the original form of φ_m on the right hand side.

6. 0 Analysis of Result.

In computing specific values of dynamic buckling load (λ_D) at different values of viscous damping and imperfection parameter using MATLAB computation platform.
The computation was done by setting damping factor at 0.01, 0.02 and 0.05 and a case where the column was not damped, hence damping factor becomes 0.00

For Figs**.** 4.1 and 4.5**.** In each case, the load required to cause buckling increases as the damping level increases and reduces as the imperfection increases.
Fig.4. 5 shows that below the imperfection level of 0.05, the load required to cause buckling in an undamped column is less than the load required to cause buckling in the column at 0.01, 0.02 and 0.05 and buckling load imperfection levels increases with increment in the damping level. At imperfection level of 0.05 and above, the load required to cause buckling in the undamped column becomes bigger compared to that of the damped column at damping levels of 0.01 and 0.02.

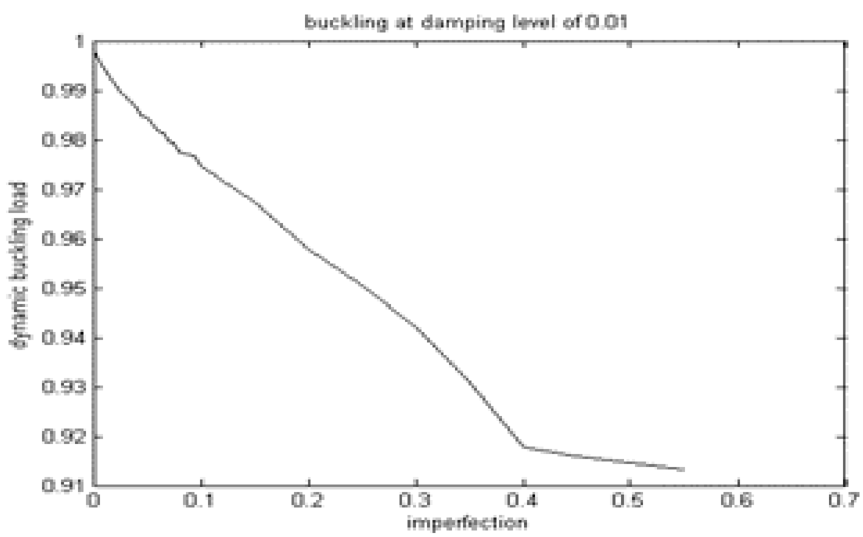

Fig 4.1 Buckling of damping at level 0.01

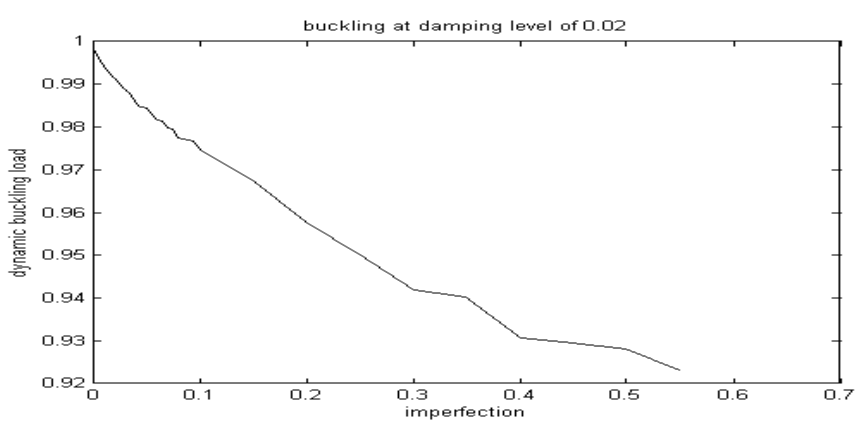

Fig 4.2 Buckling of damping at level 0.02

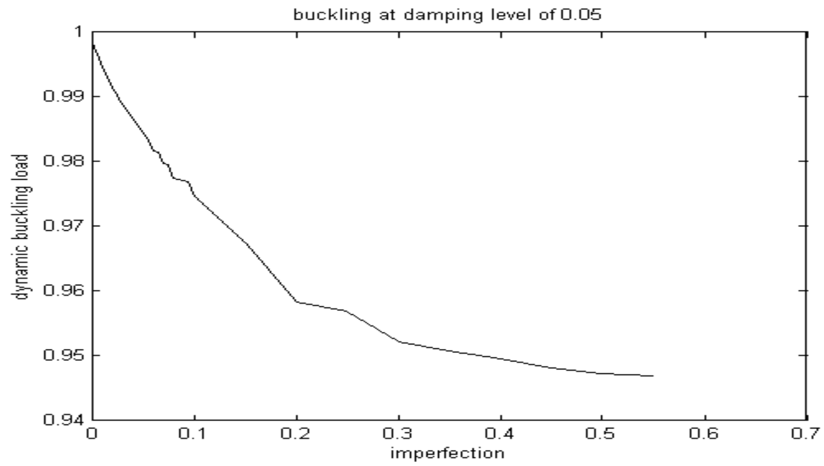

Fig 4.3 Buckling of damping at level 0.05

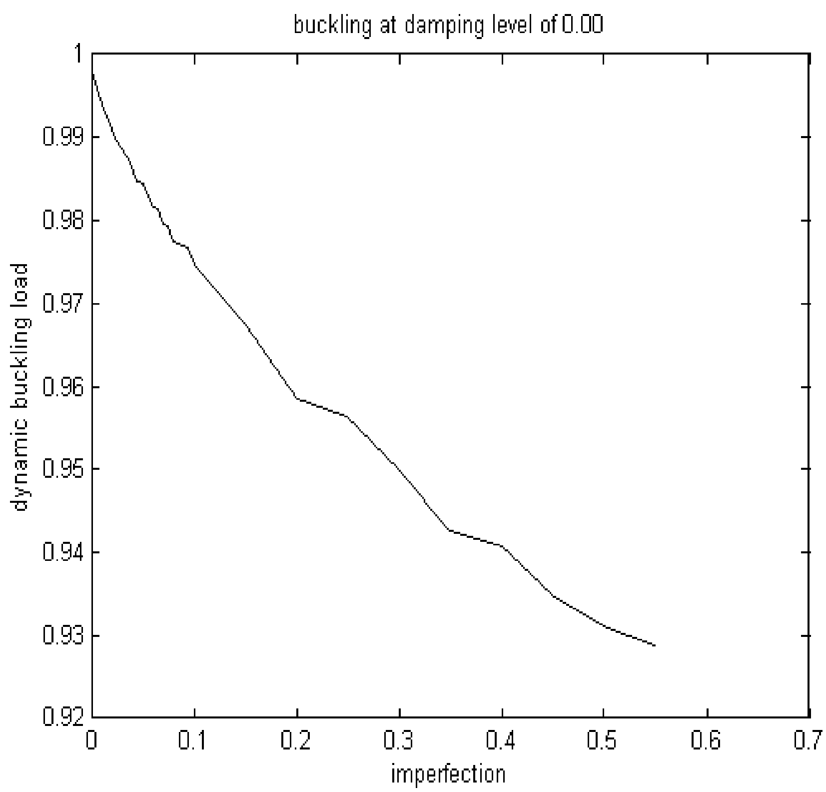

Fig 4.4 Buckling of damping at level 0.00

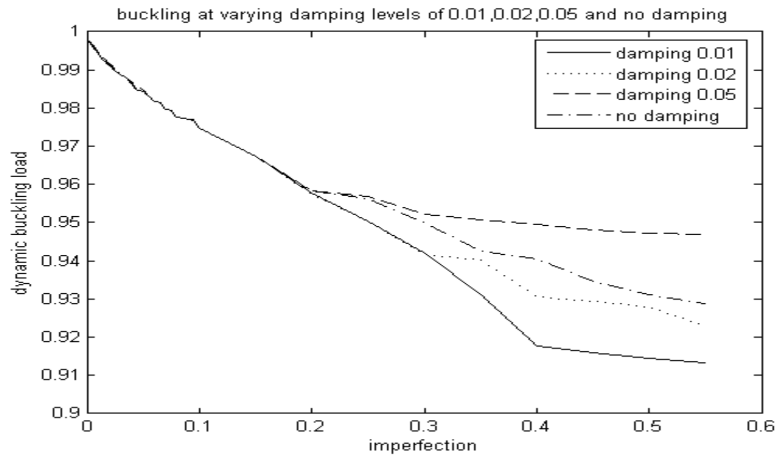

Fig 4.5 Buckling of damping at level 0.01, 0.02, 0.05 and no damping

7. 0 Conclusion.

We deduced from the observations made in chapter four that, if geometrical impurities otherwise known as imperfections are reduced, then a more stable column will be produced. Also, we determined that damping makes a column to become more stable. It is also deduced that a column can still buckle even after damping it. It is of utmost importance then, that we should know the level of imperfections of the column so as to apply the exact level of damping that will produce as table column. For optimum stability of a column, we use high level of damping on a column with high imperfection level.

Though damping strengthens a column thereby preventing early dynamic buckling, the stability or otherwise of a column on the other hand depends entirely on the ability to determine the exact imperfection level in order to apply the appropriate level of damping that will produce a stable column.

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All components of the work were carried out by the author