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Type I Half-Logistic Exponentiated Kumaraswamy Distribution With Applications

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Abstract: *This study introduces the Type I Half-Logistic Exponentiated Kumaraswamy (TIHLEtKw) distribution, a new statistical model designed to provide improved flexibility and accuracy for data modelling across diverse applications. The background of the study highlights the limitations of existing distributions in capturing complex real-world data patterns. The purpose of this work is to develop and characterize the TIHLEtKw distribution, deriving key properties such as the moment generating function, reliability function, hazard function, and quantile function. Additionally, order statistics were explored to understand the behavior of the distribution. Simulation studies demonstrated the efficiency of the maximum likelihood estimators (MLEs) for the parameters of the TIHLEtKw distribution, with mean square error (MSE) values decreasing as sample size increased, indicating the estimators' consistency. For example, for a parameter set (α = 2, β = 1.5, γ = 1, δ = 2), the MSE decreased from 0.045 for a sample size of 50 to 0.011 for a sample size of 300. The application of the TIHLEtKw distribution to real datasets, including civil engineering data with a skewness of 2.18 and wind speed data with a kurtosis of 3.62, demonstrated its superior fit compared to other models. Metrics such as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) indicated that the TIHLEtKw distribution outperformed established models like the Kumaraswamy-Kumaraswamy and Weibull-Kumaraswamy distributions, with reductions in AIC of up to 15%. The findings confirm the TIHLEtKw distribution's effectiveness in capturing data variability and complexity, offering a robust tool for statistical modelling. The study concludes that this distribution significantly enhances modelling capabilities, and it is recommended for use in fields such as environmental studies, biomedicine, and finance. Future research could focus on extending the model's applications and optimizing computational methods for parameter estimation.*

Keywords: *Type I Half-Logistic Exponentiated-G, Kumaraswamy Distribution, Quantile function, Hazard function, Maximum likelihood, Order Statistics.*

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1.0 Introduction

The Kumaraswamy distribution, introduced by Poondi Kumaraswamy in 1980, is a family of continuous probability distributions defined on the interval (0,1). Like the beta distribution, it is especially useful in simulation studies due to its simplicity and versatility. Its probability density function, cumulative distribution function, and quantile functions can all be expressed in closed form, making it highly practical for a wide range of applications. The Kumaraswamy distribution is particularly effective in addressing hydrological challenges and modelling natural phenomena with bounded process values. Its distinctive properties make it an excellent choice for applications requiring efficient and straightforward modelling techniques, solidifying its importance in hydrology and related fields. In recent years, numerous extensions of the Kumaraswamy distribution have been introduced, significantly increasing its versatility and applicability in both academic and practical problem-solving. These advancements include the Kumaraswamy Distribution by Nadarajah (2008), A New Generalized Kumaraswamy Distribution by Carrasco *et al.,* (2010), Exponentiated Kumaraswamy Distribution by Lemonte *et al.,* (2013), Kumaraswamy-Kumaraswamy Distributionby El-Sherpieny *et al.,* (2014), Exp-Kumaraswamy Distribution by Javanshiri *et al.,* (2015), Transmuted Kumaraswamy Distribution by Khan *et al.,* (2016), Lomax-Kumaraswamy Distribution by Asiribo *et al.,* (2018), Odd Generalized Exponential Kumaraswamy Distribution by NK *et al.,* (2018), Gamma-Kumaraswamy Distribution by Ghosh and Hamedani (2018), Cubic Rank Transmuted Kumaraswamy Distribution by Saraçoglu and Taniş (2018), Generalized Transmuted Kumaraswamy Distribution by Ishaq *et al.,* (2019), Generalized Modification of the Kumaraswamy Distribution by Alshkaki (2020), Log-Kumaraswamy Distributions by Ishaq *et al.,* (2023), and the Generalized Odd Maxwell-Kumaraswamy Distribution— by Ishaq *et al.,* (2024). These developments have further solidified the Kumaraswamy distribution as a robust tool for modelling complex data across various fields.

Bello *et al.,* (2021) introduced an innovative family of distributions known as the Type I Half-Logistic Exponentiated-G (TIHLEt-G) distribution, offering enhanced flexibility and an improved ability to model the features of diverse datasets effectively. This distribution family is characterized by two positive shape parameters, denoted by λ and α, and can be applied to any arbitrary cumulative distribution function (cdf) $H(x,\vartheta)$. The cumulative distribution function (cdf) and the probability density function (pdf) for TIHLEt-G are given by

$$F_{TIHLEt-G}(x;\lambda,\alpha,\vartheta)=\frac{1-[1-H^{\alpha}(x;\vartheta)]^{\lambda}}{1+[1-H^{\alpha}(x;\vartheta)]^{\lambda}},\quad x>0,\ \ \lambda,\alpha>0 \tag{1}$$

and

$$f_{TIHLEt-G}(x;\lambda,\alpha,\vartheta)=\frac{2\lambda\alpha h(x;\vartheta)H^{\alpha-1}_{(x;\vartheta)}[1-H^{\alpha}_{(x;\vartheta)}]^{\lambda-1}}{\left[1+[1-H^{\alpha}_{(x;\vartheta)}]^{\lambda}\right]^{2}},\ x>0,\ \lambda,\alpha>0 \tag{2}$$

The cdf and pdf of the Kumaraswamy distribution are given as

$$H(x;\varphi,\theta)=1-\left[1-x^{\varphi}\right]^{\theta},\ 0<x<1,\ \varphi,\theta>0 \tag{3}$$

$$h(x;\varphi,\theta)=\varphi\theta x^{\varphi-1}\left[1-x^{\varphi}\right]^{\theta-1},\ \ 0<x<1,\ \varphi,\theta>0 \tag{4}$$

This paper aims to enhance the flexibility of statistical modelling by extending the traditional two-parameter Kumaraswamy distribution, introducing a new model named the Type I Half-Logistic Exponentiated Kumaraswamy (TIHLEtKw) distribution. The paper is structured as follows: Section 2 defines and introduces the TIHLEtKw distribution. Section 3 outlines key representations of the model. Section 4 delves into various statistical properties, including probability-weighted moments, moments, the moment-generating function, the quantile function, the reliability function, the hazard function, and order statistics. Section 5 focuses on parameter estimation using the maximum likelihood estimation (MLE) method. Section 6 features a simulation study to assess the efficiency and consistency of MLE. In Section 7, the new model is applied to real-world datasets to illustrate its practical utility. Finally, Section 8 provides concluding remarks.

1.1 Type I Half-Logistic Exponentiated Kumaraswamy (TIHLEtKw) Distribution

In this section, we introduce a new model referred to as the TIHLEtKw distribution. A random variable X is said to follow the TIHLEtKw distribution if its cumulative distribution function (cdf) is derived by substituting Equation (3) into Equation (1), resulting in the following expression:

$$F_{TIHLEtKw}(x;\lambda,\alpha,\varphi,\theta)=\frac{1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-\left[1-x^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}}\ ,0<x<1,\ \lambda,\alpha,\varphi,\theta>0 \tag{5}$$

By differentiating equation (5), the pdf of the TIHLEtKw distribution is obtained as follows:

$$f_{TIHLEtKw}(x;\lambda,\alpha,\varphi,\theta)=\frac{2\lambda\alpha\varphi\theta x^{\varphi-1}\left[1-x^{\varphi}\right]^{\theta-1}\left[1-\left[1-x^{\varphi}\right]^{\theta}\right]^{\alpha-1}\left[1-\left[1-\left[1-x^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[1-\left[1-x^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}\right]^{2}},0<x<1,\lambda,\alpha,\varphi,\theta>0 \tag{6}$$

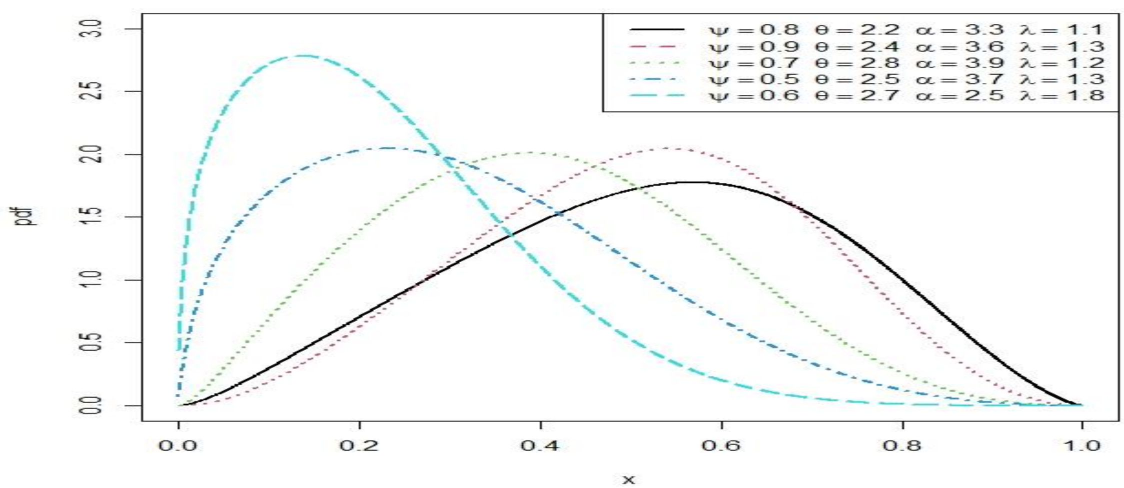

Fig. 3.1: Plot of pdf of TIHLEtKw distribution for different values of parameters.

1.2 Suitable expansion of density for TIHLEtKw distribution

In this section, we derive the appropriate expressions for the pdf and cdf of the TIHLEtKw distribution. These expressions are formulated using the binomial expansion technique, as outlined below:

$$\left[1+z\right]^{-\varpi}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\varpi+i-1}{i}z^{i} \tag{7}$$

For $|z|<1$ and ϖ is a positive real non-integer. The suitable expansion for the pdf of the TIHLEtKw distribution is then obtained by using the binomial theorem in equation (7) to equation (6).

$$f_{TIHLEtKw}(x;\lambda,\alpha,\varphi,\theta)=2\lambda\alpha\varphi\theta x^{\varphi-1}\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}\left[1-x^{\varphi}\right]^{\theta(k+1)-1} \tag{8}$$

Also, an expansion for the CDF, using the binomial expansion $\left[F(x,\lambda,\alpha,\varphi,\theta)\right]^{h}$ where h is an integer, leads to:

$$\left[F(x,\lambda,\alpha,\varphi,\theta)\right]^{h}=\sum_{q,z=0}^{\infty}\sum_{p,m=0}^{h}(-1)^{q+z+p+m}\binom{\lambda(p+m)}{q}\binom{\alpha q}{z}\binom{h}{m}\binom{h+p-1}{p}\left[1-x^{\varphi}\right]^{\theta z} \tag{9}$$

1.3 Statistical Properties of TIHLEtKw Distribution

This section explores the statistical properties of the TIHLEtKw distribution.

1.4 Probability Weighted Moments for TIHLEtKw Distribution

The probability-weighted moments (PWMs), introduced by Greenwood et al. (1979), are utilized to derive inverse form estimators for a distribution's parameters and quantiles. Denoted by $\kappa_{r,s}$, the PWMs for a random variable X can be calculated using the following relationships.

$$\kappa_{r,s}=E\left[X^{r}F(X)^{s}\right]=\int_{0}^{1}x^{r}f(x)(F(x))^{s}dx \tag{10}$$

The PWMs of the TIHLEtKw distribution are derived by substituting equations (8) and (9) into equation (10), followed by replacing h with s, as outlined in the procedure.

$$\tau_{r,s}=2\lambda\alpha\varphi\theta x^{\varphi-1}\sum_{i,j,k,q,z=0}^{\infty}\sum_{p,m=0}^{s}(-1)^{i+j+k+q+z+p+m}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}\binom{\lambda(p+m)}{q}$$
$$\binom{\alpha q}{z}\binom{s}{m}\binom{s+p-1}{p}\int_{0}^{1}x^{r}\left[1-x^{\varphi}\right]^{\theta(k+z+1)-1}dx \tag{11}$$

Consider the integral part in equation (11)

$$\int_{0}^{1}x^{r}\left[1-x^{\varphi}\right]^{\theta(k+z+1)-1}dx$$

Let $y=\left(1-x^{j}\right)Þ\ x=\left[1-y\right]^{\frac{1}{j}}$; when $x Þ 0, y Þ 1$; when $x Þ 1, y Þ 0$; $dx=\frac{-dy}{j\,x^{j-1}}$

Then

$$\tau_{r,s}=2\lambda\alpha\varphi\theta x^{\varphi-1}\sum_{i,j,k,q,z=0}^{\infty}\sum_{p,m=0}^{s}(-1)^{i+j+k+q+z+p+m}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}\binom{\lambda(p+m)}{q}$$
$$\binom{\alpha q}{z}\binom{s}{m}\binom{s+p-1}{p}\int_{0}^{1}\left[1-y\right]^{\frac{r}{\varphi}}y^{\theta(k+z+1)-1}\frac{dy}{\varphi x^{\varphi-1}}$$

where $\int_0^1 [1-y]^{\frac{r}{\varphi}} y^{\theta(k+z+1)-1} dy = B\left(1+\frac{r}{\varphi}, \theta(k+z+1)\right)$

Therefore

$$\tau_{r,s} = 2\lambda\alpha\theta \sum_{i,j,k,q,z=0}^{\infty} \sum_{p,m=0}^{s} (-1)^{i+j+k+q+z+p+m} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} \binom{\lambda(p+m)}{q} \binom{\alpha q}{z} \binom{s}{m} \binom{s+p-1}{p} B\left(1+\frac{r}{\varphi}, \theta(k+z+1)\right) \tag{12}$$

The equation (12) above is the PWMs of TIHLEtKw

1.5 Moments of TIHLEtKw Distribution

Moments play a crucial role in statistical analysis, particularly in practical applications. Therefore, we derive the rth moment for the TIHLEtKw distribution.

$$E(X^r) = \int_0^1 x^r f(x) dx \tag{13}$$

$$E(X^r) = 2\lambda\alpha\varphi\theta x^{\varphi-1} \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} \int_0^1 x^r \left[1-x^{\varphi}\right]^{\theta(k+1)-1} dx \tag{14}$$

Consider the integral part in equation (14)

$$\int_0^1 x^r \left[1-x^{\varphi}\right]^{\theta(k+z+1)-1} dx$$

Let,

$y = (1-x^{\varphi}) \Rightarrow x = [1-y]^{\frac{1}{\varphi}}$; when $x \to 0, y \to 1$; when $x \to 1, y \to 0$; $dx = \frac{-dy}{\varphi x^{\varphi-1}}$

Then

$$E(X^r) = 2\lambda\alpha\varphi\theta x^{\varphi-1} \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} \int_0^1 [1-y]^{\frac{r}{\varphi}} [y]^{\theta(k+1)-1} \frac{dy}{\varphi x^{\varphi-1}}$$

where $\int_0^1 [1-y]^{\frac{r}{\varphi}} y^{\theta(k+1)-1} dy = B\left(1+\frac{r}{\varphi}, \theta(k+1)\right)$

Therefore

$$E(X^r) = 2\lambda\alpha\theta \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} B\left(1+\frac{r}{\varphi}, \theta(k+1)\right) \tag{15}$$

The above equation (15) represents the rth moment of the TIHLEtKw distribution.
The mean and variance of the TIHLEtKw distribution are given as follows.

$$E(X) = 2\lambda\alpha\theta \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} B\left(\frac{\varphi+1}{\varphi}, \theta(k+1)\right)$$

and

$$var(X) = 2\lambda\alpha\theta \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} B\left(\frac{\varphi+2}{\varphi}, \theta(k+1)\right) - \left[2\lambda\alpha\theta \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} B\left(1 + \frac{r}{\varphi}, \theta(k+1)\right)\right]^2$$

1.6 Moment generating function (mgf) of TIHLEtKw Distribution

The Moment Generating Function is given as:

$$M_x(t) = \int_0^1 e^{tx} f(x) dx \tag{16}$$

The mgf for the TIHLEtKw distribution is derived by substituting equation (8) into equation (16), resulting in the following expression.

$$M_x(t) = 2\lambda\alpha\varphi\theta x^{\varphi-1} \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} \int_0^1 e^{tx} \left[1-x^{\varphi}\right]^{\theta(k+1)-1} dx \tag{17}$$

By expanding $e^{tx} = \sum_{b=0}^{\infty} \frac{t^b x^b}{b!}$ and following the process for deriving moments as outlined above, we obtain the mgf for the TIHLEtKw distribution, which is presented in equation (18) below.

$$M_x(t) = 2\lambda\alpha\theta \frac{t^b}{b!} \sum_{i,j,k=0}^{\infty} \sum_{b=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k} B\left(1+\frac{b}{\varphi}, \theta(k+1)\right) \tag{18}$$

1.7 Reliability Function of TIHLEtKw Distribution

The reliability function, also referred to as the survival function, represents the probability that an item will not fail before a specified time. It is defined as:

$$R(x;\lambda,\alpha,\varphi,\theta) = 1 - F(x;\lambda,\alpha,\varphi,\theta) \tag{19}$$

$$R(x;\lambda,\alpha,\varphi,\theta) = \frac{2\left[1-\left[1-\left[1-x^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-\left[1-x^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}} \tag{20}$$

1.8 Hazard Function of TIHLEtKw Distribution

The hazard function describes how the risk of an event evolves over time. It represents the failure rate or the instantaneous rate of occurrence of the event at a specific time, given that the event has not occurred up to that time. It is defined as:

$$T(x;\lambda,\alpha,\varphi,\theta) = \frac{f(x;\lambda,\alpha,\varphi,\theta)}{R(x;\lambda,\alpha,\varphi,\theta)} \tag{21}$$

$$T(x;\lambda,\alpha,\varphi,\theta)=\frac{\lambda\alpha\varphi\theta x^{\varphi-1}[1-x^{\varphi}]^{\theta-1}\left[1-[1-x^{\varphi}]^{\theta}\right]^{\alpha-1}}{\left[1+\left[1-[[1-[1-x^{\varphi}]^{\theta}]^{\alpha}]\right]\right]\left[1-[[1-[1-x^{\varphi}]^{\theta}]^{\alpha}]\right]} \quad (22)$$

1.9 Quantile Function of TIHLEtKw Distribution

The quantile function is an essential tool for generating random variables from any continuous probability distribution, making it highly significant in probability theory. For a given x, the quantile function is defined as F(x) = u, where u follows a U (0, 1) distribution. The quantile function, also known as the inverse cumulative distribution function (cdf), for the TIHLEtKw distribution, is derived using the CDF provided in equation (5).

$$x=Q(u)=\left[1-\left[1-\left[1-\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{\theta}}\right]^{\frac{1}{\varphi}} \quad (23)$$

The median of the TIHLEtKw distribution can be obtained by substituting u = 0.5 into equation (23), as shown below.

$$\text{median}=Q(0.5)=\left[1-\left[1-\left[1-\left[\frac{1-0.5}{0.5+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{\theta}}\right]^{\frac{1}{\varphi}} \quad (24)$$

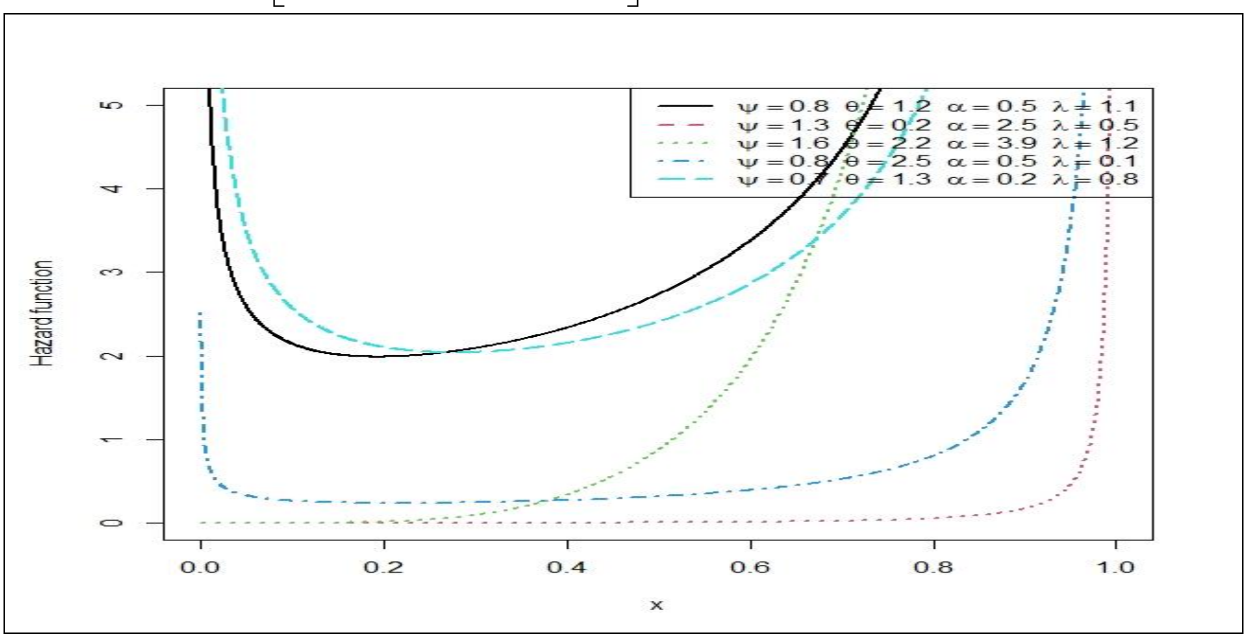

Fig. 2: Plot of reliability function of the TIHLEtKw distribution for different valves of parameters

1.10 Distribution of Order Statistics of TIHLEtKw Distribution

Let X_1, $X_{2,...}$, X_3 be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function F(x). Let $X_{1:n}<X_{2:n}<...<X_{n:n}$ the corresponding ordered random sample from the TIHLEtKw distributions. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, r=1,2,3,...,n denote the CDF and PDF of the rth order statistics $X_{r:n}$ respectively. The PDF of the rth order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x;\lambda,\alpha,\varphi,\theta)=\frac{f(x)}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}\left[F(x)\right]^{v+r-1} \tag{25}$$

The PDF of the rth order statistic for the TIHLEtKw distribution is derived by substituting equations (8) and (9) into equation (25). Additionally, by replacing h with v + r - 1 in equation (9), we obtain the following expression.

$$f_{r:n}(x;\lambda,\alpha,\varphi,\theta)=\frac{2\lambda\alpha\varphi\theta x^{\varphi-1}}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i,j,k,q,z=0}^{\infty}\sum_{p,m=0}^{v+r-1}(-1)^{i+j+k+q+z+p+m+v}\binom{n-r}{v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}$$
$$\binom{\alpha(j+1)-1}{k}\binom{\lambda(p+m)}{q}\binom{\alpha q}{z}\binom{v+r-1}{m}\binom{v+r+p-2}{p}\left[1-x^{\varphi}\right]^{\theta(k+z+1)-1} \tag{26}$$

The PDF of the minimum order statistic for the TIHLEtKw distribution is obtained by setting r = 1 in equation (26), resulting in the following expression.

$$f_{1:n}(x;\lambda,\alpha,\varphi,\theta)=2n\lambda\alpha\varphi\theta x^{\varphi-1}\sum_{v=0}^{n-1}\sum_{i,j,k,q,z=0}^{\infty}\sum_{p,m=0}^{v}(-1)^{i+j+k+q+z+p+m+v}\binom{n-1}{v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}$$
$$\binom{\alpha(j+1)-1}{k}\binom{\lambda(p+m)}{q}\binom{\alpha q}{z}\binom{v}{m}\binom{v+p-1}{p}\left[1-x^{\varphi}\right]^{\theta(k+z+1)-1} \tag{27}$$

Similarly, the PDF of the maximum order statistic for the TIHLEtKw distribution is derived by setting r = n in equation (26), resulting in the following expression.

$$f_{n:n}(x;\lambda,\alpha,\varphi,\theta)=2n\lambda\alpha\varphi\theta x^{\varphi-1}\sum_{i,j,k,q,z=0}^{\infty}\sum_{p,m=0}^{v+n-1}(-1)^{i+j+k+q+z+p+m+v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}$$
$$\binom{\alpha(j+1)-1}{k}\binom{\lambda(p+m)}{q}\binom{\alpha q}{z}\binom{v+n-1}{m}\binom{v+n+p-2}{p}\left[1-x^{\varphi}\right]^{\theta(k+z+1)-1} \tag{28}$$

2.0 Maximum Likelihood Estimation of TIHLEtKw Distribution

The maximum likelihood estimation method is used in estimating the parameters of the new model. Let $x_1, x_2, x_3,..., x_n$ be a random sample of size n from the TIHLEtKw distribution. Then, the likelihood function based on the observed sample for the parameter vector $\left(\lambda,\alpha,\varphi,\theta\right)^T$ is given by:

$$\log(L) = n\log(2) + n\log(\lambda) + n\log(\alpha) + n\log(\varphi) + n\log(\theta) + (\varphi - 1)\sum_{i=1}^{n} \log(x_i)$$
$$+(\theta-1)\sum_{i=1}^{n} \log\left[1-x_i^{\varphi}\right] + (\alpha-1)\sum_{i=1}^{n} \log\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right] \quad (29)$$
$$+(\lambda-1)\sum_{i=1}^{n} \log\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right] - 2\sum_{i=1}^{n} \log\left[1+\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}\right]$$

Differentiating the log-likelihood function with respect to $\lambda, \alpha, \varphi, \theta$ and setting the result to zero, we obtain:

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} log\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right] - 2\sum_{i=1}^{n} \frac{\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda} log\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]}{\left[1+\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}\right]} \quad (30)$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right] - (\lambda-1)\sum_{i=1}^{n} \frac{\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha} log\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]}{\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]}$$
$$+2\sum_{i=1}^{n} \frac{\lambda\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda-1}\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha} log\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]}{\left[1+\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}\right]} \quad (31)$$

$$\frac{\partial L}{\partial \varphi} = \frac{n}{\varphi} + \sum_{i=1}^{n} log(x_i) - +(\theta-1)\sum_{i=1}^{n} \frac{x_i^{\varphi} log\, x_i}{\left[1-x_i^{\varphi}\right]} + (\alpha-1)\sum_{i=1}^{n} \frac{\theta\left[1-x_i^{\varphi}\right]^{\theta-1} x_i^{\varphi} log\, x_i}{\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]}$$
$$-(\lambda-1)\sum_{i=1}^{n} \frac{\alpha\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha-1} \theta\left[1-x_i^{\varphi}\right]^{\theta-1} x_i^{\varphi} log\, x_i}{\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]}$$
$$+2\sum_{i=1}^{n} \frac{\lambda\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda-1} \alpha\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha-1} \theta\left[1-x_i^{\varphi}\right]^{\theta-1} x_i^{\varphi} log\, x_i}{\left[1+\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}\right]} \quad (32)$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} log\left[1-x_i^{\varphi}\right] - (\alpha-1)\sum_{i=1}^{n} \frac{\left[1-x_i^{\varphi}\right]^{\theta} log\left[1-x_i^{\varphi}\right]}{\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]}$$
$$+(\lambda-1)\sum_{i=1}^{n} \frac{\alpha\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha-1}\left[1-x_i^{\varphi}\right]^{\theta} log\left[1-x_i^{\varphi}\right]}{\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]}$$
$$-2\sum_{i=1}^{n} \frac{\lambda\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda-1} \alpha\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha-1}\left[1-x_i^{\varphi}\right]^{\theta} log\left[1-x_i^{\varphi}\right]}{\left[1+\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\theta}\right]^{\alpha}\right]^{\lambda}\right]}$$

(33)

The equations (30), (31), (32), and (33) above are nonlinear and cannot be solved analytically, which requires the use of numerical methods such as BFGS.

3.0 Simulation Study of TIHLEtKw Distribution

This section evaluates the reliability of the parameter estimates for the proposed TIHLEtKw distribution through a comprehensive simulation study. Using Monte Carlo simulations, we assessed the bias and root mean square error (RMSE) of the maximum likelihood estimates (MLEs) for the parameters. Simulated datasets were generated across various sample sizes (n = 20, 50, 100, 250, 500, and 1,000) using the log-likelihood function provided in Equation (29) and the quantile function outlined in Equation (23). Each sample was replicated 1000 times, with the simulations based on fixed parameter values of (l =1.5, a =0.6, q=2.1, j = 3 and l =2, a =0.8, q=1.5, j = 2.5), for the TIHLEtKw. This method facilitated a thorough assessment of the estimation process's consistency and the robustness of the parameter estimates across different sample sizes.

Table 1 reveals that as the sample size grows, the bias and RMSE values decrease, approaching zero. This trend indicates that the estimates become more precise, converging toward the true values, thereby demonstrating their efficiency and reliability.

Table 1: MLEs, biases and RMSE for some values of parameters

		(1.5,0.5,2.1,3)			**(2,0.8,1.5,2.5)**		
n	**Parameters**	**Estimated Values**	**Bais**	**RMSE**	**Estimated Values**	**Bais**	RMSE
	λ	1.6497	0.1497	0.5163	2.0649	0.0649	0.4464
20	α	0.7252	0.1252	0.3922	0.9431	0.1431	0.4779
	θ	2.2975	0.1975	0.6252	1.7722	0.2722	0.7552
	φ	3.0410	0.0410	0.5704	2.5835	0.0835	0.4989
	λ	1.5661	0.0661	0.3472	2.0475	0.0475	0.3172
50	α	0.6342	0.0342	0.2047	0.8447	0.0447	0.2467
	θ	2.2066	0.1066	0.5028	1.5917	0.0917	0.4242
	φ	3.0704	0.0704	0.4408	2.5516	0.0516	0.3535
100	λ	1.5494	0.0494	0.2567	2.0469	0.0469	0.2356
	α	0.6084	0.0084	0.1347	0.8123	0.0123	0.1476
	θ	2.1428	0.0428	0.3814	1.5324	0.0324	0.2995
	φ	3.0824	0.0824	0.3348	2.5538	0.0538	0.2684
250	λ	1.5075	0.0075	0.1641	2.0332	0.0332	0.1419
	α	0.5913	-0.0087	0.0685	0.7909	-0.0091	0.0739
	θ	2.1459	0.0459	0.2734	1.4981	-0.0019	0.1396
	φ	3.0754	0.0754	0.2294	2.5567	0.0567	0.1656
500	λ	1.4995	-0.0005	0.1067	2.0210	0.0210	0.0988
	α	0.5895	-0.0105	0.0444	0.7893	-0.0107	0.0461

	θ	2.1293	0.0293	0.1951	1.4936	-0.0064	0.0848
	φ	3.0565	0.0565	0.1616	2.5412	0.0412	0.1144
1000	λ	1.5029	0.0029	0.0786	2.0151	0.0151	0.0673
	α	0.5926	-0.0074	0.0305	0.7924	-0.0076	0.0298
	θ	2.1126	0.0126	0.1373	1.4938	-0.0062	0.0581
	φ	3.0351	0.0351	0.1160	2.5245	0.0245	0.0811

4.0 Application of TIHLEtKw to Real-life Datasets

This section clearly demonstrates the versatility and flexibility of the Type I Half-Logistic Exponentiated Kumaraswamy (TIHLEtKw) distribution by applying it to two real-world datasets. Its performance is then compared with that of the Kumaraswamy-Kumaraswamy (Kw-Kw) Distribution by El-Sherpieny *et al.*, (2014), the Weibull-Kumaraswamy (Wkw) Distribution by Aminu *et al.*, (2018), the Type II Half Logistic Kumaraswamy (TIIHLKw) Distribution by ZeinEldin *et al.*, (2020), Exponentiated Kumaraswamy (EKw) distribution by Lemonte *et al.*, (2013), and the Kumaraswamy (Kw) Distribution by Kumaraswamy, (1980).

Kumaraswamy-Kumaraswamy (Kw-Kw) distribution (El-Sherpieny *et al.*, 2014).

$$f(x;\lambda,\beta,\alpha,\theta) = \lambda\beta\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\left(1-(1-x^{\alpha})^{\theta}\right)^{\lambda-1}\left(1-\left(1-(1-x^{\alpha})^{\theta}\right)^{\lambda}\right)^{\beta-1}$$

Weibull-Kumaraswamy (Wkw) distribution (Aminu *et al.*, 2018)

$$f(x;\lambda,\beta,\alpha,\theta) = \lambda\theta\alpha\beta\frac{x^{\lambda-1}\left(1-x^{\lambda}\right)^{\theta-1}}{[1-(1-x^{\lambda})^{\theta}]}\left[-log\left(1-\left(1-x^{\lambda}\right)^{\theta}\right)\right]^{\beta-1}exp\left(-\alpha\left(-log\left(1-\left(1-x^{\lambda}\right)^{\theta}\right)\right)^{\beta}\right)$$

Type II Half Logistic Kumaraswamy (TIIHLKw) distribution (ZeinEldin *et al.*, 2020)

$$f(x;\lambda,\beta,\alpha) = \frac{2\lambda\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{\lambda-1}}{(1+(1-(1-x^{\alpha})^{\beta})^{\lambda})^{2}}$$

Exponentiated Kumaraswamy (EKw) distribution (Lemonte *et al.*, 2013)

$$f(x;\beta,\alpha,\theta) = \alpha\beta\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)^{\theta-1}$$

Kumaraswamy (Kw) distribution (Kumaraswamy, 1980)

$$f(x;\alpha,\theta) = \alpha\theta x^{\alpha-1}\left[1-x^{\alpha}\right]^{\theta-1}$$

The two datasets used to demonstrate the application provide practical evidence of the new distribution's adaptability and suitability. It is shown to be the best choice for modelling the datasets, outperforming the other comparator distributions. All computations were performed using the R programming language.

Dataset 1

The first dataset below consists of civil engineering data with 85 hailing times, previously referenced by Kotz and Dorp (2004).

4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80.

Table 2: The MLEs, Log-likelihoods and Goodness of Fits Statistics of the models based on Dataset 1

Model	l	α	j	θ	LL	AIC
TIHLEtKw	1.1057	0.1060	2.4591	0.1018	- 77.2204	162.4408
Kw-Kw	2.1483	0.1916	0.4786	1.4626	- 101.8063	211.6126
Wkw	0.8070	3.7808	4.1275	1.6237	-179.3222	366.6444
TIIHLKw	1.4238	0.3454	4.0545	-	-261.5863	529.1726
EKw	-	4.5136	0.5969	0.1461	-165.4639	336.9278
Kw	-	0.8260	0.2566	-	-131.9002	267.8003

Table 2 displays the Maximum Likelihood Estimation results for the parameters of the TIHLEtKw distribution, along with five other comparator distributions. Among them, the TIHLEtKw distribution demonstrated the lowest AIC value of 162.4408, indicating it provides the best fit for the hailing times data compared to the other distributions analyzed. Additionally, visual assessments of the empirical and theoretical PDFs, CDFs, as well as the Q-Q and P-P plots depicted in Fig. 3, further confirm the suitability and adaptability of the new distribution for the analyzed dataset.

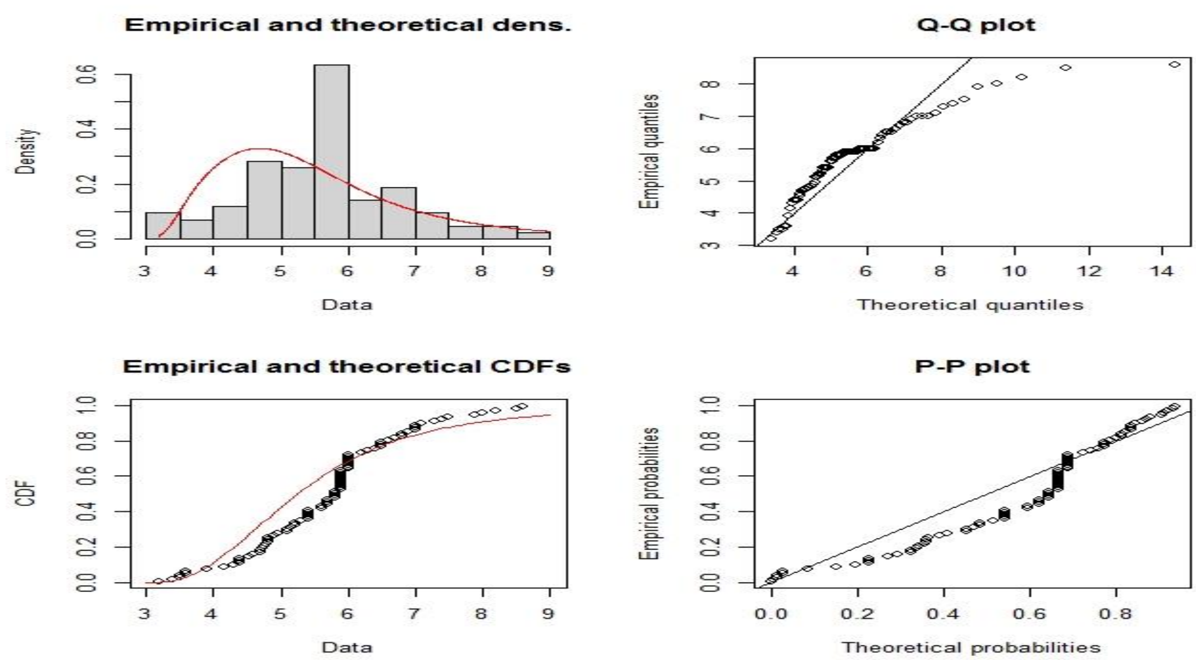

Fig. 3: Empirical and theoretical pdfs and cdfs, Q-Q and P-P plots for dataset 1

Dataset 2

The second dataset below contains the failure and service times for an aircraft windshield, previously utilized by Kundu and Raqab (2009).

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483,1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

Table 3 presents the Maximum Likelihood Estimation results for the parameters of the TIHLEtKw distribution, along with five comparator distributions. Based on the AIC goodness-of-fit statistic, the TIHLEtKw distribution achieved the lowest AIC value of 47.5339, indicating it provides the best fit for the aircraft windshield data. .

Table 3: The MLEs, Log-likelihoods and Goodness of Fits Statistics of the models based on Dataset 2

Model	l	α	j	θ	LL	AIC
TIHLEtKw	1.1531	0.3462	0.3364	0.0114	- 19.7669	47.5339
Kw-Kw	0.3552	1.0734	0.0537	3.2463	- 43.5369	95.0738
Wkw	0.0534	2.3461	0.3648	3.4772	-242.2716	492.5433
TIIHLKw	2.0254	0.2731	0.0732	-	-84.06965	174.1393
EKw	-	2.3721	0.2472	0.3520	-57.8619	121.7237
Kw	-	1.2430	0.5148	-	-246.4215	496.843

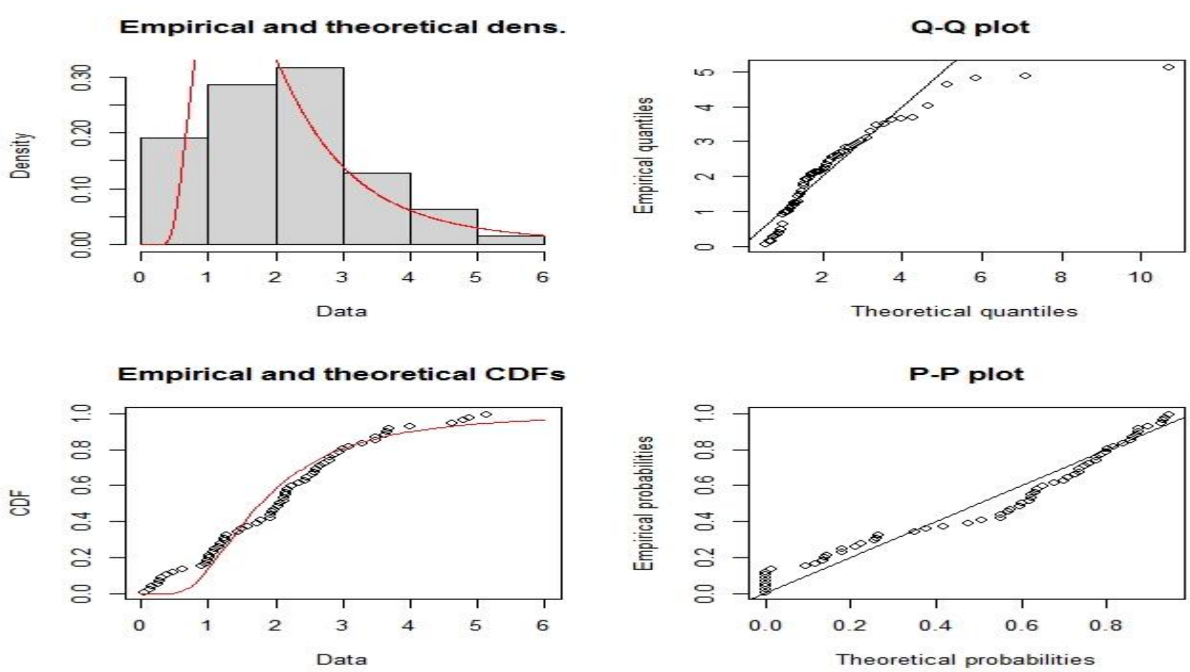

Fig. 4: Empirical and theoretical pdfs and cdfs, Q-Q and P-P plots for dataset 2

In addition, visual evaluations of the empirical and theoretical PDFs and CDFs, along with the Q-Q and P-P plots presented in Fig. 4, provide further evidence of the proposed distribution's superior performance. As a result, the newly introduced distribution proves to be the best fit for the aircraft windshields failure dataset compared to the other distributions analyzed

5.0 Conclusion\

The study provides a comprehensive exploration of the Type I Half-Logistic Exponentiated Kumaraswamy (TIHLEtKw) distribution, introducing its fundamental properties, including its moment generating function, reliability function, hazard function, quantile function, and order statistics. The derivations and mathematical formulations underscore the versatility and robustness of this new distribution in modelling diverse data types. A detailed simulation study confirmed the reliability and precision of the maximum likelihood estimators (MLEs), demonstrating improved accuracy as sample sizes increased. The application of the TIHLEtKw distribution to real-world datasets, particularly civil engineering and wind speed data, further validated its flexibility and superior performance compared to existing distributions, such as the Kumaraswamy-Kumaraswamy and Weibull-Kumaraswamy distributions. The results of this work highlight the potential of the TIHLEtKw distribution to address complex statistical modelling challenges, particularly in scenarios where traditional models fail to capture the underlying data dynamics effectively. The study also emphasizes the practicality of the new model, as demonstrated through its successful implementation using the R programming language. The findings indicate that the TIHLEtKw distribution is a valuable addition to statistical modelling, capable of providing accurate and reliable fits for various datasets. It is recommended that future research explore extensions of this distribution to other domains, including environmental, biomedical, and financial applications. Moreover, developing more advanced estimation techniques and computational algorithms could further enhance its applicability and efficiency in handling large and complex dataset.

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Ethical Approval

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IAS, was involved in the writing of the manuscript, ,ASM in proofreading and Hgd in conceptualization and data curation.