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The Effects of External Toxicants on Competitive Environment: A Mathematical Modeling Approach

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Abstract: *The presence of microplastics in aquatic environments has become a critical global problem. These tiny particles called microplastics less than 5mm in size pose severe risks to ecosystems and human health via the food chain due to the presence of heat and sunlight acting on these disposed plastics into streams and rivers, then flow into the seas and oceans in particular. Sources of microplastic pollution include the disposal of plastics into aquatic environments daily, the constant radiation of sunlight acting on larger disposed plastics leads to the frequent emission of micrometers of plastic into the aquatic environment. Once in aquatic systems, microplastics are ingested by marine life, entering the food chain and causing significant health hazards. Assessing the ecological risks of microplastics is essential, but few works have been done on the effects of microplastics as an external toxicant. This dissertation modified and analyzed a nonlinear mathematical model to study the effects of toxicant concentration leaks from external sources on competing species environments. The system's stability is examined using the tools of the theory of differential equations and computer simulations. The analysis results indicated a sharp increase in species one concentration from the initial value of 0.1 to a maximum of 23.7789 within a month with the toxicant influx at* $Q = 30$*, after that decreasing to a stable minimum of 23.7786, for the rest of the months. It is further observed that the increased toxicant flux reduces the concentration of species one. The more toxicant influx increases, the more the effects are felt by species one and two and the resource biomass over the investigated time intervals.*

Keywords*: Effects, External Toxicants, Competing Species, Modeling, Microplastics, Concentration, Competitive Environment, Stability.*

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1.0. Introduction:

Our surroundings are filled with interconnected relationships, prompting many researchers to explore these real-world phenomena. This study focuses on the interactions between two biological species within a competitive setting, which may occur over extended or brief periods. Freedman & Shukla (1991), developed a model to study single species and predator-prey systems in a closed polluted environment. During their investigation, they consider three cases and the steady state for small constant toxicants in the considered environment. Chattopadhyaya (1996), investigated the effect of toxicants on two species competitive systems by a suitable Lyapunov function.

A well-known outcome of competitive diversification is inter-specific character displacement, where species that coexist differentiate their resource utilization to reduce the impact of competition (Simberloff, 2005). Dubey *et al*., (2006),), investigated a model for the survival of resource-dependent populations to see the effect of toxicants emitted from external sources as well as formed by their precursors. They discovered that the densities of resources and the population decrease as the cumulative emission rate of environmental toxicants increases. See also Shukla *et al*., (2009); Naresh *et al*., (2006); Jiwei *et al*., (2009), and Dubey *et al*., (2010). It was reported that if two species must co-exist, then intra-specific competition must be greater than inter-specific competition according to (Gotelli, 2008). Ran *et al*., (2012), proposed a nonlinear model to investigate the effect of intermediate toxic products on the survival of a resource-dependent species in a polluted ecosystem. Also see Mitteliberg (2012), Vandermeer & Goldberg (2013).

Dike and George (2020) examined the inter-competition coefficients concerning the Resource Biomass of Resource-Dependent Interacting Biological Species by employing a three-dimensional continuous system of nonlinear first-order ordinary differential equations. Their findings indicate that an increase in the inter-competition coefficient corresponds with a rise in the availability of resource biomass in the environment.

Anuj *et al*., (2016), studied the effect of an external toxicant on biological species in case of deformity. The findings from their model indicated that an increase in the emission of external toxicants leads to a reduction in overall population density. In contrast, the density of the deformed subclass rises. Numerous researchers have employed mathematical models to examine and forecast the growth of biological species in toxic conditions. Their investigations have included various scenarios, such as the impact of a single toxicant or multiple toxicants on biological species, cases of allelopathy, and deformities within a subclass of species, among others. (Akpan *et al*. 2022; Dike & George 2020; Ram & Shyam 2012; Agrawal & Shukla 2012; Isobeye et al. 2018; Awortu & George 2024), to provide important insights for the effect of toxicants on biological species. In particular, Awortu and George (2024) have investigated a model analysis of the effect of public enlightenment and behavioral change with treatment on Neisseria gonorrhea dynamics. Their results showed that a decrease in the rate at which susceptible individuals know their gonorrhea status increases the value of the basic reproduction number. Also see Shangge *et al*., (2022), Haomimg *et al*., (2023); Samanta & Matti (2004).

2.0.Mathematical Formulation

In this paper, we shall be formulating a system of mathematical models that represents the effect of toxicants from an external source on concentration and the competing species density in an environment. And before we delve into formulating the mathematical models, let's consider the following assumptions:

2.1. Assumptions:

(i) The Species one grows unboundedly without interference with species two with the toxicants.

(ii) The species two grows unboundedly without interference with species one and the toxicants.

(iii) The resource biomass density grows without interference with species one and two densities and the toxicants.

(iv) There is a constant supply of toxicants in the environment.

(v) The inter and intra-specific interactions of species lead to the depletion of their densities.

(vi) The interaction of the species and the toxicants leads to the depletion of the species' densities.

(vii) The interaction between the resource biomass and the toxicants depletes the biomass quantity.

(viii) The interaction between the biomass resource and the species densities reduces the resource of the biomass.

(ix) The interaction between the biomass resource and species' densities improves the species' densities.

(x) Interaction between the toxicants and the species' densities leads to depletion of the toxicants due to the absorption by the species.

Given the aforementioned assumptions in section 2.1, the models representing the effect of toxicants from external sources on toxicant concentration and the densities of competing species in an environment are:

$$\frac{dx_1}{dt} = \alpha_1 x_1 - \alpha_2 x_1^2 - \alpha_3 x_1 x_2 + \alpha_4 x_1 R - \alpha_5 x_1 T \quad (1)$$

$$\frac{dx_2}{dt} = \beta_1 x_2 - \beta_2 x_2^2 - \beta_3 x_1 x_2 + \beta_4 x_2 R - \beta_5 x_2 T \quad (2)$$

$$\frac{dR}{dt} = \gamma_1 R - \gamma_2 R^2 - \gamma_3 x_1 R - \gamma_4 x_2 R - \gamma_5 RT \quad (3)$$

$$\frac{dT}{dt} = Q - \delta_0 T - \delta_1 x_1 T - \delta_2 x_2 T - \gamma RT \quad (4)$$

The corresponding initial conditions are:

$$\left.\begin{aligned} x_1(0) &= x_{10} \geq 0 \\ x_2(0) &= x_{20} \geq 0 \\ R(0) &= R_0 \geq 0 \\ T(0) &= T_0 \geq 0 \end{aligned}\right\} \quad (5)$$

2.2. Definition of Nomenclature:

x_1 The species one density.

x_2 The species two density.

R The resource biomass.

T The toxicant concentration level.

α_1 The growth rate of species one without any other species.

α_2 The depletion rate of intra-specific interaction of species one.

α_3 The depletion rate of inter-specific interaction of species one and two.

α_4 The growth rate of inter-specific interaction of species one and resource biomass.

α_5 The depletion rate of species one interacting with toxicants.

β_1 Is the growth rate of species two without interference with any other species.

β_2 The depletion rate of intra-specific interaction of species two.

β_3 Is the depletion rate of inter-specific interaction of species one and two respectively.

β_4 The growth rate of inter-specific interaction of species two and resource biomass.

β_5 The depletion rate of species two interacting with toxicants.

γ_1 The growth rate of the resource biomass.

γ_2 The depletion rate of intra-specific interaction of the resource biomass.

γ_3 The depletion rate of inter-specific interaction of species one with resource biomass.

γ_4 The depletion rate of interspecific interaction of species two with resource biomass.

γ_5 The depletion rate of resource biomass interacting with toxicants

Q The constant supply of toxicants.

δ_0 The depletion rate of toxicants.

δ_1 The depletion rate of toxicants with the interference of species one.

δ_2 The depletion rate of toxicants with interference of species two.

γ The depletion rate of resource biomass interacting with toxicants.

If there is no interaction between species, then the depletion rates are considered zero in the environment. That is, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \gamma_2 = \gamma_3 = \delta_1 = \delta_2 = \gamma = 0,$ then equations (1) to (4) reduce to:

$$\frac{dx_1}{dt} = \alpha_1 x_1 \quad (6)$$

$$\frac{dx_2}{dt} = \beta_1 x_2 \quad (7)$$

$$\frac{dR}{dt} = \gamma_1 R \quad (8)$$

$$\frac{dT}{dt} = Q - \delta_0 T \quad (9)$$

Simplifying equations (6) to (9), to obtain the toxicant free solutions with the initial conditions given in equation (5) above by using the method of separation of variables results to:

$$x_1 = x_{10} e^{\alpha_1 t} \quad (10)$$

$$x_2 = x_{20} e^{\beta_1 t} \quad (11)$$

$$R = R_0 e^{\gamma_1 t} \quad (12)$$

$$T = T_0 e^{-\delta_0 t} + \frac{Q}{\delta_0}\left(1 - e^{-\delta_0 t}\right) \quad (13)$$

Equation (10) – (13) above reveals that as $x_1(t)$, $x_2(t)$ and $R(t)$ will grow unboundedly while $T(t)$ will be left with the amount of resources as $t \to \infty$. This conclusion is mathematically valid; however, the scientific perspective presents a different scenario. The rationale behind this is that, in general, population growth is not limitless, as factors such as spatial constraints, limited resource biomass, and the presence of toxins in the competitive environment can restrict such expansion.

3.0. Method of Solution

In this section, we shall let equations (1) to (4) be zero as $(x_1, x_2, T, R) \to (x_{1e}, x_{2e}, T_e, R_e)$, then we have:

$$\frac{dx_1}{dt} = x_1\left(\alpha_1 - \alpha_2 x_1 - \alpha_3 x_2 + \alpha_4 R - \alpha_5 T\right) \geq 0 \quad (14)$$

$$\frac{dx_2}{dt} = x_2\left(\beta_1 - \beta_2 x_2 - \beta_3 x_1 + \beta_4 R - \beta_5 T\right) \geq 0 \quad (15)$$

$$\frac{dR}{dt} = R\left(\gamma_1 - \gamma_2 R - \gamma_3 x_1 - \gamma_4 x_2 - \gamma_5 T\right) \geq 0 \quad (16)$$

$$\frac{dT}{dt} = T\left(-\delta_0 - \delta_1 x_1 - \delta_2 x_2 - \gamma R\right) \geq -Q \quad (17)$$

3.1. Evaluation of the Trivial Steady State Solution (TSSS)

Let the trivial steady state solution be as $E_1(x_{1e}, x_{2e}, R_e, T_e)$ so that equations (14) to (17) can becomes:

$$x_{1e} = 0, \left(\alpha_1 - \alpha_2 x_{1e} - \alpha_3 x_{2e} + \alpha_4 R_e - \alpha_5 T_e\right) \neq 0 \quad (18)$$

$$x_{2e} = 0, \left(\beta_1 - \beta_2 x_{2e} - \beta_3 x_{1e} + \beta_4 R_e - \beta_5 T_e\right) \neq 0 \quad (19)$$

$$R_e = 0, \left(\gamma_1 - \gamma_2 R_e - \gamma_3 x_{1e} - \gamma_4 x_{2e} - \gamma_5 T_e\right) \neq 0 \quad (20)$$

Inserting equations (18) – (20) into equation (17), we obtain:

$$T_e \geq \frac{Q}{\delta_0} \quad (21)$$

Therefore, the trivial steady-state solution **(TSSS)** becomes:

$$E_0\left(0,0,0,\frac{Q}{\delta_0}\right) \tag{22}$$

3.2. Characterization of the Steady-State Solution of the Interaction Functions (CSSSIF)

Let us equate equations (23) – (26) below with the model equations given in equation (1) – (4). That is;

$$F_1(x_1,x_2,R,T)=\alpha_1x_1-\alpha_2x_1^2-\alpha_3x_1x_2+\alpha_4x_1R-\alpha_5x_1T \tag{23}$$

$$F_2(x_1,x_2,R,T)=\beta_1x_2-\beta_2x_2^2-\beta_3x_1x_2+\beta_4x_2R-\beta_5x_2T \tag{24}$$

$$F_3(x_1,x_2,R,T)=\gamma_1R-\gamma_2R^2-\gamma_3x_1R-\gamma_4x_2R-\gamma_5RT \tag{25}$$

$$F_4(x_1,x_2,R,T)=Q-\delta_0T-\delta_1x_1T-\delta_2x_2T-\gamma RT \tag{26}$$

Let $D\subseteq\mathbb{R}^n$ be an open region or domain of attraction. A steady state solution $(x_{1e},x_{2e},R_e,T_e)\in D$ of (31) to (4). Then by analysis $\mathbb{R}^n$, we have that the interacting functions are continuous and partially differentiable. i.e; $F_i\in C^n(D)$, where $i=1,2,3,4$. $\left[C^n(D)\right]$: Space of all continuously differentiable functions.

Evaluating the above partial derivatives i.e., equations (23) – (26) at a steady-state solution $E_0\left(0,0,0,\frac{Q}{\delta_0}\right)$, the following results are achieved:

$$\left.\begin{array}{l}
J_{11}=\alpha_1-\dfrac{\alpha_5Q}{\delta_0},\quad J_{12}=0,\quad J_{13}=0,\quad J_{14}=0\\
J_{21}=0,\quad J_{22}=\beta_1-\dfrac{\beta_5Q}{\delta_0},\quad J_{23}=0,\quad J_{24}=0\\
J_{31}=0,\quad J_{32}=0,\quad J_{33}=\gamma_1-\dfrac{\gamma_5Q}{\delta_0},\quad J_{34}=0\\
J_{41}=\dfrac{-\delta_1Q}{\delta_0},\quad J_{42}=\dfrac{-\delta_2Q}{\delta_0},\quad J_{43}=\dfrac{-\gamma Q}{\delta_0},\quad J_{44}=-\delta_0.
\end{array}\right\} \tag{27}$$

The Jacobian matrix is:

$$J_0\left(E_0\left(0,0,0,\frac{Q}{\delta_0}\right)\right)=\begin{bmatrix}\alpha_1-\dfrac{\alpha_5Q}{\delta_0} & 0 & 0 & 0\\ 0 & \beta_1-\dfrac{\beta_5Q}{\delta_0} & 0 & 0\\ 0 & 0 & \gamma_1-\dfrac{\gamma_5Q}{\delta_0} & 0\\ \dfrac{-\delta_1Q}{\delta_0} & \dfrac{-\delta_2Q}{\delta_0} & \dfrac{-\gamma Q}{\delta_0} & -\delta_0\end{bmatrix}$$

(28)

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1.1 Assumptions:

(xi) Species one grows unboundedly without interference with species two with the toxicants.

(xii) Species two grows unboundedly without interference with species one and the toxicants.

(xiii) The resource biomass density grows without interference with species one and two densities and the toxicants.

(xiv) There is a constant supply of toxicants in the environment.

(xv) The inter and intra-specific interactions of species lead to the depletion of their densities.

(xvi) The interaction of the species and the toxicants leads to the depletion of the species' densities.

(xvii) The interaction between the resource biomass and the toxicants depletes the biomass quantity.

(xviii) The interaction between the biomass resource and the species densities reduces the resource of the biomass.

(xix) he interaction between the biomass resource and species’ densities improves the species’ densities.

(xx) Interaction between the toxicants and the species’ densities leads to depletion of the toxicants due to the absorption by the species.

Given the aforementioned assumptions in section 2.1, the models representing the effect of toxicants from external sources on toxicant concentration and the densities of competing species in an environment are:

$$\frac{dx_1}{dt} = \alpha_1 x_1 - \alpha_2 x_1^2 - \alpha_3 x_1 x_2 + \alpha_4 x_1 R - \alpha_5 x_1 T \quad (1)$$

$$\frac{dx_2}{dt} = \beta_1 x_2 - \beta_2 x_2^2 - \beta_3 x_1 x_2 + \beta_4 x_2 R - \beta_5 x_2 T \quad (2)$$

$$\frac{dR}{dt} = \gamma_1 R - \gamma_2 R^2 - \gamma_3 x_1 R - \gamma_4 x_2 R - \gamma_5 RT \quad (3)$$

$$\frac{dT}{dt} = Q - \delta_0 T - \delta_1 x_1 T - \delta_2 x_2 T - \gamma RT \quad (4)$$

The corresponding initial conditions are:

$$\left.\begin{aligned} x_1(0) &= x_{10} \geq 0 \\ x_2(0) &= x_{20} \geq 0 \\ R(0) &= R_0 \geq 0 \\ T(0) &= T_0 \geq 0 \end{aligned}\right\} \quad (5)$$

4.1. *Definition of Nomenclature*

x_1 The species one density.

x_2 The species two density.

R The resource biomass.

T The toxicant concentration level.

α_1 The growth rate of species one without any other species.

α_2 The depletion rate of intra-specific interaction of species one.

α_3 The depletion rate of inter-specific interaction of species one and two.

α_4 The growth rate of inter-specific interaction of species one and resource biomass.

α_5 The depletion rate of species one interacting with toxicants.

β_1 Is the growth rate of species two without interference with any other species.

β_2 The depletion rate of intra-specific interaction of species two.

β_3 Is the depletion rate of inter-specific interaction of species one and two respectively.

β_4 The growth rate of inter-specific interaction of species two and resource biomass.

β_5 The depletion rate of species two interacting with toxicants.

γ_1 The growth rate of the resource biomass.

γ_2 The depletion rate of intra-specific interaction of the resource biomass.

γ_3 The depletion rate of inter-specific interaction of species one with resource biomass.

γ_4 The depletion rate of interspecific interaction of species two with resource biomass.

γ_5 The depletion rate of resource biomass interacting with toxicants

Q The constant supply of toxicants.

δ_0 The depletion rate of toxicants.

δ_1 The depletion rate of toxicants with the interference of species one.

δ_2 The depletion rate of toxicants with interference of species two.

γ The depletion rate of resource biomass interacting with toxicants.

If there is no interaction between species, then the depletion rates are considered zero in the environment. That is, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \gamma_2 = \gamma_3 = \delta_1 = \delta_2 = \gamma = 0,$ then equations (1) to (4) reduce to:

$$\frac{dx_1}{dt} = \alpha_1 x_1 \quad (6)$$

$$\frac{dx_2}{dt} = \beta_1 x_2 \quad (7)$$

$$\frac{dR}{dt} = \gamma_1 R \quad (8)$$

$$\frac{dT}{dt} = Q - \delta_0 T \quad (9)$$

Simplifying equations (6) to (9), to obtain the toxicant free solutions with the initial conditions given in equation (5) above by using the method of separation of variables results to:

$$x_1 = x_{10}e^{\alpha_1 t} \quad (10)$$

$$x_2 = x_{20}e^{\beta_1 t} \quad (11)$$

$$R = R_0 e^{\gamma_1 t} \quad (12)$$

$$T = T_0 e^{-\delta_0 t} + \frac{Q}{\delta_0}\left(1 - e^{-\delta_0 t}\right) \quad (13)$$

Equation (10) – (13) above reveals that as $x_1(t)$, $x_2(t)$ and $R(t)$ will grow unboundedly while $T(t)$ will be left with the amount of resources as $t \to \infty$. This conclusion is mathematically valid; however, the scientific perspective presents a different scenario. The rationale behind this is that, in general, population growth is not limitless, as factors such as spatial constraints, limited resource biomass, and the presence of toxins in the competitive environment can restrict such expansion.

5.0. Method of Solution:

In this section, we shall let equations (1) to (4) be zero as $(x_1, x_2, R, T) \to (x_{1e}, x_{2e}, R_e, T_e)$, then we have:

$$\frac{dx_1}{dt} = x_1\left(\alpha_1 - \alpha_2 x_1 - \alpha_3 x_2 + \alpha_4 R - \alpha_5 T\right) \geq 0 \quad (14)$$

$$\frac{dx_2}{dt} = x_2\left(\beta_1 - \beta_2 x_2 - \beta_3 x_1 + \beta_4 R - \beta_5 T\right) \geq 0 \quad (15)$$

$$\frac{dR}{dt} = R\left(\gamma_1 - \gamma_2 R - \gamma_3 x_1 - \gamma_4 x_2 - \gamma_5 T\right) \geq 0 \quad (16)$$

$$\frac{dT}{dt} = T\left(-\delta_0 - \delta_1 x_1 - \delta_2 x_2 - \gamma R\right) \geq -Q \quad (17)$$

5.1. Evaluation of the Trivial Steady State Solution (TSSS)

Let the trivial steady state solution be as $E_0(x_{1e}, x_{2e}, R_e, T_e)$ so that equations (14) to (17) can becomes:

$$x_{1e} = 0, \left(\alpha_1 - \alpha_2 x_{1e} - \alpha_3 x_{2e} + \alpha_4 R_e - \alpha_5 T_e\right) \neq 0 \quad (18)$$

$$x_{2e} = 0, \left(\beta_1 - \beta_2 x_{2e} - \beta_3 x_{1e} + \beta_4 R_e - \beta_5 T_e\right) \neq 0 \quad (19)$$

$$R_e = 0, \left(\gamma_1 - \gamma_2 R_e - \gamma_3 x_{1e} - \gamma_4 x_{2e} - \gamma_5 T_e\right) \neq 0 \quad (20)$$

Inserting equations (18) – (20) into equation (17), we obtain:

$$T_e \geq \frac{Q}{\delta_0} \quad (21)$$

Therefore, the trivial steady-state solution **(TSSS)** becomes:

$$E_0\left(0, 0, 0, \frac{Q}{\delta_0}\right) \quad (22)$$

3.2. Characterization of the Steady-State Solution of the Interaction Functions (CSSSIF):

Let us equate equations (23) – (26) below with the model equations given in equation (1) – (4). That is;

$$F_1(x_1,x_2,R,T)=\alpha_1x_1-\alpha_2x_1^2-\alpha_3x_1x_2+\alpha_4x_1R-\alpha_5x_1T \quad (23)$$
$$F_2(x_1,x_2,R,T)=\beta_1x_2-\beta_2x_2^2-\beta_3x_1x_2+\beta_4x_2R-\beta_5x_2T \quad (24)$$
$$F_3(x_1,x_2,R,T)=\gamma_1R-\gamma_2R^2-\gamma_3x_1R-\gamma_4x_2R-\gamma_5RT \quad (25)$$
$$F_4(x_1,x_2,R,T)=Q-\delta_0T-\delta_1x_1T-\delta_2x_2T-\gamma RT \quad (26)$$

Let $D\subseteq\mathbb{R}^n$ be an open region or domain of attraction. A steady state solution $(x_{1e},x_{2e},R_e,T_e)\in D$ of (1) to (4). Then by analysis $\mathbb{R}^n$, we have that the interacting functions are continuous and partially differentiable. i.e; $F_i\in C^n(D)$, where $i=1,2,3,4$. $\left[C^n(D)\right]$: Space of all continuously differentiable functions.

Evaluating the above partial derivatives i.e., equations (23) – (26) at a steady-state solution $E_0\left(0,0,0,\frac{Q}{\delta_0}\right)$, the following results are achieved:

$$\left.\begin{array}{l} J_{11}=\alpha_1-\frac{\alpha_5Q}{\delta_0},\quad J_{12}=0,\quad J_{13}=0,\quad J_{14}=0 \\ J_{21}=0,\quad J_{22}=\beta_1-\frac{\beta_5Q}{\delta_0},\quad J_{23}=0,\quad J_{24}=0 \\ J_{31}=0,\quad J_{32}=0,\quad J_{33}=\gamma_1-\frac{\gamma_5Q}{\delta_0},\quad J_{34}=0 \\ J_{41}=\frac{-\delta_1Q}{\delta_0},\quad J_{42}=\frac{-\delta_2Q}{\delta_0},\quad J_{43}=\frac{-\gamma Q}{\delta_0},\quad J_{44}=-\delta_0. \end{array}\right\} \quad (27)$$

The Jacobian matrix is:

$$J_0\left(E_0\left(0,0,0,\frac{Q}{\delta_0}\right)\right)=\begin{bmatrix} \alpha_1-\frac{\alpha_5Q}{\delta_0} & 0 & 0 & 0 \\ 0 & \beta_1-\frac{\beta_5Q}{\delta_0} & 0 & 0 \\ 0 & 0 & \gamma_1-\frac{\gamma_5Q}{\delta_0} & 0 \\ \frac{-\delta_1Q}{\delta_0} & \frac{-\delta_2Q}{\delta_0} & \frac{-\gamma Q}{\delta_0} & -\delta_0 \end{bmatrix} \quad (28)$$

where,

$$J_{11} = \alpha_1 - \frac{\alpha_5 Q}{\delta_0}$$

$$J_{22} = \beta_1 - \frac{\beta_5 Q}{\delta_0}$$

$$J_{33} = \gamma_1 - \frac{\gamma_5 Q}{\delta_0}$$

$$J_{44} = -\delta_0$$

Contribute to the decaying behaviour of the solution trajectories as $t \to \infty$.

On the other hand, the steady-state solution $E_0\left(0,0,0,\frac{Q}{\delta_0}\right)$ system is stable if

$$\left.\begin{aligned} J_{11} &= \left(\alpha_1 - \frac{\delta_1 Q}{\delta_0}\right) \prec 0 \\ J_{22} &= \left(\beta_1 - \frac{\delta_2 Q}{\delta_0}\right) \prec 0 \\ J_{33} &= \left(\gamma_1 - \frac{\gamma Q}{\delta_0}\right) \prec 0 \\ J_{44} &= -\delta_0 \end{aligned}\right\} \tag{29}$$

That is,
if all the eigenvalues in equation (29) have negative real parts. Otherwise, it is unstable.

3.3. Effect of intrinsic growth rate and concentration rate of external toxicant on the type of stability

The area where stability is lost is of concern to ecologists and environmentalists for proper planning. The center of interest of this section is to identify the parameter values where this took place. As a result of this, the nonlinear mathematical model (1) – (4) is considered to assess the effect of the intrinsic growth rate α_1, and the concentration rate of leaks of the same toxicant, Q, on the type of stability (TOS). The Mathematica function has been employed for the simulations to make it easier to investigate of the behaviour of stability of the steady-state solution $E_0\left(0,0,0,\frac{Q}{\delta_0}\right)$ for varying values of α_1.

3.4. Accessing the impact of variation of the intra-competition and inter-competition coefficients on the stabilization analysis.

For a dynamical system that is continuous and partially differentiable, a numerical simulation based on a **Mathematica** programming function has been used to facilitate the investigation of the impact of variation of the intra-competition coefficients α_2 and β_2, the analysis of stabilization for fixed values of another model parameter as $t \to \infty$. The effect of variation of the degradation factors, γ_2, in the environment affecting the density of resource biomass, R, is also considered.

To facilitate the interpretation of the mathematical analysis, the following parameter values given by Agarwal *et al.*, (2011) are used in the simulations for the dynamical system (1) to (4):

$$\alpha_1 = 5,\ \alpha_2 = 0.22,\ \alpha_3 = 0.007,\ \alpha_4 = 0.02,\ \alpha_5 = 0.1$$
$$\beta_1 = 3,\ \beta_2 = 0.26,\ \beta_3 = 0.008,\ \beta_4 = 0.04,\ \beta_5 = 0.2$$
$$\gamma_1 = 10,\ \gamma_2 = 0.3,\ \gamma_3 = 0.02,\ \gamma_4 = 0.04,\ \gamma_5 = 0.1$$
$$Q = 30,\ \delta_0 = 7,\ \delta_1 = 0.05,\ \delta_2 = 0.04,\ \gamma = 0.3$$

6.0. Results and Discussion

This section outlines the numerical results obtained from our research. The outcomes are displayed in both tabular and graphical formats, following analysis with Mathematica to enhance the readability and comprehension of the study. This investigation focused on modeling the effect of external toxicants within a competitive environment.

6.1. Simulation and Presentation Behavior of the Toxicant Concentration on Specie one, Specie two, and Resource Biomass.

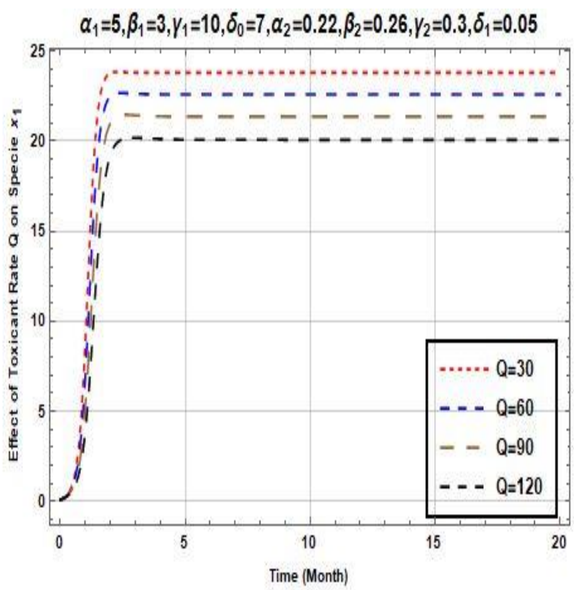

Fig. 1: 1 Effect of Toxicant Flux on Specie One Concentration

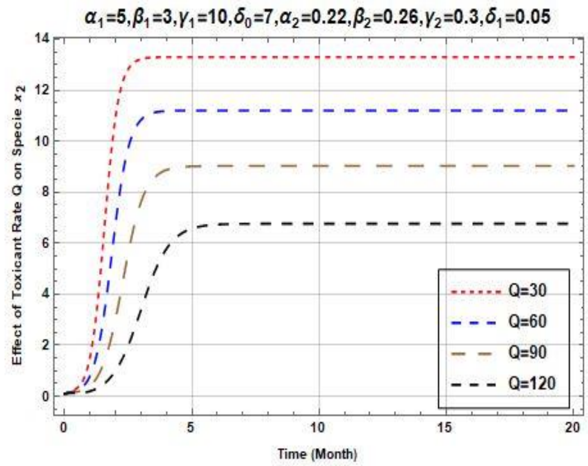

Fig. 2 Effect of Toxicant Flux on Specie Two Concentration

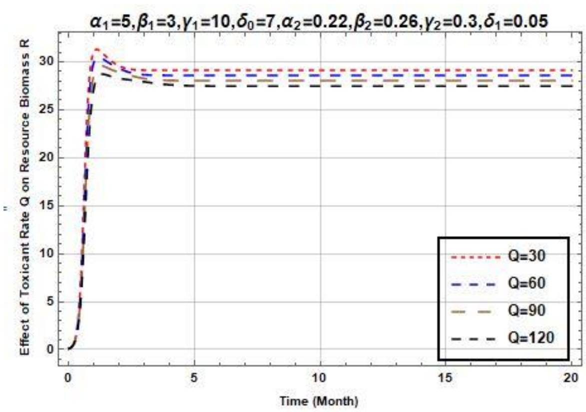

Fig. 3: Effect of Toxicant Flux on Resource Biomass Concentration

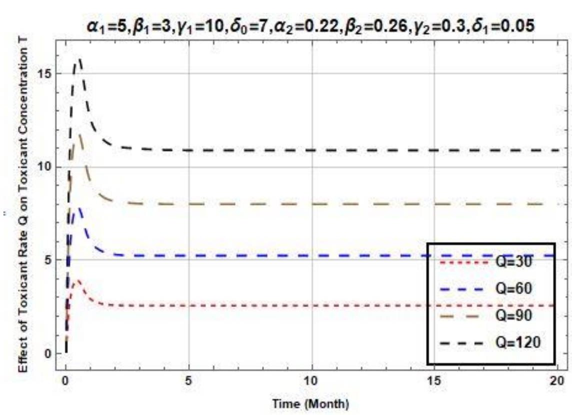

Fig.4: Effect of Toxicant Flux on Toxicant Concentration

Table 1:

Time	$x_2, Q=30$	$x_2, Q=60$	$x_2, Q=90$	$x_2, Q=120$
0	0.1	0.1	0.1	0.1
1	1.53649	0.80753	0.410851	0.203837
2	10.9417	6.7491	3.02334	1.00075
3	13.2198	10.8345	7.60858	3.39321
4	13.2998	11.1856	8.876	5.78051
5	13.3021	11.2043	9.0164	6.57698
6	13.3022	11.2053	9.02972	6.73625
7	13.3022	11.2053	9.03096	6.76432
8	13.3022	11.2053	9.03108	6.76915
9	13.3022	11.2053	9.03109	6.76998
10	13.3022	11.2053	9.03109	6.77012
11	13.3022	11.2053	9.03109	6.77015
12	13.3022	11.2053	9.03109	6.77015
13	13.3022	11.2053	9.03109	6.77015
14	13.3022	11.2053	9.03109	6.77015
15	13.3022	11.2053	9.03109	6.77015
16	13.3022	11.2053	9.03109	6.77015
17	13.3022	11.2053	9.03109	6.77015
18	13.3022	11.2053	9.03109	6.77015
19	13.3022	11.2053	9.03109	6.77015
20	13.3022	11.2053	9.03109	6.77015

Table 2

Time	$x_1, Q=30$	$x_1, Q=60$	$x_1, Q=90$	$x_1, Q=120$
0	0.1	0.1	0.1	0.1
1	8.76669	6.81306	5.16936	3.83746
2	23.7527	22.464	20.9657	19.1772
3	23.7882	22.6149	21.4222	20.1712
4	23.7789	22.5866	21.3578	20.1053
5	23.7786	22.5846	21.3476	20.0693
6	23.7786	22.5845	21.3465	20.0608
7	23.7786	22.5845	21.3464	20.0593
8	23.7786	22.5845	21.3464	20.059
9	23.7786	22.5845	21.3464	20.0589
10	23.7786	22.5845	21.3464	20.0589
11	23.7786	22.5845	21.3464	20.0589
12	23.7786	22.5845	21.3464	20.0589
13	23.7786	22.5845	21.3464	20.0589
14	23.7786	22.5845	21.3464	20.0589
15	23.7786	22.5845	21.3464	20.0589
16	23.7786	22.5845	21.3464	20.0589
17	23.7786	22.5845	21.3464	20.0589
18	23.7786	22.5845	21.3464	20.0589
19	23.7786	22.5845	21.3464	20.0589
20	23.7786	22.5845	21.3464	20.0589

Table 3:

Time	$T, Q=30$	$T, Q=60$	$T, Q=90$	$T, Q=120$
0	0.1	0.1	0.1	0.1
1	2.91478	5.96832	9.16357	12.5083
2	2.59533	5.31247	8.15849	11.1149
3	2.57934	5.25068	8.03965	10.9803
4	2.57899	5.24682	8.01549	10.9153
5	2.57898	5.24664	8.01302	10.8942
6	2.57898	5.24663	8.01279	10.89
7	2.57898	5.24663	8.01276	10.8893
8	2.57898	5.24663	8.01276	10.8892
9	2.57898	5.24663	8.01276	10.8892
10	2.57898	5.24663	8.01276	10.8892
11	2.57898	5.24663	8.01276	10.8892
12	2.57898	5.24663	8.01276	10.8892
13	2.57898	5.24663	8.01276	10.8892
14	2.57898	5.24663	8.01276	10.8892
15	2.57898	5.24663	8.01276	10.8892
16	2.57898	5.24663	8.01276	10.8892
17	2.57898	5.24663	8.01276	10.8892
18	2.57898	5.24663	8.01276	10.8892
19	2.57898	5.24663	8.01276	10.8892
20	2.57898	5.24663	8.01276	10.8892

Table 4:

Time	$R, Q=30$	$R, Q=60$	$R, Q=90$	$R, Q=120$
0	0.1	0.1	0.1	0.1
1	31.0859	29.9956	28.7511	27.3529
2	29.564	29.303	28.9039	28.2731
3	29.1319	28.6532	28.2606	27.9194
4	29.1153	28.5885	28.0614	27.6091
5	29.1148	28.585	28.0377	27.4931
6	29.1148	28.5848	28.0354	27.4689
7	29.1148	28.5848	28.0352	27.4646
8	29.1148	28.5848	28.0352	27.4638
9	29.1148	28.5848	28.0352	27.4637
10	29.1148	28.5848	28.0352	27.4637
11	29.1148	28.5848	28.0352	27.4637
12	29.1148	28.5848	28.0352	27.4637
13	29.1148	28.5848	28.0352	27.4637
14	29.1148	28.5848	28.0352	27.4637
15	29.1148	28.5848	28.0352	27.4637
16	29.1148	28.5848	28.0352	27.4637
17	29.1148	28.5848	28.0352	27.4637
18	29.1148	28.5848	28.0352	27.4637
19	29.1148	28.5848	28.0352	27.4637
20	29.1148	28.5848	28.0352	27.4637

4.2. Discussion of Results

Fig. 1 illustrates the impact of toxicant flux on the concentration of specie two with the values of

$$\alpha_1 = 5, \beta_1 = 3, \gamma_1 = 10, \delta_0 = 7, \alpha_2 = 0.22, \beta_2 = 0.26, \gamma_2 = 0.3,$$

remained the same. The result indicated a sharp increase in species one concentration from the initial value of 0.1 to a maximum of 23.7789 within a month period with the toxicant influx at $Q = 30$, thereafter decreasing to a stable minimum of 23.7786, for the rest of the months. It is further observed that the increased toxicant flux reduces the concentration of species one.

Fig. 2 illustrates the impact of toxicant flux on the concentration of species 2 with the values of

$$\alpha_1 = 5, \beta_1 = 3, \gamma_1 = 10, \delta_0 = 7, \alpha_2 = 0.22, \beta_2 = 0.26, \gamma_2 = 0.3,$$

remained the same. The result indicated a sharp increase in species two concentration from the initial value of 0.1 to a maximum of 13.2198 within a month with the toxicant influx at $Q = 30$, thereafter decreasing to a stable minimum of 13.3022, for the rest of the months. It is further observed that the increased toxicant flux reduces the concentration of species two.

Fig. 3 illustrates the impact of a constant supply of toxicants on Resource biomass for the values of $\alpha_1 = 5, \beta_1 = 3, \gamma_1 = 10, \delta_0 = 7, \alpha_2 = 0.22, \beta_2 = 0.26, \gamma_2 = 0.3, \delta_1 = 7$. The fig. shows an increase in resource biomass at the initial stage of with a value 0.1 and grows to a maximum of 31.0859 units within a month with the toxicant influx at $Q = 30$, thereafter decreases to a stable minimum of 29.1148 units, for the rest of the months. The fig. further showed that, as the toxicant flux increases, the resource biomass reduces from 31.0859, 29.9956, 28.7511, and 27.3529 to a minimum of 29.1148, 28.5848, 28.0352, and 27.4637respecticely. This result is of the view that an increase in toxicity reduces the quality and quantity of the resource biomass.

Fig. 4 illustrates the impact of toxicant flux on the overall toxicant concentration for the different values of

$$\alpha_1 = 5, \beta_1 = 3, \gamma_1 = 10, \delta_0 = 7, \alpha_2 = 0.22, \beta_2 = 0.26, \gamma_2 = 0.3, \delta_1 = 7$$

. The shows a sharp increase in concentration from the initial value of 0.1 to a maximum of 2.91478 within a month with the toxicant influx at $Q = 30$, thereafter decreases to a minimum of 2.57934, then stabilizes at 2.57898 for the rest of the months. It is further observed that the toxicant concentration increases to different peaks for an increase in toxicant influx at different levels before reducing to stable levels of 2.57898, 5.24663, 8.01276, and 10.8892 respectively. This figure is of the view that the increase in toxicant flux increases the concentration of the toxicant in the system.

7.0. Conclusion

This study explored the effects of external toxicants (Microplastics) on the dynamics of two competing species using a mathematical modeling approach. The results indicate that the introduction of toxicants into the environment significantly impacts the density of both species and the resource biomass. Specifically, as the concentration of toxicants increases, the population densities of the competing species decrease, leading to a reduction in overall resource biomass. The mathematical analysis and simulations reveal that the stability of the ecosystem is closely linked to the intrinsic growth rates of the species and the concentration of toxicants, highlighting the critical thresholds where stability is lost.

The findings underscore the importance of considering external toxicants in environmental management and ecological modeling. The study concludes that excessive toxicant influx can destabilize competitive environments, reducing species diversity and biomass. The results suggest that regulatory measures should be implemented to control toxicant emissions in ecosystems. Further research is recommended to refine the models by incorporating more interactions and data, which will improve the accuracy and applicability of the predictions in environmental conservation efforts.

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Compliance with Ethical Standards Declarations:

The authors declare that they have no conflict of interest.

Data availability

All data used in this study will be readily available to the public.

Consent for publication

Not Applicable.

Availability of data and materials

The publisher can make the data public.

Competing interests

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Author Contributions

Obonin, Samuel Sabastine carried out 55% aspect of the work. 20% aspect of the work were done by Dr. Mrs. Amadi, Ugwulo Chinyere and 25% aspect of the work were done by Mr. Sylvanus, Kupongoh Samaila.