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A New Symmetric Bimodal Extension of Ailamujia Distribution: Properties and Application to Time Series Data

Kingsley Uchendu, Emmanuel Wilfred Okereke and Joy Chioma Nwabueze
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Abstract: *A new symmetric bimodal extension of the Ailamujia distribution (SBEOAD) has been introduced in this study. pdf, cdf, survival function, hazard rate function, moments, absolute moments, mean residual life function, and quantile function were among the statistical characteristics of the distribution that were studied extensively. The maximum likelihood and method of moments approaches to estimating the SBEOAD's parameters were considered. The consistency property of the maximum likelihood estimates of the parameters has been empirically illustrated through a simulation investigation. The fits of the SBEOAD, the Laplace, Normal, and double Lindley distributions to two time series data were achieved using the maximum likelihood technique. When judged from the standpoint of the minimum AIC and BIC values, the SBEOAD provides the best fit to each of the data compared to the other three distributions fitted to the data.*

Keywords: *Ailamujia distribution, absolute moments, symmetric bimodal distributions, consistency property, maximum likelihood method.*

Kingsley Uchendu
Department of Statistics, Michael Okpara University of Agriculture, Umudike, Nigeria.
Email: uchendu.kingsley@mouau.edu.ng
Orcid id: 0000-0002-5986-5557

Emmanuel Wilfred Okereke
Department of Statistics, Michael Okpara University of Agriculture, Umudike, Nigeria.
Email: okereke.emmanuel@mouau.edu.ng
Orcid id: 0000-0002-6578-8324

Joy Chioma Nwabueze
Department of Statistics
Michael Okpara University of Agriculture, Umudike, Nigeria
Email :

1.0 Introduction

The choice of a probability distribution is often a critical issue in statistical modelling (Shanker, 2015; Okereke, 2019; Awodutire, 2022). It is obvious that the structural properties of data, namely, symmetry, tail behaviour, modality, and central density, determine which distribution to fit to the data (Semary *et al.*, 2025). Several distributions may have different structural properties, making them suitable for certain kinds of data. For example, the Ailamujia distribution of Lv *et al.*(2002) is unimodal, positively skewed and possesses the increasing hazard rate function. Interestingly, it is useful in modelling skewed data pertaining to reliability engineering. Though the distribution has tractable properties, it still suffers some setbacks. First, owing to its unimodal shape, it is not suitable for modelling multi modal data. Second, the possession of a single parameter by the distribution limits the flexibility and adaptability of the distribution in modelling data across different domains. Third, the distribution is applicable to continuous data comprising nonnegative values only.

Quite a number of studies in the literature deal with the extensions of the Ailamujia distribution. Specifically, the area biased weighted Ailamujia distribution was introduced by Jayakumar and Elangovan (2019). Jan et al. (2020) examined the statistical properties and real-world applications of the power Lindley Ailmujia distribution. Rather et al. (2022) proposed the exponentiated Ailamujia distribution, illustrating the superiority of the model to several comparable distributions when fitted to medical data. The theoretical framework and applications of the power Ailamujia distribution were explored by Jamal et al. (2021). Other extensions of the distribution include the Alpha power Ailamujia distribution (Gomaa et al., 2023) and type II half-logistic Ailamujia (Ragab and Elgarhy 2025).

From the foregoing, there is no work on the symmetric extension of the Ailamujia distribution. Despite wide applicability of unimodal symmetric distributions such as the Gaussian and Laplace models due to their analytical convenience, they fall short in situations where the data exhibit bimodal behaviour, sharp changes near the center, or non-zero likelihoods of extreme deviations. For such data, a more nuanced distributional form is required, which retains symmetry, tractability and introduces features like zero density at the mean and moderate kurtosis. This paper aims at introducing a symmetric bimodal extension of the Ailamujia distribution (SBEOAD). The remaining components of this article are arranged as follows. In Section 2, the pdf and other properties of the distribution are considerably determined. Section 3 deal with the approaches to estimating the parameters of the distribution. Simulation results are presented in Section 4 so as to enable investigate the consistency properties of the distribution. We demonstrate the applicability of the distribution in Section 5. This work is concluded in Section 6.

2.0 Derivation and Basic Properties of SBEOAD

This section is predicated on the derivation of the pdf of SBEOAD from Ailamujia distribution using the reflection method. Consequently, properties, namely, cumulative distribution function (cdf), moments, moment generating, characteristic function (mgf) and others are also determined in this section.

Consider the Ailamujia distribution with pdf

$$g(x) = 4\theta^2\, x e^{-2\theta x}, x > 0, \theta > 0. \quad (1)$$

A crucial aspect of the derivation of the new distribution has to do with the determination of value k for which (2.1) is a valid pdf.

$$f * (x) = kg\,(|x|), -\infty < x < \infty, \quad (2)$$

where k is a normalizing constant. To find k, we proceed as follows

$$k \int_{-\infty}^{\infty} g|x|\, dx = 4k\theta^2 \int_{-\infty}^{\infty} |x| e^{-2\theta|x|} dx = 1$$

$$\Rightarrow 4k\theta^2 \left[1 - \int_{-\infty}^{0} x\, e^{2\theta x} dx + \int_{0}^{\infty} x\, e^{-2\theta x} dx \right] = 1$$

$$4k\theta^2 \left[\frac{1}{4\theta^2} + \frac{1}{4\theta^2}\right] = 1$$

$$2k = 1$$

$$\therefore K = {}^{1}/_{2}$$

From the foregoing,

$$f * (x) = 2\theta^2 |x| e^{-2\theta|x|}\,, -\infty < \infty,\ \theta > 0 \quad (3)$$

The distribution whose pdf is defined in (2.3) can be addressed as the double Ailamujia distribution. If we do the substitution $\theta = 1$ in (2.3), the resulting distribution becomes the standard

Double Ailamujia distribution (SDAD). Introducing the location parameter (μ) and scale parameter (σ) into the pdf of the SDAD leads the (SBEOA) distribution with pdf

$$f(x)=\frac{2}{\sigma^2}|x-\mu|e^{-\frac{2}{\sigma}|x-\mu|},-\infty<x<\infty,-\infty<\mu<\infty,\sigma>0. \quad (4)$$

In Fig. 1, we that the pdf of the SBEOAD is M shaped, indicating that it is symmetric bimodal.

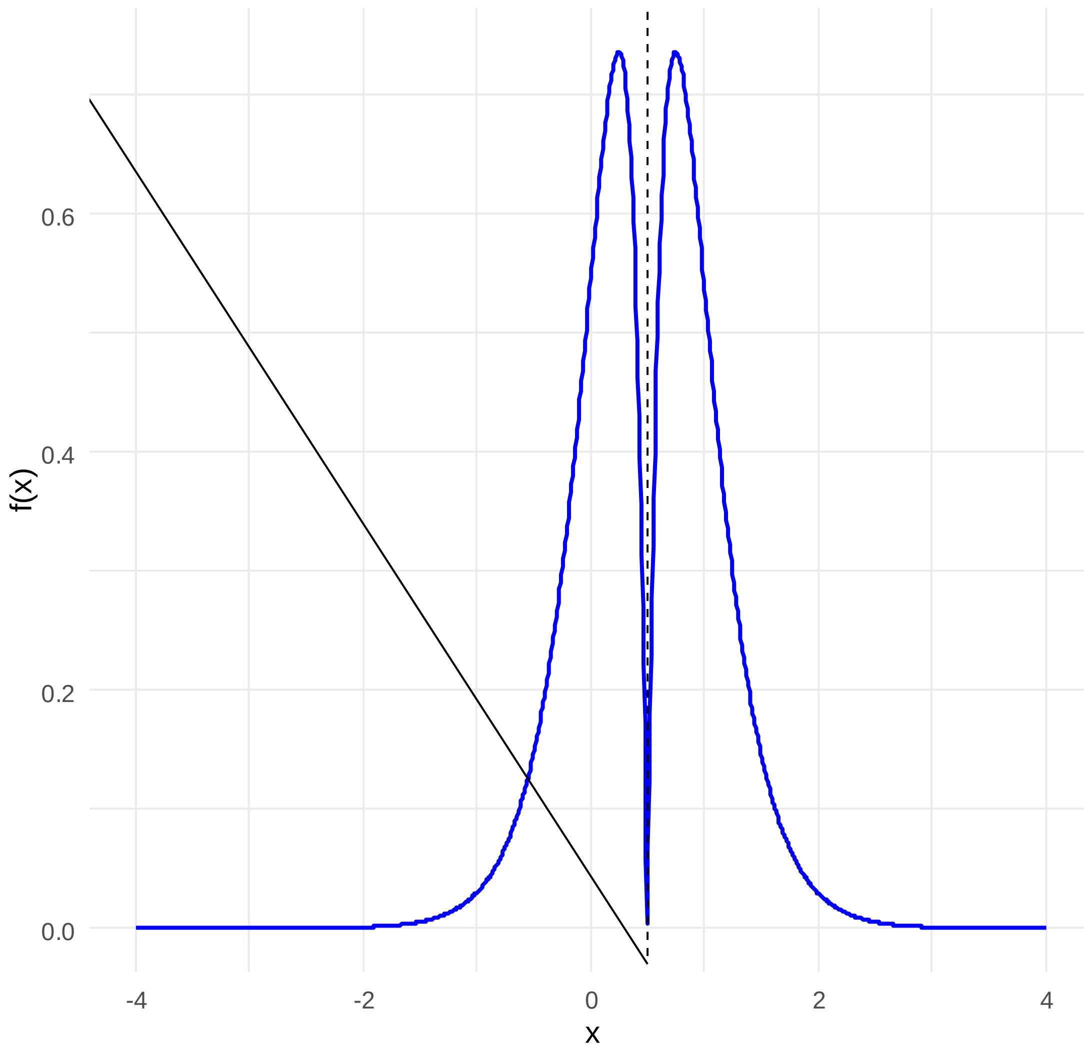

Fig.1:pdf plot of the SBEOAD with $\mu=0.5$ and $\sigma=0.5$

We present and prove Theorem 2.1 for the sake of establishing theoretically that the distribution is arguably bimodal.

Theorem 2.1: The SBEOAD is bimodal with two symmetric modes at $x=\mu\pm\frac{\sigma}{2}$.

Proof: To provide a formal proof of Theorem 2.1, let $y=x-\mu$ in (2.4). When $y\geq 0$, we have the derivative

$$f'(y)=\frac{2}{\sigma^2}e^{-\frac{2y}{\sigma}}\left(1-\frac{2y}{\sigma}\right).$$

If $f'(y)=0$, then $1-\frac{2y}{\sigma}=0$.

That is $y=\frac{\sigma}{2}$.

Consequently,

$$f''(y)=\frac{2}{\sigma^2}\left[-\frac{2}{\sigma}e^{-\frac{2y}{\sigma}}\left(1-\frac{2y}{\sigma}\right)-\frac{2}{\sigma}e^{-\frac{2y}{\sigma}}\right].$$

It follows that

$$f''\left(\frac{\sigma}{2}\right)=-\frac{4}{\sigma^3}e^{-1},$$

Which is less than zero provided that $\sigma>0$. Hence, $x=\mu+\frac{\sigma}{2}$ is one of the modes of the distribution.

Similarly, if $y<0$, then

$$f'(y)=-\frac{2}{\sigma^2}e^{\frac{2y}{\sigma}}\left(1+\frac{2y}{\sigma}\right).$$

As a consequence, $f'(y)=0$ indicates that $y=-\frac{\sigma}{2}$ and $x=\mu-\frac{\sigma}{2}$.

Also, $f''(y)=-\frac{8}{\sigma^3}\left[1+\frac{y}{\sigma}\right]e^{\frac{2y}{\sigma}}$.

If $y=-\frac{\sigma}{2}$, we obtain

$$f''\left(-\frac{\sigma}{2}\right)=-\frac{4}{\sigma^3}e^{-1},$$

detailing that $x=\mu-\frac{\sigma}{2}$ is another mode of the distribution.

In summary, we have established that the distribution is symmetric bimodal with the modes $x=\mu+\frac{\sigma}{2}$ and $x=\mu-\frac{\sigma}{2}$.

To determine the associated cumulative distribution function (cdf), we refer to two cases, which are $x<\mu$ and $x\geq\mu$. If $x<\mu$, then the cdf is

$$F(x)=\int_{-\infty}^{x}f(t)dt=\frac{2}{\sigma^2}\int_{-\infty}^{x}(\mu-t)e^{-\frac{2(\mu-t)}{\sigma}}dt.$$

Let $s=\frac{2(\mu-t)}{\sigma}$. Then $t=\mu-\frac{\sigma s}{2}$ and $dt=-\frac{\sigma}{2}ds$.

Hence,

$$F(x)=\frac{1}{2}\int_{\frac{2(\mu-x)}{\sigma}}^{\infty} se^{-s}ds.$$

Applying the concept of integration by parts leads to

$$F(x)=\left[\frac{(\mu-x)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(\mu-x)}{\sigma}\right).$$

When $x\geq\mu$,

$$F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{\mu} f(t)dt+\int_{\mu}^{x} f(t)dt$$

$$=\frac{2}{\sigma^2}\left[\int_{-\infty}^{\mu}(t-\mu)e^{-\frac{2(t-\mu)}{\sigma}}dt+\int_{\mu}^{\infty}(t-\mu)e^{-\frac{2(t-\mu)}{\sigma}}dt\right]=1-\left[\frac{(x-\mu)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(x-\mu)}{\sigma}\right).$$

It follows that the cdf of the SBEOA Distribution is

$$F(x)=\begin{cases}\left[\frac{(\mu-x)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(\mu-x)}{\sigma}\right), \text{if } x<\mu\\ 1-\left[\frac{(x-\mu)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(x-\mu)}{\sigma}\right), \text{if } x\geq\mu.\end{cases} \quad (5)$$

The cdf of the SBEOAD is graphed in Figure 2.2.

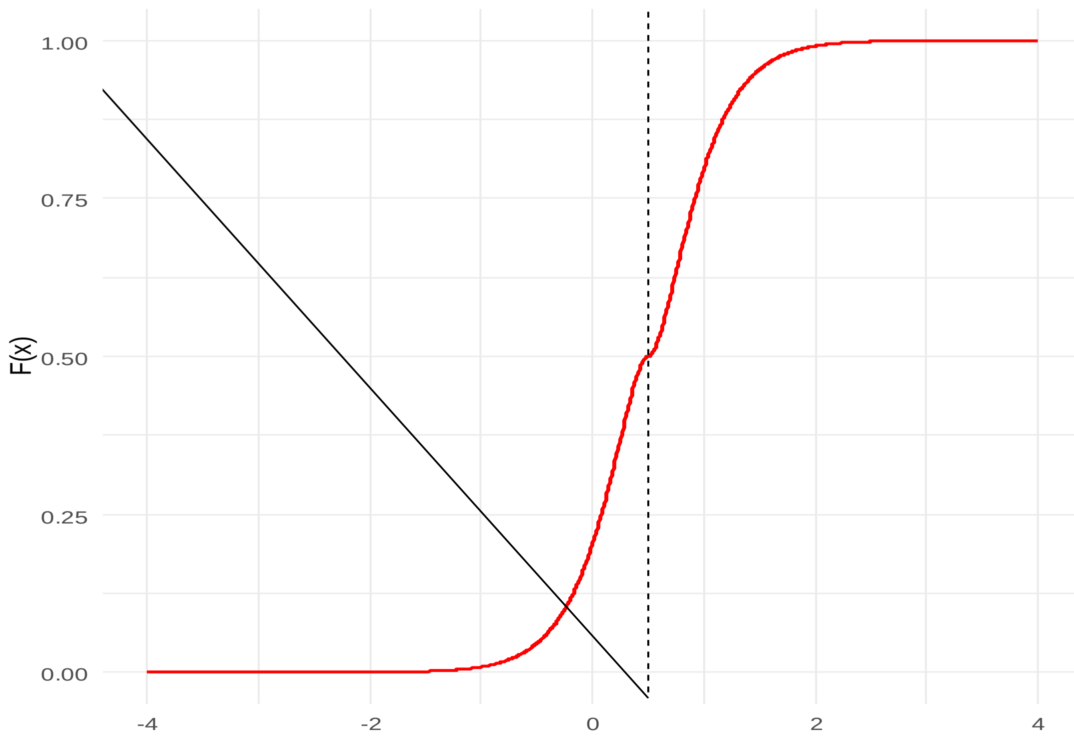

Fig..2: cdf plot of the SBEOAD with $\mu=0.5$ **and** $\sigma=0.5$

2.2 Moments of the SBEOAD

Moments of a distribution are important because they can be used to find notable properties of the distribution, among which are mean, variance, skewness and kurtosis. The SBEOAD is a continuous distribution. Hence, its rth raw moment has the form

$$\mathrm{E}\left(X^{\mathrm{r}}\right)=\int_{-\infty}^{\infty} x^{\mathrm{r}} f(x) dx.$$

For simplicity, we evaluate the moments of Y and employ the relationship between the two variables to derive the moments of X. Now, the rth raw moment of Y is

$$\mathrm{E}\left(Y^{\mathrm{r}}\right)=\frac{2}{\sigma^{2}} \int_{-\infty}^{\infty} y^{\mathrm{r}}|y| \mathrm{e}^{-\frac{2}{\sigma}|y|} dy,-\infty<y<\infty, \sigma>0.$$

For the effective evaluation of the moments, two cases are worthy of consideration. These are when r is an odd number say 2k+1 and when it is even, say 2k. For odd-order moments, we have

$$\mathrm{E}\left(Y^{2\mathrm{k}+1}\right)=\frac{2}{\sigma^{2}} \int_{-\infty}^{\infty} y^{2\mathrm{k}+1}|y| \mathrm{e}^{-\frac{2}{\sigma}|y|} dy=\frac{2}{\sigma^{2}}\left[\int_{-\infty}^{0} y^{2\mathrm{k}+1}(-y) \mathrm{e}^{\frac{2}{\sigma} y} dy+\int_{0}^{\infty} y^{2\mathrm{k}+2} \mathrm{e}^{-\frac{2}{\sigma} y} dy\right].$$

Let $m=-\frac{2y}{\sigma}$. $y=-\frac{\sigma m}{2}$ and $dy=-\frac{\sigma}{2} dm$. Let $v=\frac{2y}{\sigma}$. $y=\frac{\sigma v}{2}$ and $dy=\frac{\sigma}{2} dv$.

Hence,

$$\begin{aligned}\mathrm{E}\left(Y^{2\mathrm{k}+1}\right)&=\frac{2}{\sigma^{2}}\left[-\left(\frac{\sigma}{2}\right)^{2k+3} \int_{0}^{\infty} m^{2\mathrm{k}+2} \mathrm{e}^{-m} dm+\left(\frac{\sigma}{2}\right)^{2k+3} \int_{0}^{\infty} v^{2\mathrm{k}+2} \mathrm{e}^{-v} dv\right] \\ &=\frac{2}{\sigma^{2}}\left[-\left(\frac{\sigma}{2}\right)^{2k+3} \Gamma(2k+3)+\left(\frac{\sigma}{2}\right)^{2k+3} \Gamma(2k+3)\right]=0.\end{aligned}$$

For even-order raw moments, we have

$$\begin{aligned}\mathrm{E}\left(Y^{2\mathrm{k}}\right)&=\frac{2}{\sigma^{2}}\left[\left(\frac{\sigma}{2}\right)^{2k+2} \int_{0}^{\infty} m^{2\mathrm{k}+1} \mathrm{e}^{-m} dm+\left(\frac{\sigma}{2}\right)^{2k+2} \int_{0}^{\infty} v^{2\mathrm{k}+1} \mathrm{e}^{-v} dv\right] \\ &=\frac{4}{\sigma^{2}}\left[\left(\frac{\sigma}{2}\right)^{2k+2} \Gamma(2k+2)\right]=\left(\frac{\sigma}{2}\right)^{2k}(2k+1)!.\end{aligned}$$

Thus, the r central moment of X is

$$\mathrm{E}\left((X-\mu)^{\mathrm{r}}\right)=\begin{cases}0, \text{ if r is odd} \\ \left(\frac{\sigma}{2}\right)^{\mathrm{r}}(\mathrm{r}+1)!, \text{ if r is even.}\end{cases} \tag{6}$$

From (2.5), it is clear that $\mathrm{E}((X-\mu))=0$, implying that $\mathrm{E}(X)=\mu=0$. Also,

$$\operatorname{var}(X)=\mathrm{E}\left((X-\mu)^{2}\right)=3!\left(\frac{\sigma}{2}\right)^{2}=\frac{3\sigma^{2}}{2},$$

$$\mathrm{E}\left((X-\mu)^{3}\right)=0$$

and

$$\mathrm{E}\left(\left(X-\mu\right)^4\right)=5!\left(\frac{\sigma}{2}\right)^4=\frac{15\sigma^4}{2}.$$

Since $\mathrm{E}\left(\left(X-\mu\right)^3\right)=0$, the coefficient of skewness for distribution is zero. As a consequence, it is a symmetric distribution. The coefficient of kurtosis for the distribution is given by

$$\mathrm{K}=\frac{\mathrm{E}\left(\left(X-\mu\right)^4\right)}{\left[\mathrm{E}\left(\left(X-\mu\right)^2\right)\right]^2}. \tag{7}$$

Substituting $\mathrm{E}\left(\left(X-\mu\right)^2\right)=\frac{3\sigma^2}{2}$ and $\mathrm{E}\left(\left(X-\mu\right)^4\right)=\frac{15\sigma^4}{2}$ into (2.6) yields $\mathrm{K}=3\frac{1}{3}$.

2.3 Moment Generating Function and Characteristic Function of the SBEOAD

The moment generating function (mgf) and characteristic function (cf) are useful in generating moments of random variables. Each of them uniquely determines a distribution. Again, the mgf does not exist for some random variables. However, the characteristic function exists for all distributions. It is worthy of note that when both functions exist, it is possible to obtain the cf of a distribution from the corresponding mgf. The mgf of the SBEOA distribution is

$$\mathrm{M}_X\left(t\right)=\mathrm{E}\left(\mathrm{e}^{tX}\right)=\int_{-\infty}^{\infty}\mathrm{e}^{tX}f(x)dx.$$

In order to simplify the derivation of the mgf of the distribution, we first determine that of the random variable Y. Thereafter, we use the relationship between X and Y to derive the requisite mgf and the accompanying cf. For the mgf of Y, we have

$$\mathrm{M}_Y\left(t\right)=\frac{2}{\sigma^2}\int_{-\infty}^{\infty}\mathrm{e}^{ty}\left|y\right|\mathrm{e}^{-\frac{2}{\sigma}\left|y\right|}dy=\frac{2}{\sigma^2}\left[\int_0^{\infty}y\mathrm{e}^{\left(\frac{2}{\sigma}+t\right)y}dy+\int_0^{\infty}y\mathrm{e}^{-\left(\frac{2}{\sigma}-t\right)y}dy\right]$$

$$=\frac{2}{\sigma^2}\left[\left(\frac{2}{\sigma}+t\right)^{-2}+\left(\frac{2}{\sigma}-t\right)^{-2}\right]=\frac{4\left(4+\sigma^2t^2\right)}{\left(4-\sigma^2t^2\right)^2},t<\frac{2}{\sigma}.$$

Since $x=\mu+y$, the mgf of X is

$$\mathrm{M}_X\left(t\right)=\mathrm{e}^{t\mu}\mathrm{M}_Y\left(t\right)=\frac{4\mathrm{e}^{t\mu}\left(4+\sigma^2t^2\right)}{\left(4-\sigma^2t^2\right)^2},t<\frac{2}{\sigma}.$$

The related characteristic function has the form

$$\phi_X\left(t\right)=\mathrm{E}\left(\mathrm{e}^{itX}\right)=\frac{4\mathrm{e}^{it\mu}\left(4-\sigma^2t^2\right)}{\left(4+\sigma^2t^2\right)^2}.$$

2.4 Quantile Function for SBEOAD

The quantile function is well known for its applications in statistical science. It is being used to generate random numbers from a distribution. Other uses of this function include quantile regression, nonparametric hypothesis testing, data visualization and determination of risk measures like value at risk (VaR) and income inequality measures. Let $q\in\left(0,1\right)$. Here, the quantile function x_q for the distribution is obtained by finding the solution of the equation

$$F\left(x_q\right)=q.$$

If $x < \mu$, we consider the equation

$$\left[\frac{\left(\mu - x_q\right)}{\sigma} + \frac{1}{2}\right]\exp\left(-\frac{2\left(\mu - x_q\right)}{\sigma}\right) = q. \tag{8}$$

Solving for x_q in (2.7) yields

$$x_q = \mu + \frac{\sigma}{2}\left[1 + W_{-1}\left(-2q\mathrm{e}^{-1}\right)\right].$$

Similarly, when $x \geq \mu$, the quantile function satisfies the equation

$$1 - \left[\frac{\left(x_q - \mu\right)}{\sigma} + \frac{1}{2}\right]\exp\left(-\frac{2\left(x_q - \mu\right)}{\sigma}\right) = q. \tag{9}$$

We solve for x_q in (2.8) to obtain

$$x_q = \mu - \frac{\sigma}{2}\left[1 + W_{-1}\left(-2(1-q)\mathrm{e}^{-1}\right)\right].$$

From the foregoing, the quantile function for the SBEOAD is

$$x_q = \begin{cases} \mu + \frac{\sigma}{2}\left[1 + W_{-1}\left(-2q\mathrm{e}^{-1}\right)\right], 0 < q < \frac{1}{2} \\ \mu - \frac{\sigma}{2}\left[1 + W_{-1}\left(-2(1-q)\mathrm{e}^{-1}\right)\right], \frac{1}{2} \leq q < 1, \end{cases} \tag{10}$$

where $W_{-1}(.)$ is the negative branch of the Lambert W function.

2.3 Absolute moments

Given that $X \sim \mathrm{SBEOA}(\mu,\sigma)$ and $r \geq 1$, the rth absolute moment of $|X-\mu|$ is

$$\delta \mathrm{r} = \frac{2}{\sigma^2}\int_{-\infty}^{\infty} |x-\mu|^{\mathrm{r}}\, \mathrm{e}^{-\frac{2}{\sigma}|x-\mu|}.$$

The evaluation of the integral above is made simple by utilizing the distribution of $Z = |X-\mu|$. Proposition 2.2 deals with this distribution.

Proposition 2.2: Suppose that $X \sim \mathrm{SBEOA}(\mu,\sigma)$. Then $Z = |X-\mu|$ is gamma distributed with shape parameter 2 and scale parameter $\frac{2}{\sigma}$. That is $Z \sim G\left(2, \frac{2}{\sigma}\right)$.

Proof: Let $W(z)$ represent the cdf of Z. Consequently,

$$W(z) = \mathrm{P}\left(Z \leq z\right) = \mathrm{P}\left(|X-\mu| \leq z\right)$$
$$= \mathrm{P}\left(\mu - z \leq X \leq \mu + z\right) = F\left(\mu + z\right) - F\left(\mu - z\right).$$

Applying (2.5) when $x \geq \mu$ leads to

$$F\left(\mu + z\right) = 1 - \left[\frac{z}{\sigma} + \frac{1}{2}\right]\exp\left(-\frac{2z}{\sigma}\right).$$

Similarly, if we apply (2.5) based on $x < \mu$, we obtain

$$F\left(\mu - z\right) = \left[\frac{z}{\sigma} + \frac{1}{2}\right]\exp\left(-\frac{2z}{\sigma}\right).$$

It follows that

$$W(z)=1-2\left[\frac{z}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2z}{\sigma}\right).$$

Differentiating $W(z)$ with respect to z yields the pdf of Z

$$w(z)=\frac{4z}{\sigma^2}\exp\left(-\frac{2z}{\sigma}\right), z\geq 0,$$

indicating that $Z\sim G\left(2,\frac{2}{\sigma}\right)$.

Now, we return to the problem of finding the absolute moments. In terms of Z, the requisite rth absolute moment is

$$\delta_{\mathrm{r}}=E\left(Z^{\mathrm{r}}\right)=\frac{4}{\sigma^2}\int_{-\infty}^{\infty}\mathrm{z}^{\mathrm{r}+1}\mathrm{e}^{-\frac{2}{\sigma}z}dz=\frac{\sigma^{\mathrm{r}}\left(\mathrm{r}+1\right)!}{2^{\mathrm{r}}}.$$

In particular, the mean absolute deviation about the mean of the distribution is

$$\delta_1=E\left(Z\right)=\sigma.$$

2.4 Reliability Concepts

In view of the importance of probability distributions in reliability analysis, we deem it fit to derive expressions for reliability concepts pertaining to the distribution, especially the survival function, hazard rate function and mean residual life function. The survival function $S(x)$ of the SBEOAD is

$$S(x)=1-F\left(x\right).$$

$$S\left(x\right)=\begin{cases}1-\left[\frac{\left(\mu-x\right)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2\left(\mu-x\right)}{\sigma}\right), \text{if } x<\mu\\ \left[\frac{\left(x-\mu\right)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2\left(x-\mu\right)}{\sigma}\right), \text{if } x\geq\mu.\end{cases} \quad (11a)$$

Again, the hazard rate function (hrf) of the SBEOAD is

$$h(x)=\begin{cases}\dfrac{2\left(\mu-x\right)\mathrm{e}^{-\frac{2}{\sigma}\left(\mu-x\right)}}{\sigma^2\left(1-\left[\frac{\left(\mu-x\right)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2\left(\mu-x\right)}{\sigma}\right)\right)}, \text{if } x<\mu\\ \dfrac{2\left(x-\mu\right)\mathrm{e}^{-\frac{2}{\sigma}\left(x-\mu\right)}}{\sigma^2\left[\frac{\left(x-\mu\right)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2\left(x-\mu\right)}{\sigma}\right)}, \text{if } x\geq\mu.\end{cases} \quad (11b)$$

The graphical representation of the hrf of SBEOAD in Fig. 2 is an indication that the function is unimodal when $x<\mu$ and increasing provided $x\geq\mu$.

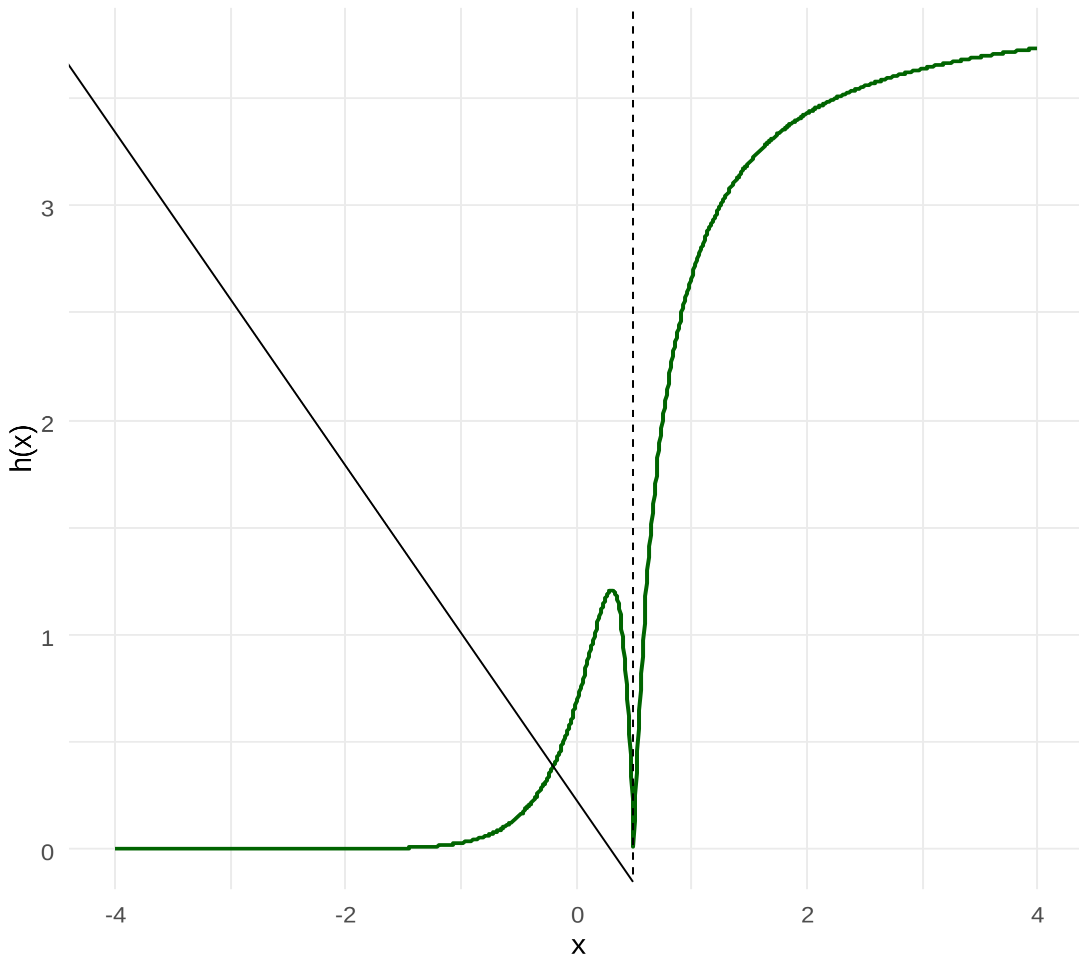

Fig. 2: hrf plot of the SBEOAD with $\mu = 0.5$ and $\sigma = 0.5$

The mean residual life function (MRL) is

$$m(x) = \frac{1}{S(x)}\int_x^{\infty} S(t)dt.$$

When $x \geq \mu$, the requisite MRL is

$$\int_x^{\infty} S(t)dt = \int_x^{\infty}\left[\frac{(t-\mu)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(t-\mu)}{\sigma}\right)dt. \quad (12)$$

Let $w = \frac{2(t-\mu)}{\sigma}$. Then (2.11) becomes

$$\int_x^{\infty} S(t)dt = \frac{\sigma}{2}\int_{\frac{2(x-\mu)}{\sigma}}^{\infty}\left[\frac{w}{2}+\frac{1}{2}\right]\exp(-w)\,dt. \quad (13)$$

$$=\left[\frac{x-\mu+\sigma}{2}\right]\exp\left(-\frac{2(x-\mu)}{\sigma}\right).$$

In the light of the above,

$$m(x)=\frac{\left[\frac{x-\mu+\sigma}{2}\right]\exp\left(-\frac{2(x-\mu)}{\sigma}\right)}{\left[\frac{(x-\mu)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(x-\mu)}{\sigma}\right)}. \qquad (14)$$

When $x<\mu$, the MRL is

$$\int_x^{\infty} S(t)dt=\int_x^{\mu}\left(1-\left[\frac{(\mu-t)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(\mu-t)}{\sigma}\right)\right)dt+\int_{\mu}^{\infty}\left[\frac{(t-\mu)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(t-\mu)}{\sigma}\right)dt.$$

Suppose that $m=\frac{2(\mu-t)}{\sigma}$. Then

$$\int_x^{\mu}\left(1-\left[\frac{(\mu-t)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(\mu-t)}{\sigma}\right)\right)dt$$

$$=\frac{(\sigma-2\mu+2x)+(x-\mu-\sigma)\exp\left(-\frac{2(\mu-t)}{\sigma}\right)}{2}. \qquad (15)$$

Changing the lower limit of the integral in (2.13) to μ, we obtain

$$\int_{\mu}^{\infty}\left[\frac{(t-\mu)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(t-\mu)}{\sigma}\right)dt=\frac{\sigma}{2}. \qquad (16)$$

Combining (2.15) and (2.16) leads to

$$\int_x^{\infty} S(t)dt=\frac{2(\sigma-\mu+x)+(x-\mu-\sigma)\exp\left(-\frac{2(\mu-x)}{\sigma}\right)}{2}.$$

The related MRL is

$$m(x)=\frac{(\sigma-\mu+x)+(x-\mu-\sigma)\exp\left(-\frac{2(\mu-x)}{\sigma}\right)}{2\left(1-\left[\frac{(\mu-x)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(\mu-x)}{\sigma}\right)\right)}. \qquad (17)$$

Using (2.14) and (2.17), the MRL for the SBEOAD is found to be

$$m(x)=\begin{cases}\dfrac{(\sigma-\mu+x)+(x-\mu-\sigma)\exp\left(-\frac{2(\mu-x)}{\sigma}\right)}{2\left(1-\left[\frac{(\mu-x)}{\sigma}+\frac{1}{2}\right]\exp\left(-\frac{2(\mu-x)}{\sigma}\right)\right)}, & \text{if } x<\mu\\[2ex] \dfrac{\sigma(x-\mu+\sigma)}{2(x-\mu)+\sigma}, & \text{if } x\geq\mu.\end{cases} \qquad (18)$$

3.0 Estimation

The two point estimation procedures that are of considerable interest in the article include the method of moments and maximum likelihood method.

3.1 Method of Moments

Given a random sample $X_1, X_2, ..., X_n$ from SBEOA**(** μ, σ **).** The method of moments of μ and σ are obtained by equating the theoretical mean and variance to their sample counterparts to obtain (19) and (20) respectively.

$$\mu = \frac{\sum_{i=1}^{n} X_i}{n} = \overline{X} \quad (19)$$

$$\frac{3\sigma^2}{2} = \frac{\sum_{i=1}^{n}\left(X_i - \mu\right)}{n} \quad (20)$$

Let the method of $\hat{\mu}$ and $\hat{\sigma}$ denote the method moments estimators of μ and σ respectively. Solving (19) and (20) simultaneously, we have

$$\hat{\mu} = \overline{X}. \quad (21)$$

$$\hat{\sigma} = \sqrt{\frac{2\sum_{i=1}^{n}\left(X_i - \overline{X}\right)}{3n}}. \quad (22)$$

Though the estimators obtained above may be as good as the associated maximum likelihood estimators, the corresponding estimates can be useful in obtaining the maximum likelihood estimates of the parameters of the distribution through a numerical approach.

3.2 Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$, denote a random sample of size n from SBEOA**(** μ, σ **).** The related likelihood function is

$$L = \prod_{i=1}^{n} f(x_i) = \left(\frac{2}{\sigma^2}\right)^n \prod_{i=1}^{n} \left|x_i - \mu\right| \exp\left(-\frac{2}{\sigma}\left|x_i - \mu\right|\right) \quad (23)$$

Taking natural log on both sides yields the log-likelihood function.

$$\ln L = n\ln 2 - 2n\ln\sigma + \sum_{i=1}^{n} \ln\left|x_i - \mu\right| - \frac{2}{\sigma}\sum_{i=1}^{n}\left|x_i - \mu\right|. \quad (24)$$

The partial derivative of the log-likelihood function with respect to each of the parameters is given below:

$$\frac{\partial \ln L}{\partial \mu} = -\sum_{i=1}^{n} \frac{\operatorname{sgn}\left(x_i - \mu\right)}{\left|x_i - \mu\right|} + \frac{2}{\sigma}\sum_{i=1}^{n} \operatorname{sgn}\left(x_i - \mu\right).$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{2n}{\sigma} + \frac{2}{\sigma^2}\sum_{i=1}^{n}\left|x_i - \mu\right|.$$

Equating each partial derivative to zero results in (25) and (26).

$$-\sum_{i=1}^{n} \frac{\operatorname{sgn}\left(x_i - \mu\right)}{\left|x_i - \mu\right|} + \frac{2}{\sigma}\sum_{i=1}^{n} \operatorname{sgn}\left(x_i - \mu\right) = 0, x_i \neq \mu. \quad (25)$$

$$-\frac{2n}{\sigma} + \frac{2}{\sigma^2}\sum_{i=1}^{n}\left|x_i - \mu\right| = 0. \quad (26)$$

Let $\hat{\mu}$ and $\hat{\sigma}$ be the maximum likelihood estimates of μ and σ respectively.

Using (25), we obtain

$$\sum_{i=1}^{n} \frac{1}{x_i - \mu} = \frac{2}{\sigma}\sum_{i=1}^{n} \operatorname{sgn}\left(x_i - \mu\right) = 0, x_i \neq \mu.$$

From (26),

$$\hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}\left|x_i - \hat{\mu}\right|.$$

Notably, (25) is a nonlinear equation and implicit in μ. Hence, $\hat{\mu}$ can be estimated numerically using the Newton -Raphson approach. Since $\hat{\sigma}$ depends on $\hat{\mu}$**,** an iterative procedure is essential**.**

4.0 Simulation

In this section, a Monte Carlo simulation experiment is carried out the investigate the consistency property of the maximum likelihood estimators of the parameters of the SBEOAD. In each of N=1000 replications of the experiment, the quantile function of the SBEOAD is used to generate data from the distribution based on the sample sizes n=20,50,100,500 and the three sets of parameter values $(\mu,\sigma) = (0.5,2),(2,0.5),(2,3)$. For each simulated data, the maximum likelihood estimate of each of the parameters μ and σ is obtained. Let $\hat{\mu}_j$ and $\hat{\sigma}_j$, respectively, denote the maximum likelihood estimates of μ and σ that correspond to the jth replication of the Monte Carlo simulation experiment. We compute the average estimate (AE), average bias (AB) and mean squared error (MSE) for each sample size and a set of parameter values. Symbolically,

$$AE(\hat{\mu}) = \frac{1}{1000}\sum_{j=1}^{1000}\hat{\mu}_j;$$

$$AB(\hat{\mu}) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\mu}_j - \mu\right);$$

$$MSE(\hat{\mu}) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\mu}_j - \mu\right)^2;$$

$$AE(\hat{\sigma}) = \frac{1}{1000}\sum_{j=1}^{1000}\hat{\sigma}_j;$$

$$AB(\hat{\sigma}) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\sigma}_j - \sigma\right);$$

$$MSE(\hat{\sigma}) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\sigma}_j - \sigma\right)^2.$$

The requite simulation results are enshrined in Table 4.1. The results indicate that the MSE of each of the maximum likelihood estimators decreases as the sample size increases, detailing that the estimators are consistent. It is also noteworthy that the average bias tends to zero as the sample size increases.

Table 4.1:

N	**μ**	σ	$\hat{\mu}$	**B(**$\hat{\mu}$**)**	**MSE(**$\hat{\mu}$**)**	$\hat{\sigma}$	**B(**$\hat{\sigma}$**)**	**MSE(**$\hat{\sigma}$**)**
20	0.5	2	0.5429	0.0429	0.1061	0.8822	-1.1178	1.2537
50	0.5	2	0.5557	0.0557	0.0918	0.9099	-1.0901	1.1912
100	0.5	2	0.5408	0.0408	0.0639	0.9163	-1.0837	1.1768
500	0.5	2	0.5095	0.0095	0.0153	0.9181	-1.0819	1.1712
20	2	0.5	1.8885	-0.1115	0.0601	0.2914	-0.2086	0.0490
50	2	0.5	1.9050	-0.0950	0.0648	0.2986	-2014	0.0444
100	2	0.5	1.8873	-0.1127	0.0663	0.2991	-0.2009	0.0436
500	2	0.5	1.8926	-0.1074	0.0690	0.2980	-0.2021	0.0434
20	2	3	1.9863	-0.0137	0.3812	1.3246	-1.6754	2.8184
50	2	3	2.0190	0.0190	0.3030	1.3624	-1.6376	2.6903
100	2	3	2.0364	0.0364	0.3145	1.3937	-1.6063	2.5962
500	2	3	1.9459	-0.0541	0.1981	1.4026	-1.5974	2.5681

5.0 Application

This section is dedicated to the illustration of the applicability of the SBEOD. As a consequence, we fit the model to two time series data and compare its fits to the data with the fits of the normal distribution, Laplace distribution and two-parameter Lindley distribution. The first data (Data I) refers to the annual maximum rainfall in Jakarta in the last 20 years: The data are reported as (Kurniawan *et al*., 2019)
147.2,94.8,82.2,168.5,199.7,129.3,124.1,72.0,234.7,192.7,122.5,93.0,119.2,105.2,193.4,147.9,277.5,124.5, 179.7,104. The autocorrelation function (ACF) and partial autocorrelation function (PACF) graphed in Figure 4.1.

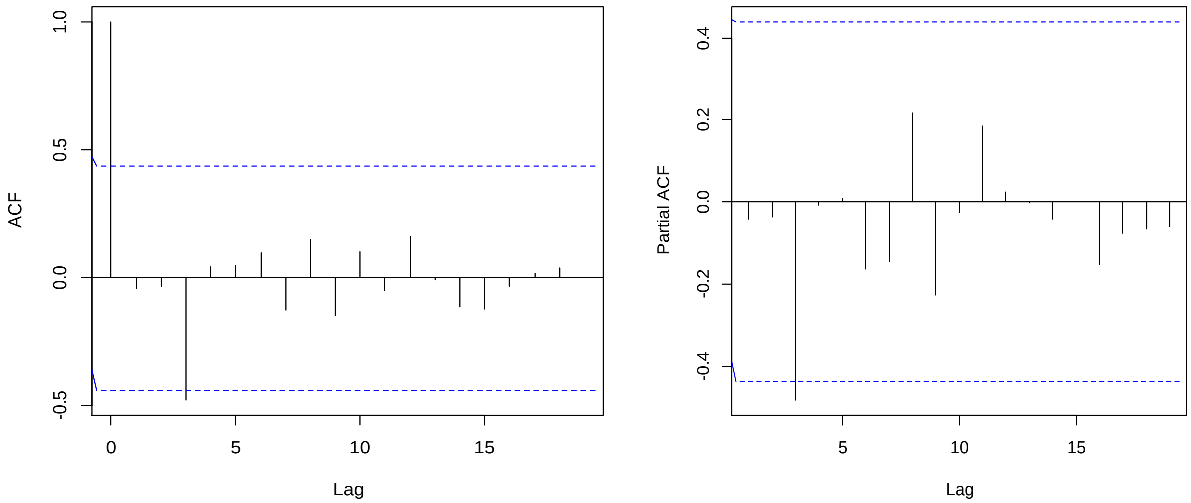

Figure 4.1: ACF and PACF plots for Data 1.

The second data (Data II) constitute the monthly actual tax revenue Egypt from January 2006 to November 2010. The actual taxes revenue data (in 1000 million Egyptian pounds) are (Owoloko *et al*., 2015)
5.9, 20.4, 14.9, 16.2, 17.2,7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1,6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1,36, 8.5, 8, 9.2, 26.2, 21.9,16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7,8.6,12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.
Figure 4.2 comprises the sample correlogram and sample partial correlogram for Data II.

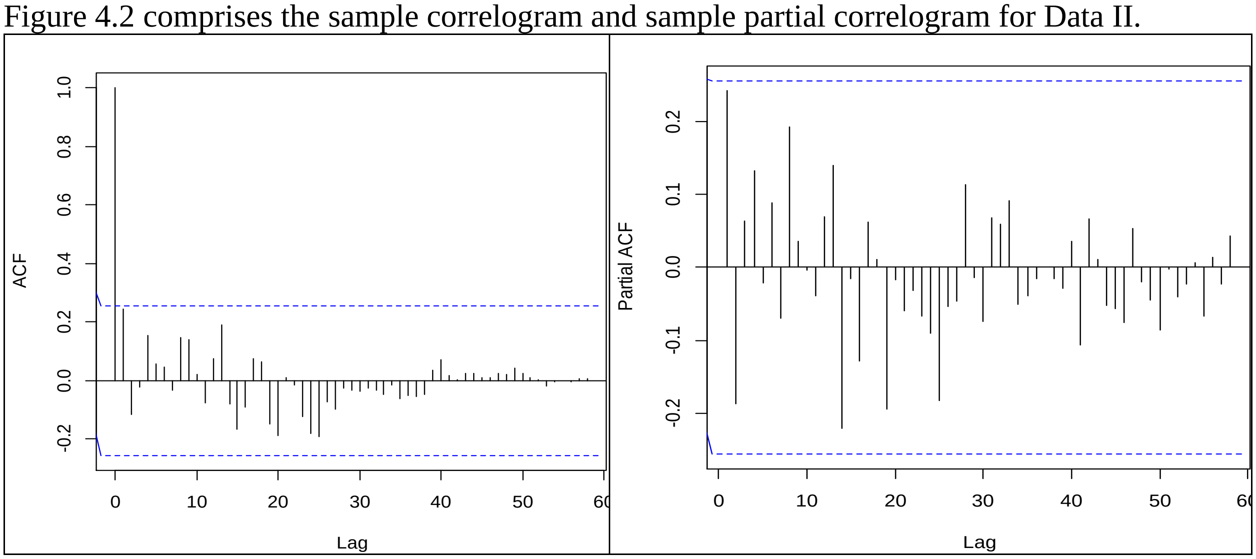

Fig. 4.2: ACF and PACF plots for Data II.

From Figures 4.1 and 4.2, it is easy to deduce that both data are purely random. Next, we present Table 4.3 which contains descriptive statistics for the data.

Table 4.3: Descriptive statistics for Data I and II

Data	Mean	Standard Deviation	Median	Skewness	Kurtosis
Data I	145.61	53.81	126.9	0.86	3.36
Data II	13.49	8.05	10.6	1.65	5.57

Obviously, the data are slightly positively skewed. However, the justification for fitting the SBEOAD to the data as well as the normal distribution, Laplace distribution and two-parameter double Lindley distribution is the closeness of the coefficients to the theoretical coefficients of kurtosis of the distributions being considered. Let $f_1(x)$ and $f_2(x)$, respectively, represent the pdfs of Laplace distribution and two-parameter double Lindley distribution. Then

$$f_1(x) = \frac{1}{2\sigma}\exp\left[-\frac{1}{\sigma}|x-\mu|\right], -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

and

$$f_2(x) = \frac{\theta^2}{2(1+\theta)}\left(1+|x-\mu|\right)\exp\left[-\theta|x-\mu|\right], -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0 .$$

For effective comparison of the fits of the distributions to the data, we consider the Akaike information criterion (AIC) and Bayesian information criterion (BIC). Accordingly, the distribution that corresponds to the smallest AIC and BIC values provides the best to the data. Table 4.4 consists of the maximum likelihood estimates of the parameters of the abovementioned distributions and the accompanying results based on Data I and II. On the basis of the results in Table 4.4, the SBEOAD outperforms the three other distributions.

Table 4.4: Maximum Likelihood estimates of the parameters of the distributions fitted to Data I and II and the associated results.

DATA	DISTRIBUTION	ESTIMATE	LOGLIKEHOOD	AIC	BIC
Data I	LAPLACE	$\hat{\mu}$ =128.4 $\hat{\sigma}$ =41.5	-108.3551	220.7102	222.7017
	SBEOA	$\hat{\mu}$ =139.5 $\hat{\sigma}$ =42.5	-106.7077	217.4155	219.4069
	TWO-PARAMETER DOUBLE LINDLEY	$\hat{\mu}$ =139.6 $\hat{\sigma}$ =0.1	-106.7955	217.5911	219.5826
	NORMAL	$\hat{\mu}$ =145.6 $\hat{\sigma}$ =52.5	-107.5745	219.1490	221.1405
	LAPLACE	$\hat{\mu}$ =10.6 $\hat{\sigma}$ =5.6	-201.1088	406.2176	410.3726

Data II	SBEOA	$\hat{\mu}$ =13.7 $\hat{\sigma}$ =6.1	-200.2568	404.5137	408.6688
	TWO-PARAMETER DOUBLE LINDLEY	$\hat{\mu}$ =13.0 $\hat{\sigma}$ =0.3	-201.4667	406.9333	411.0884
	NORMAL	$\hat{\mu}$ =13.5 $\hat{\sigma}$ =8.0	-206.2787	416.5574	420.7125

6.0 Conclusion

A two two-parameter distribution called the symmetric bimodal extension of Ailamujia distribution (SBEOAD) is developed in this scholarly work via the introduction of location and scale parameters in the standard double Ailamujia distribution (SDAD). Several properties of the distribution, among which are its pdf, cdf, moments, absolute moments, moment generating function, characteristic function, modes, survival function, hazard rate function and mean residual life function are determined. The proposed distribution is symmetric and bimodal. Its coefficient of skewness is 0 while the corresponding coefficient of kurtosis is $3\frac{1}{3}$, making it applicable to symmetric bimodal data that is slightly leptokurtic. Though the mean and median of this symmetric distribution are equal, they are never equal to the mode of the distribution. The distribution of the absolute deviation of the mean of the SBEOAD from a random variable that follows the SBEOAD is found to be a gamma distribution with shape parameter 2 and scale parameter $\frac{2}{\sigma}$. Simulation results based on maximum likelihood approach to estimating the parameters of the SBEOAD reveal the consistency of the concerned maximum likelihood estimates. We have established the capability of the SBEOAD to outperform the normal, Laplace and two-parameter double Lindley distributions by comparing the fits of the four distributions to two time series data based on minimum AIC and BIC values.

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Declaration

Consent for publication

Not Applicable

Availability of data and materials

The publisher has the right to make the data public

Ethical Considerations

The authors declare that all research and development described in this manuscript were conducted with the highest standards of integrity. The project was carried out as a collaborative effort, and all authors involved in the physical construction were voluntary participants who have been appropriately acknowledged.

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