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On Some Properties and Applications of the Exponentiated Type II Generalised Topp-Leone Inverse Exponential Distribution

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Abstract: *The literature calls for new distribution models with greater flexibility in distribution theory. As a result, a new four-parameter lifetime distribution, the Exponentiated Type II Generalized Topp-Leone Inverse Exponential distribution, is introduced. An expansion for the probability distribution functions and cumulative density function was carried out, which was used to derive some mathematical and statistical properties of the distribution, such as the moments, moment generating function, quantile function, survival function, hazard function, and probability weighted Moment. The estimation of the parameters by the maximum likelihood method was discussed. Its potential was illustrated by applying it to two real-life datasets to demonstrate its fit and flexibility relative to some lifetime distributions in the literature. The results showed that the new distribution developed fits the two datasets used better than the comparators.*

Keywords: *Inverse, Exponentiated, Type II Topp-Leone, PWM, ET$_2$GTLIEx*

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1.0 Introduction

Distribution theory plays a fundamental role in modern statistical analysis. Selecting the right statistical model for a phenomenon, application, or data validity can be somewhat challenging. In many fields of pure science and social sciences, conventional statistical distributions have been widely used to model data and draw inferences. . The emergence of large and complex datasets has necessitated the development of more flexible and robust distributions capable of capturing intricate data structures. Probability distributions are used in biological statistics to model the uncertainty surrounding measurements, clinical trials, and other research investigations (Kolawole *et al*., 2023).

Although the exponential distribution is widely applied in Poisson processes and reliability engineering, it has many appealing qualities; however, its constant failure rate is a drawback because it makes the distribution inappropriate for simulating real-world scenarios with bathtub and inverted bathtub failure rates (Lemonte, 2013). Furthermore, in real-world applications, memorylessness—another characteristic of the exponential distribution—is rarely achievable. To address these limitations of the exponential distribution, Keller & Kamath (1982) proposed a modified version of the exponential distribution, which resulted in the inverse exponential distribution. A key advantage of this distribution is its ability to model non-constant failure rates. This distribution has been studied by various researchers, among them Mushtaq et al. (2025), who introduced the Exponentiated Inverse Exponential Distribution (EIED), a novel probability model developed within the power inverse exponential distribution framework. They derived comprehensive statistical properties of the distribution, including the reliability and hazard functions, moments, characteristic and quantile functions, moment generating function, mean deviations, Lorenz and Bonferroni curves, and various entropy measures. Additionally, they compared the model with two real datasets—medical and flood datasets—demonstrating that EIED offers a better fit compared to competing models. Recent works in this area have focused on extending existing probability distributions to improve their modeling versatility. Thus, the inverse exponential distribution has been extended using various families of distributions in several works, such as those by Mushtaq et al. (2025), Ismail et al. (2022a). However, despite these developments, many existing models still lack sufficient flexibility to adequately capture diverse hazard rate behaviors and complex data patterns observed in real-life applications.

To further enhance the modeling flexibility of classical distributions, several authors have proposed new families of distributions. , . Notable examples include the Exponentiated Type II Generalized Topp-Leone-G family (Kolawole et al., 2024a), the Kumaraswamy Type II Generalized Topp-Leone-G family (Kolawole et al., 2024b), a new generalized exponentiated family of continuous distributions (Sule *et al*., 2025), and the Type I Half-Logistic Exponentiated-G family of distributions (Bello et al., 2021), among others. In this context, we propose a generalization of the inverse exponential distribution based on Kolawole et al. (2024a), which introduced the Exponentiated Type II Generalized Topp-Leone-G (ET$_2$GTL-G) family of distributions with cumulative distribution function (cdf) and probability density function (pdf) given respectively as equation 1

;

$$F_{\mathrm{ET_2GTL-G}}(t,\beta,\alpha,\theta;\xi)=\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta} \tag{1}$$

And its pdf is derived as

$$f_{\mathrm{ET_2GTL-G}}(t,\beta,\alpha,\theta,\xi)=2\alpha\beta\theta h(t;\xi)H^{2\beta-1}\left(t;\xi\right)\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha-1}\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta-1} \tag{2}$$

where $H\left(t;\xi\right)$ is the cdf of the baseline distribution with parameter vector ξ.

Let us consider the Inverse Exponential distribution, which is the baseline distribution with a scale parameter σ with cumulative distribution and probability density functions given respectively by equations 3 and 4

$$F_{IEx}(t;\sigma)=e^{-\left(\frac{\sigma}{t}\right)} \tag{3}$$

$$F_{IEx}\left(t;\sigma\right)=\frac{\sigma}{t^{2}}e^{-\left(\frac{\sigma}{t}\right)} \tag{4}$$

$t>0,\theta>0$. This study proposes a new four-parameter continuous distribution that generalizes the inverse exponential distribution using the family of distributions derived by Kolawole et al. (2024a). This is intended to improve the flexibility of the baseline distribution, enabling it to better model lifetime data arising in fields such as reliability engineering, biomedical studies, and environmental sciences.

2.0 Methodology

2.1 The Exponentiated Type II Generalized Topp-Leone Inverse Exponential (ET2GTLIEx) Distribution

The cumulative distribution function (cdf) and probability density function (pdf) of the proposed four-parameter ET$_2$GTLIEx distribution are obtained by substituting equation (3) into equation (1). These are given respectively as equations 5 and 6

$$F_{ET_2GTLIEx}(t;\beta,\alpha,\theta,\sigma)=\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta} \tag{5}$$

$$f_{ET_2GTLIEx}(t;\beta,\alpha,\theta,\sigma)=2\beta\alpha\theta\frac{\sigma}{t^2}e^{-\left(\frac{\sigma}{t}\right)}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta-1}\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha-1}\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta-1} \tag{6}$$

where x >0 and α, β ,λ, θ > 0 are shape and scale parameters of the distribution.

The plots of the pdf illustrating the behavior of the distribution for different parameter values are presented in Fig. 1 below.

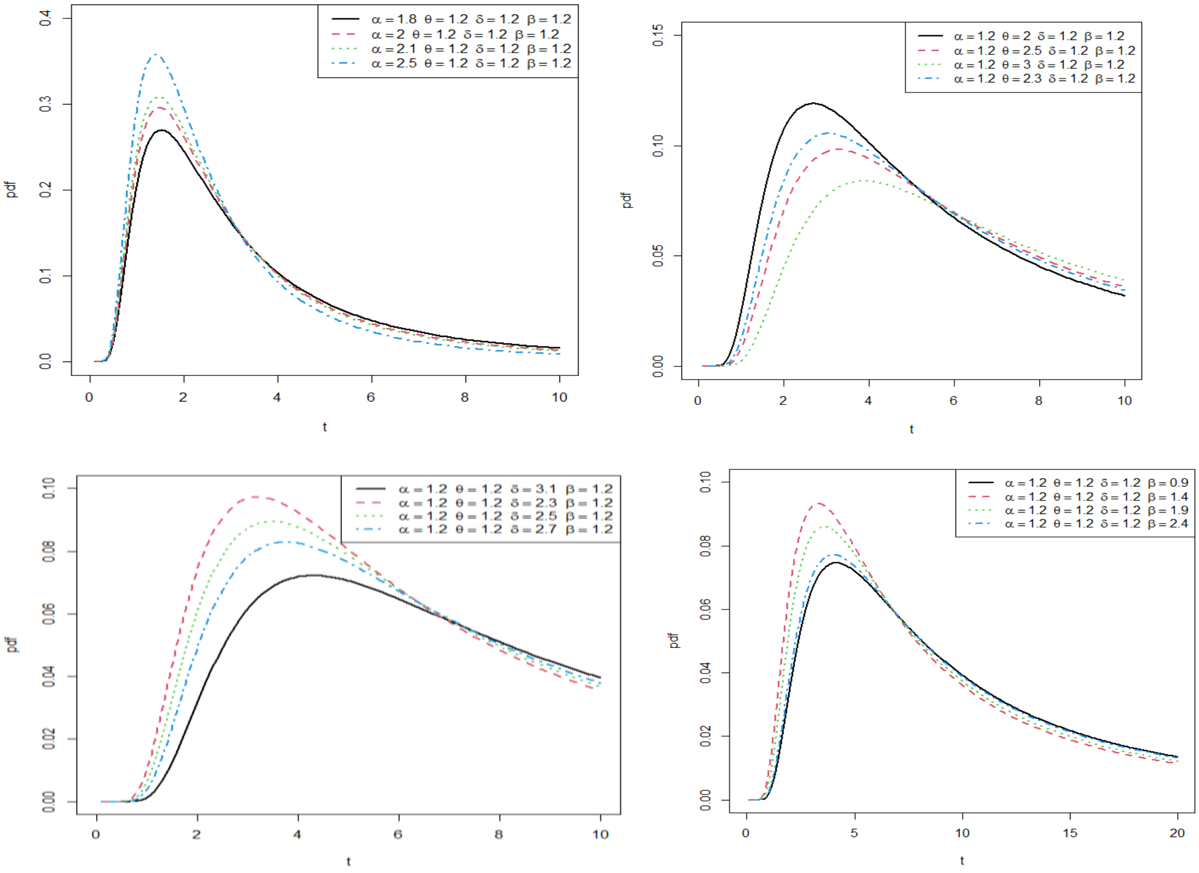

Fig. 1: Probability density function of the ET2GTLIEx distribution for selected parameter values

2.2 Expansion of the Probability Density Function of the ET2GTLIEx Distribution

Using the binomial expansion given by equation 7

$$(1-t)^{\theta-1}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma\theta}{i!\Gamma(\theta-i)}t^{i} \tag{7}$$

Considering the expansion term in equation (7), we obtain:

$$f_{ET_2GTLIEx}(t;\beta,\alpha,\theta,\sigma)=2\beta\alpha\theta\frac{\sigma}{t^2}e^{-\left(\frac{\sigma}{t}\right)}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta-1}\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha-1}\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta-1}$$

Substituting into the pdf expression and simplifying yields:

$$\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta-1}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma\theta}{i!\Gamma(\theta-i)}\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha i} \tag{8}$$

$$\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha(i+1)-1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma\alpha(i+1)}{j!\Gamma\left(\alpha(i+1)-j\right)}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta_j} \tag{9}$$

$$\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta(j+1)-1}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma2\beta(j+1)}{k!\Gamma\left(2\beta(j+1)-k\right)}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{k} \tag{10}$$

$$f(t;\alpha,\beta,\theta,\sigma)=2\beta\alpha\theta\frac{\sigma}{t^2}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma\theta\Gamma\alpha(i+1)\Gamma2\beta(j+1)}{i!j!k!\Gamma(\theta-i)\Gamma\left(\alpha(i+1)-j\right)\Gamma\left(2\beta(j+1)-k\right)}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{k+1} \tag{11}$$

Hence, the pdf can be expressed in the expanded form as:

$$f(t;\alpha,\beta,\theta,\sigma)=\sum_{i,j,k=0}^{\infty}\tau_q\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{k+1} \tag{12}$$

where i=0,1,2,…, and the coefficients depend on the binomial expansion parameters.

$$\tau_q=2\beta\alpha\theta\frac{\sigma}{t^2}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma\theta\Gamma\alpha(i+1)\Gamma2\beta(j+1)}{i!j!k!\Gamma(\theta-i)\Gamma\left(\alpha(i+1)-j\right)\Gamma\left(2\beta(j+1)-k\right)}$$

Replace with:

Similarly, the cumulative distribution function (cdf) can be expanded using the binomial series ($\left[F_{ET_2GTLIEx}(t;\beta,\alpha,\theta,\sigma)\right]^{h}$ where h is an integer). For an integer k, this leads to:

$$\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta h}=\sum_{m=0}^{\infty}(-1)^{m}\binom{\theta h}{m}\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha m}$$

$$\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha m}=\sum_{n=0}^{\infty}(-1)^{n}\binom{\alpha m}{n}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta n}$$

$$\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta n}=\sum_{p=0}^{\infty}(-1)^{p}\binom{2\beta n}{p}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p}$$

Therefore, the expanded form of the cdf is given by:

$$\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta h}=\sum_{m,n,p=0}^{\infty}(-1)^{m+n+p}\binom{\theta h}{m}\binom{\alpha m}{n}\binom{2\beta n}{p}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p}$$

$$\left[F_{ET_2GTLIEx}(t;\beta,\alpha,\theta,\delta\right]^{h}=\sum_{m,n,p}^{\infty}\mathrm{H}_q\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p} \tag{13}$$

Where:

$$\mathrm{H}_q=(-1)^{m+n+p}\binom{\theta h}{m}\binom{\alpha m}{n}\binom{2\beta n}{p}$$

3.0 Mathematical Properties of the ET2GTLIEx Distribution

This section presents some important statistical and mathematical properties of the ET2GTLIEx distribution.

3.1. Probability Weighted Moments (PWM)

$$\tau r,s=E\left[T^{r}F(t)^{s}\right]=\int_{-\infty}^{\infty}T^{r}f(t)\left(F(t)\right)^{s}\,dt \tag{14}$$

The PWMs are derived by substituting equation (12) and equation (13) into equation (14) replacing x^r with the required moment term, we obtain:

$$\tau r,s=2\beta\alpha\theta\frac{\sigma}{t^{2}}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma\theta\Gamma\alpha(i+1)\Gamma2\beta(j+1)}{i!j!k!\Gamma(\theta-i)\Gamma(\alpha(i+1)-j)\Gamma(2\beta(j+1)-k)}\sum_{m,n,p=0}^{s}(-1)^{m+n+p}\binom{\theta h}{m}\binom{\alpha m}{n}\binom{2\beta n}{p}$$

$$\int_{-\infty}^{\infty}T^{r}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p+2}dt \tag{15}$$

Consider the integral component of equation (15), which can be expressed as:

$$\int_{0}^{\infty}T^{r}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p+k+1}dt$$

Let $v=(p+k+1)\frac{\sigma}{t}\Rightarrow t=\frac{(p+k+1)\sigma}{v};dv=\frac{-dvt^{2}}{(p+k+1)\sigma}$

then,

$$\int_{0}^{\infty}\left[\frac{(p+k+1)\sigma}{v}\right]^{r}e^{-v}\frac{dvt^{2}}{(p+k+1)\sigma}$$

$$\frac{\left[\left(p+k+1\right)\sigma\right]^{r}t^{2}}{\left(p+k+1\right)\sigma}\int_{0}^{\infty}v^{-r}e^{-v}dv \qquad \Rightarrow\int_{0}^{\infty}v^{-r}e^{-v}dv=\Gamma\left(1-r\right)$$

Therefore, the integral simplifies to:

$$\tau_{r,s}=2\beta\alpha\theta\frac{\sigma}{t^{2}}\sum_{m,n,p=0}^{s}\sum_{i,j,k=0}^{\infty}\frac{\left(-1\right)^{i+j+k}\left(-1\right)^{m+n+p}\Gamma\theta\Gamma\left(\alpha\left(i+1\right)\right)\Gamma\left(2\beta\left(j+1\right)\right)}{i!j!k!\Gamma\left(\theta-i\right)\Gamma\left(\alpha\left(i+1\right)-j\right)\Gamma\left(2\beta\left(j+1\right)-k\right)}\binom{\theta h}{m}\binom{\alpha m}{n}\binom{2\beta n}{p}$$

$$\frac{\left[\left(p+k+1\right)\sigma\right]^{r}t^{2}\Gamma(1-r)}{\left(p+k+1\right)\sigma}$$

$$\tau_{r,s}=2\beta\alpha\theta\sigma^{r}\sum_{m,n,p=0}^{s}\sum_{i,j,k=0}^{\infty}\frac{\left(-1\right)^{i+j+k}\left(-1\right)^{m+n+p}\Gamma\theta\Gamma\left(\alpha\left(i+1\right)\right)\Gamma\left(2\beta\left(j+1\right)\right)}{i!j!k!\Gamma\left(\theta-i\right)\Gamma\left(\alpha\left(i+1\right)-j\right)\Gamma\left(2\beta\left(j+1\right)-k\right)}\binom{\theta h}{m}\binom{\alpha m}{n}\binom{2\beta n}{p}$$

$$\left[\left(p+k+1\right)\right]^{r-1}\Gamma(1-r) \tag{16}$$

3.2. *Moments*

Moments are essential in statistical analysis, particularly in practical applications. Therefore, we derive the r^{th} moments for ET$_2$GTLIEx distribution as equation 17

$$E\left(T^{r}\right)=\int_{0}^{\infty}t^{r}f\left(t\right)dt \tag{17}$$

$$E\left(T^{r}\right)=2\beta\alpha\theta\frac{\sigma}{t^{2}}\sum_{i,j,k=0}^{\infty}\frac{\left(-1\right)^{i+j+k}\Gamma\theta\Gamma\left(\alpha\left(i+1\right)\right)\Gamma\left(2\beta\left(j+1\right)\right)}{i!j!k!\Gamma\left(\theta-i\right)\Gamma\left(\alpha\left(i+1\right)-j\right)\Gamma\left(2\beta\left(j+1\right)-k\right)}\int_{0}^{\infty}t^{r}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{k+1}dt \tag{18}$$

Consider the integral component of equation (18):

$$\int_{0}^{\infty}T^{r}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{k+1}dt \tag{19}$$

Let $y=\frac{\left(k+1\right)\sigma}{t}\Rightarrow t=\frac{\left(k+1\right)\sigma}{y};dt=\frac{-t^{2}dy}{\left(k+1\right)\sigma}$

Then;

$$\int_{0}^{\infty}\left[\frac{\left(k+1\right)\sigma}{y}\right]^{r}e^{-y}\frac{t^{2}dy}{\left(k+1\right)\sigma} \qquad \Rightarrow\frac{\left[\left(k+1\right)\sigma\right]^{r}t^{2}}{\left(k+1\right)\sigma}\int_{0}^{\infty}y^{-r}e^{-y}dy \qquad \Rightarrow\int_{0}^{\infty}y^{-r}e^{-y}dy=\Gamma\left(1-r\right)$$

Therefore;

$$E\left(T^{r}\right)=2\alpha\beta\theta\sigma^{r}\sum_{i,j,k=0}^{\infty}\frac{\left(-1\right)^{i+j+k}\Gamma\theta\Gamma\left(\alpha\left(i+1\right)\right)\Gamma\left(2\beta\left(j+1\right)\right)\left(k+1\right)\Gamma\left(1-r\right)}{i!j!k!\Gamma\left(\theta-i\right)\Gamma\left(\alpha\left(i+1\right)-j\right)\Gamma\left(2\beta\left(j+1\right)-k\right)} \tag{20}$$

The mean of the ET$_2$GTLIEx distribution is obtained by setting r = 1 in equation (20) yield equation 21

The mean of the ET$_2$GTLIEx distribution is obtained by setting $r=1$ in (20) as

$$E\left(T\right)=2\alpha\beta\theta\sigma\sum_{i,j,k=0}^{\infty}\frac{\left(-1\right)^{i+j+k}\Gamma\theta\Gamma\left(\alpha\left(i+1\right)\right)\Gamma\left(2\beta\left(j+1\right)\right)\left(k+1\right)}{i!j!k!\Gamma\left(\theta-i\right)\Gamma\left(\alpha\left(i+1\right)-j\right)\Gamma\left(2\beta\left(j+1\right)-k\right)}\tag{21}$$

3.3. Moment Generating Function of ET₂GTLIEx Distribution

The moment generating function (MGF) of x is given by equation 22

$$E\left(e^{xt}\right)=\int_{0}^{\infty}e^{xt}f\left(t\right)dt\tag{22}$$

Consider the expansion $e^{xt}=\sum_{m=0}^{\infty}\frac{\left(xt\right)^{m}}{m!}$ then, the MGF of the ET₂GTLIEx distribution follows from the moments as expressed by equation 23

$$E\left(e^{xt}\right)=2\alpha\beta\theta\sigma^{m}\sum_{i,j,k=0}^{\infty}\frac{\left(-1\right)^{i+j+k}\Gamma\theta\Gamma\left(\alpha\left(i+1\right)\right)\Gamma\left(2\beta\left(j+1\right)\right)\left(k+1\right)\Gamma\left(1-m\right)t^{m}}{i!j!k!\Gamma\left(\theta-i\right)\Gamma\left(\alpha\left(i+1\right)-j\right)\Gamma\left(2\beta\left(j+1\right)-k\right)m!}\tag{23}$$

.4 *Reliability Function) The* reliability function, also known as the survival function, represents the probability that a system survives beyond a given time, which can be represented by equation 24

$$R\left(t;\alpha,\beta,\theta,\sigma\right)=1-F\left(t;\alpha,\beta,\theta,\sigma\right)\tag{24}$$

$$R\left(t;\alpha,\beta,\theta,\sigma\right)=1-\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta}\tag{25}$$

3.5. Hazard Rate Function of ET₂GTLIEx Distribution

The hazard rate function (hrf) is defined according to equation 26

$$r\left(t;\alpha,\beta,\theta,\sigma\right)=\frac{f\left(t;\alpha,\beta,\theta,\sigma\right)}{R\left(t;\alpha,\beta,\theta,\sigma\right)}\tag{26}$$

$$r(t;\beta,\alpha,\theta,\sigma)=\frac{2\beta\alpha\theta\frac{\sigma}{t^{2}}e^{-\left(\frac{\sigma}{t}\right)}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta-1}\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha-1}\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta-1}}{1-\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta}}\tag{27}$$

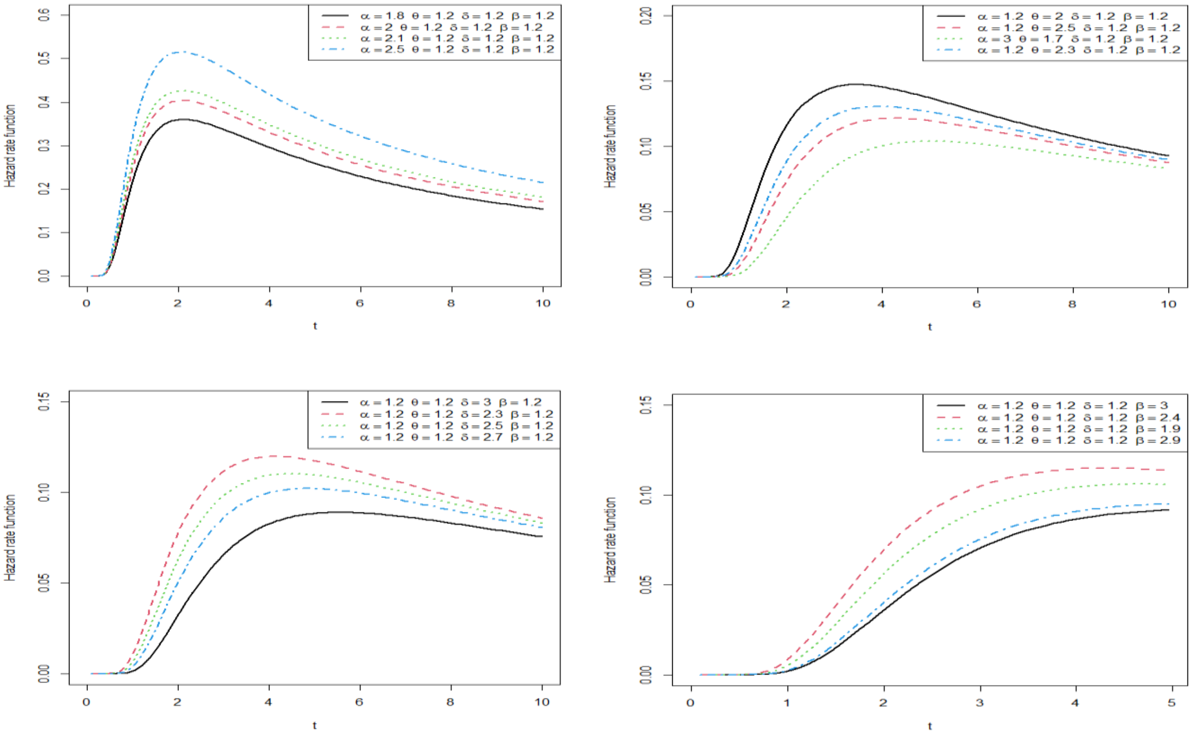

Fig. 2: Hazard rate function of the ET$_2$GTLIEx distribution for selected parameter values

3.6. Quantile Function of ET$_2$GTLIEx Distribution

The quantile function is defined as the inverse of the cdf. Using the cdf ($Q(u) = F^{-1}(u).$) of the ET$_2$GTLIEx distribution in equation (5), we obtain:

$$F(t;\alpha,\beta,\theta,\sigma) = \left[1 - \left[1 - \left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]^{\theta} = U$$

$$U^{\frac{1}{\theta}} = 1 - \left[1 - \left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}$$

$$1 - U^{\frac{1}{\theta}} = \left[1 - \left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}$$

$$\left[1 - U^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}} = 1 - \left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}$$

$$1 - \left[1 - U^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}} = \left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}$$

$$\left[1-\left[1-U^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}} = e^{-\left(\frac{\sigma}{t}\right)}$$

$$-\left(\frac{\sigma}{t}\right) = \log\left[1-\left[1-U^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}}$$

$$t = Q\left(U\right) = \frac{\sigma}{\left[-\log\left[1-\left[1-U^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}}\right]} \tag{28}$$

The median of the ET$_2$GTLIEx distribution can be derived by substituting U=0.5 in (28) as follows;

$$Median = Q\left(0.5\right) = \frac{\sigma}{\left[-\log\left[1-\left[1-U^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}}\right]} \tag{29}$$

4.0 Distribution of Order Statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $T_1, T_2 ... T_n$ be independent and identically distributed $(i.i.d)$ random variables with their corresponding continuous distribution function $F(t)$. Let $T_{1:n} < T_{2:n} < ... < T_{n:n}$ The corresponding ordered random sample from a population of size n. Let $F_{r:n}(t)$ and $f_{r:n}(t), r = 1,2,3...,n$ denote the cdf and pdf of the r^{th} order statistics $T_{r:n}$ respectively. The probability density function of $T_{r:n}$ can be expressed according to equation 30

$$f_{r:n}\left(t;\alpha,\beta,\theta,\sigma\right) = \frac{f\left(t\right)}{B\left(r,n-r+1\right)}\sum_{v=0}^{\infty}(-1)^{v}\binom{n-r}{v}\left[F(t)\right]^{v+r-1} \tag{30}$$

The PDF of r^{th} order statistics for ET$_2$GTLIEx distribution is derived by substituting (12) and (1), into (30). Also replace h with $v+r-1$, we have;

$$f_{r:n}\left(t;\alpha,\beta,\theta,\sigma\right) = \frac{1}{B\left(r,n-r+1\right)}\sum_{v=0}^{n-r}\sum_{i,j,k=0}^{\infty}\sum_{m,n,p=0}^{v+r-1}\left(-1\right)^{v}\binom{n-r}{v}\tau_{q}\mathrm{H}_{q}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p+k+1} \tag{31}$$

Now

$$H_{q} = \left(-1\right)^{m+n+p}\binom{\theta(v+r-1)}{m}\binom{\alpha m}{n}\binom{2\beta n}{p}$$

$$\tau_q = 2\alpha\beta\theta\frac{\sigma}{t^2}\frac{(-1)^{i+j+k}\,\Gamma\theta\Gamma\alpha(i+1)\Gamma2\beta(j+1)}{i!j!k!\Gamma(\theta-i)\Gamma(\alpha(i+1)-j)\Gamma(2\beta(j+1)-k)}$$

Therefore, the pdf of the minimum order statistics of the ET$_2$GTLIEx distribution is obtained by setting $r=1$ in (31) as;

$$f_{r:n}(t;\alpha,\beta,\theta,\sigma) = 2n\beta\alpha\theta\frac{\sigma}{t^2}\sum_{v=0}^{n-1}\sum_{i,j,k=0}^{\infty}\sum_{m,n,p=0}^{v+r-1}(-1)^{v}\binom{n-1}{v}\tau_q \mathrm{H}_q\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p+k+1}$$

$$f_{1:n}(t;\alpha,\beta,\theta,\sigma) = 2n\beta\alpha\theta\frac{\sigma}{t^2}*$$

$$\sum_{v=0}^{n-1}\sum_{j,k,l=0}^{\infty}\sum_{m,n,p=0}^{v}(-1)^{i+j+k}(-1)^{m+n+p}(-1)^{v}\binom{n-1)}{v}\binom{\theta v}{m}\binom{2\beta n}{p}\frac{\Gamma\theta\Gamma\alpha(i+1)\Gamma2\beta(j+1)}{i!j!k!\Gamma(\theta-i)\Gamma(\alpha(i+1)-j)\Gamma(2\beta(j+1)-k)}*$$

$$\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p+k+1} \tag{32}$$

Also, the pdf of the maximum order statistics of the ET$_2$GTLIEx distribution is obtained by setting $r=n$ in (32) as;

$$f_{n:n}(t;\alpha,\beta,\theta,\sigma) = 2n\beta\alpha\theta\frac{\sigma}{t^2}\sum_{j,k,l=0}^{\infty}\sum_{m,n,p=0}^{v+n-1}(-1)^{i+j+k}(-1)^{m+n+p}(-1)^{v}\binom{\theta(v+n-1}{m}\binom{2\beta n}{p}*$$

$$\frac{\Gamma\theta\Gamma\alpha(i+1)\Gamma2\beta(j+1)}{i!j!k!\Gamma(\theta-1)\Gamma(\alpha(i+1)-j)\Gamma(2\beta(j+1)-k)}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{p+k+1} \tag{33}$$

5.0 Theoretical Results

5.1 Parameter Estimation

In this work, the Maximum Likelihood Estimate (MLE) is used to estimate the unknown parameter of the ET$_2$GTLIEx distribution. Let $t_1, t_2 \ldots t_n$ be a random sample of size n from the ET$_2$GTLIEx Distribution. Then, the likelihood function based on the observed sample for the vector η of parameter $(\alpha,\beta,\theta,\sigma)^T$ is given by;

$$LogL = n\log2 + n\log\beta + n\log\alpha + n\log\theta + n\log\sigma -$$

$$\sum_{i=1}^{\infty}\log\left(\frac{1}{t^2}\right)-\sum_{i=1}^{\infty}\left(\frac{\sigma}{t}\right)+(2\beta-1)\sum_{i=1}^{\infty}\log\left[e^{-\left(\frac{\sigma}{t}\right)}\right]+(\alpha-1)\sum_{i=1}^{\infty}\log\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]+(\theta-1)\sum_{i=1}^{\infty}\log\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right] \tag{34}$$

Differentiating the log-likelihood with respect to each parameter ($\alpha,\beta,\theta,\sigma$) and equating the results to zero, we obtain:

$$\frac{\partial(\log L)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{\infty}\log\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]-\left(\theta-1\right)\sum_{i=1}^{\infty}\frac{\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\log\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]}{\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]} \quad (35)$$

$$\frac{\partial(\log L)}{\partial\beta}=\frac{n}{\beta}+$$

$$2\sum_{i=1}^{\infty}\log\left[e^{-\left(\frac{\sigma}{t}\right)}\right]-(\alpha-1)\sum_{i=0}^{\infty}\frac{\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\log\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2}}{\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]}+(\theta-1)\sum_{i=1}^{\infty}\frac{\alpha\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha-1}\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\log\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2}}{\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right]}$$

$$(36)$$

$$\frac{\partial(\log L)}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{\infty}\log\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t}\right)}\right]^{2\beta}\right]^{\alpha}\right] \quad (37)$$

$$\frac{\partial(\log L)}{\partial\sigma}=\frac{n}{\sigma}-\sum_{i=1}^{\infty}\frac{1}{t_i}-(2\beta-1)\sum_{i=1}^{\infty}\left[\frac{\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]\frac{1}{t_i}}{\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]}\right]+\left(\alpha-1\right)\sum_{i=1}^{\infty}\left[\frac{2\beta\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]^{2\beta-1}\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]\frac{1}{t_i}}{\left[1-\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]^{2\beta}\right]}\right]-$$

$$\left(\theta-1\right)\sum_{i=0}^{\infty}\left[\frac{\alpha\left[1-\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]^{2\beta}\right]^{\alpha-1}2\beta\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]^{2\beta-1}\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]\frac{1}{t_i}}{\left[1-\left[1-\left[e^{-\left(\frac{\sigma}{t_i}\right)}\right]^{2\beta}\right]^{\alpha}\right]}\right] \quad (38)$$

The solutions of the resulting non-linear equations (35)–(38) cannot be obtained analytically; therefore, numerical iterative methods are employed.

6.0 Simulation study

The quantile function of the $ET_2GTLIEx$ distribution, defined in equation (29), was used to generate 1000 Monte Carlo replicates. To examine the performance of the estimators under different sample sizes, simulations were conducted for n=20, 50, 100, 250, 500 and 1000 from the $ET_2GTLIEx$ distribution For each of the 1000 replicates, parameter estimates, bias, and root mean square error (RMSE) were computed. The results of which are presented in Table 1, showing the MLE parameter estimates, bias, and RMSE for the estimated parameters of $ET_2GTLIEx$ at the chosen values of $\sigma = 1.34$, $\alpha = 2$, $\beta = 1.6$ $\theta = 1.2$. The values of biases and RMSEs approach zero in the table, and the estimates tend to the true values as the sample size increases, indicating that the estimates are efficient and consistent.

Table:1. MLE Estimates of $ET_2GTLIEx$ for First selected parameter values

N	Actual Parameter Value	Estimate	Bias	RMSE
20	$s = 1.34$	1.4918	0.1518	0.5406
	$a = 2$	2.4100	0.4100	0.9475
	$b = 1.6$	1.6967	0.0967	0.4441
	$q = 1.2$	1.4078	0.2078	0.7647
50	$s = 1.34$	1.3923	0.0523	0.3527
	$a = 2$	2.1780	0.1780	0.5061
	$b = 1.6$	1.7085	0.1085	0.3952
	$q = 1.2$	1.2884	0.0884	0.5630
100	$s = 1.34$	1.3449	0.0049	0.2746
	$a = 2$	2.0922	0.0922	0.3620
	$b = 1.6$	1.6889	0.0889	0.3041
	$q = 1.2$	1.2540	0.0540	0.4057
250	$s = 1.34$	1.3341	-0.0059	0.1959
	$a = 2$	2.0356	0.0356	0.1968
	$b = 1.6$	1.6697	0.0697	0.2164
	$q = 1.2$	1.2096	0.0096	0.2727
500	$s = 1.34$	1.3183	-0.0217	0.1430
	$a = 2$	2.0105	0.0105	0.1307
	$b = 1.6$	1.6419	0.0419	0.1614
	$q = 1.2$	1.2186	0.0186	0.2019
1000	$s = 1.34$	1.3242	-0.0158	0.1096
	$a = 2$	2.0031	0.0031	0.0869
	$b = 1.6$	1.6195	0.0195	0.1161
	$q = 1.2$	1.2148	0.0148	0.1442

7.0 Application to Real-Life Dataset

The competing distributions' pdfs are as follows

Topp-Leone Kumaraswamy Inverse exponential (TLKwIEx) distribution by Ismail et al (2022b).

$$f(t) = 2\alpha\theta\lambda\left(\frac{\beta}{t^2}\right)\left[e^{-\left(\frac{\beta}{t}\right)}\right]^{\lambda}\left[1-\left[e^{-\left(\frac{\beta}{t}\right)}\right]^{\lambda}\right]^{2\alpha-1}\left[1-\left[1-\left[e^{-\left(\frac{\beta}{t}\right)}\right]^{\lambda}\right]^{2\alpha}\right]^{\theta-1}$$

- Exponentiated Kumaraswamy Inverse Exponential (EtKwIEx) distribution by Umar et al (2017)

$$f(t)=\alpha\theta\lambda\left(\frac{\beta}{t^2}\right)\left[e^{-\left(\frac{\beta}{t}\right)}\right]^{\lambda}\left[1-\left[e^{-\left(\frac{\beta}{t}\right)}\right]^{\lambda}\right]^{\alpha-1}\left[1-\left[1-\left[e^{-\left(\frac{\beta}{t}\right)}\right]^{\lambda}\right]^{\alpha}\right]^{\theta-1}$$

- Kumaraswamy Inverse exponential (KwIEx) distribution by Oguntunde (2014)

$$f(t)=\alpha\lambda\left(\frac{\beta}{t^2}\right)\left[e^{-\left(\frac{\beta}{t}\right)}\right]^{\lambda}\left[1-\left[e^{-\left(\frac{\beta}{t}\right)}\right]^{\lambda}\right]^{\alpha-1}$$

- Exponentiated Weibull-Exponential (EWEx) Distribution by Elgarhy et al (2017)

$$f(x)=\alpha\delta\beta\lambda\left[e^{\lambda x}-1\right]^{\beta-1}\exp\left[-\left[\alpha\left(e^{\lambda x}-1\right)^{\beta}-\lambda x\right]\right]\left[1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{\alpha-1}$$

Dataset 1

The first datasets shown below represents the strength of carbon fibers tested under tension at gauge lengths of 10mm, previously used by Mushtaq *et al* (2025):

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

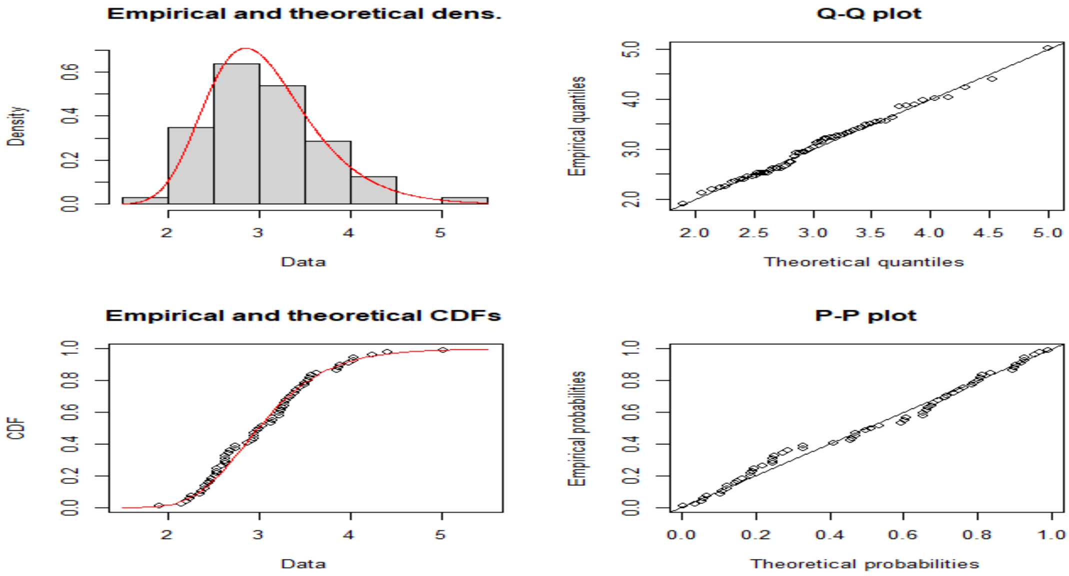

Fig.3: Plots of fitted Empirical and theoretical densities, cdf, Q-Q plot and P-P plot for Datasets

Table 2 presents the maximum likelihood estimates, log-likelihoods, and goodness-of-fit statistics for the fitted distributions. Based on the AIC values, the ET$_2$GTLIEx distribution provides a much better fit than the other four model comparators. Additionally, visual assessments of the empirical and theoretical PDFs, CDFs, as well as the Q-Q and P-P plots depicted in Figure 3, further confirm the suitability and adaptability of the new distributions for the analyzed datasets.

Dataset 2

The second dataset shown below represents the tensile strength of 100 observations of carbon fibers, previously used by (Nichols & Padgett, 2016)

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65

Table 2. MLEs, Log-Likelihoods and Goodness of fit Statistics of the models based on the Datasets 1

Distribution	β	α	θ	λ	δ	LL	AIC
ET$_2$GTLIEx	0.3932	29.9294	4.1573	-	10.6220	-56.4019	120.8037
TLKwIEx	6.1767	0.6565	24.3994	6.1503	-	-56.9560	121.9119
EtKwIEx	1.4709	3.1822	10.4322	16.9584	-	-56.6090	121.2181
KwIEx	175.3955	0.3587	-	46.8762	-	-57.9252	122.4504
EWEx	1.4496	1.8214	-	0.26451	7.9653	-57.5959	123.1918

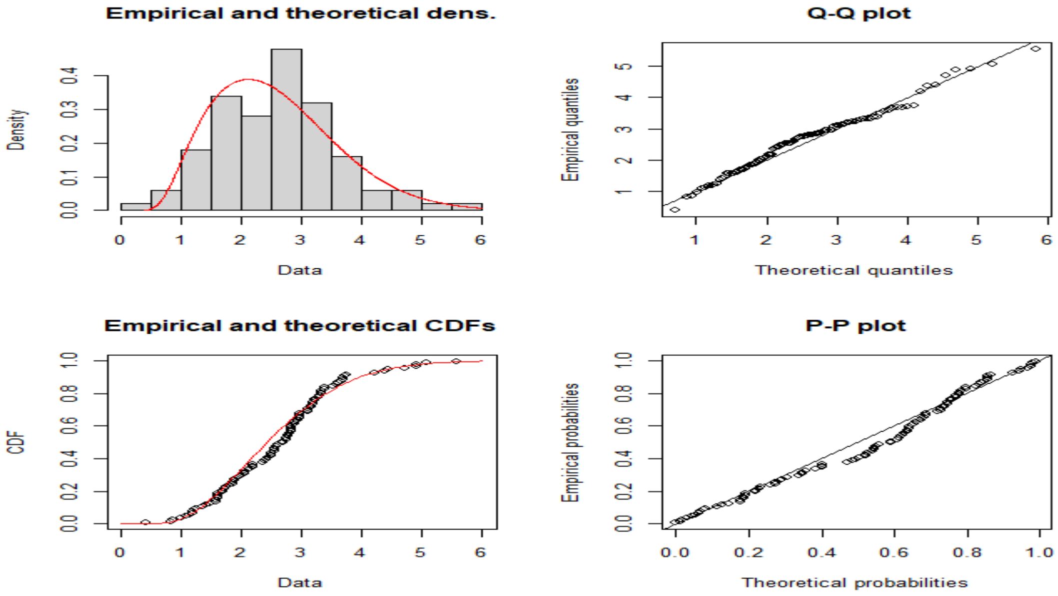

Fig.4: Plots of fitted Empirical and theoretical densities, cdf, Q-Q plot and P-P plot for Datasets 2

Table 3 shows the maximum likelihood estimates, log-likelihoods, and goodness-of-fit statistics for the fitted distributions. Based on the AIC values, the ET$_2$GTLIEx distribution provides a much better fit than the other four model comparators. Additionally, visual assessments of the empirical and theoretical PDFs, CDFs, as well as the Q-Q and P-P plots depicted in Figure 4 further confirm the suitability and adaptability of the new distributions for the analyzed datasets.

Table 3 MLEs, Log-Likelihoods and Goodness of fits Statistics of the models based on the Datasets 2

Distribution	β	α	θ	λ	δ	LL	AIC
ET$_2$GTLIEx	598.5235	45.4997	0.3819	-	0.0119	-143.9409	295.8819
TLKwIEx	0.8965	14.9156	0.3354	18.1813	-	-144.6409	299.2818
EtKwIEx	0.6946	20.3996	0.6946	43.4998	-	-143.9819	295.9638
KwIEx	9.9171	0.3563	-	17.4418	-	-150.4161	306.8322
EWEx	2.02205	0.3149	-	0.3177	0.8409	-144.9769	297.9537

8.0 Conclusion

This study introduced a new four-parameter distribution, the Exponentiated Type II Generalized Topp-Leone Inverse Exponential (ET$_2$GTLIEx) distribution, obtained by extending the inverse exponential distribution through the inclusion of additional shape parameters. Several important statistical properties of the proposed distribution were derived, including the probability weighted moments, ordinary moments, survival function, hazard rate function, quantile function, and order statistics. Graphical analyses of the probability density and hazard rate functions demonstrated the distribution's flexibility in capturing various shapes of real-life data. The model parameters were estimated using the maximum likelihood estimation (MLE) method implemented via the AdequacyModel package in R. A Monte Carlo simulation study was conducted to evaluate the performance of the estimators, and the results showed that the bias and mean square error (MSE) decrease as the sample size increases, confirming the consistency and efficiency of the estimators.

The practical applicability of the proposed model was further demonstrated using two real-life datasets. The results, presented in Tables 1 and 2, indicate that the ET2GTLIEx distribution provides a better fit compared to competing models based on goodness-of-fit measures. Additionally, graphical assessments using fitted densities, Q–Q plots, and P–P plots (Figures 3 and 4) support the superior flexibility and adaptability of the proposed distribution.

Overall, the ET$_2$GTLIEx distribution offers a useful and flexible tool for modeling lifetime and reliability data, and it has potential applications in fields such as engineering, environmental science, and biomedical studies.

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Declaration

Consent for publication

Not Applicable

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