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Construction of Symmetric Balanced Incomplete Block Designs Using Mutually Orthogonal Latin Squares and Galois Field GF(4) Methods

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Abstract: *This study presents a novel approach for the construction and verification of a Symmetric Balanced Incomplete Block Design (SBIBD) using Mutually Orthogonal Latin Squares (MOLS) derived from Galois Field GF(4). Three mutually orthogonal Latin squares of order 4 were generated using finite field transformations and modular arithmetic principles, followed by column and row permutation techniques to preserve orthogonality and balance. The constructed design produced an SBIBD satisfying the standard combinatorial conditions of v = b = 16, constant block size k = r = 4, and pairwise occurrence parameter λ=1\lambda =1λ=1, thereby ensuring uniform treatment allocation and balanced replication. The orthogonality verification confirmed that all ordered treatment pairs occurred exactly once across the superimposed structures, validating the mathematical consistency of the design. The proposed method reduced the complexity associated with manual BIBD construction while maintaining symmetry, balance, and structural flexibility. The results demonstrate that the GF(4)-based MOLS framework provides a statistically robust and computationally efficient approach for constructing SBIBDs applicable to agricultural experiments, industrial process optimization, and other combinatorial experimental designs. The study, therefore contributes to the advancement of experimental design through the integration of combinatorial mathematics and statistical design theory.*

Keywords: *Mutually Orthogonal Latin Square, Symmetric BIBD, Galois Field, Modular arithmetic, Superimposed Latin squares.*

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1.0 Introduction

In the world of Statistics, "Design of Experiment (DOE) is a powerful tool for achieving significant improvements in product quality and process efficiency, which involves the application of statistical methods to improve experimental performance and process efficiency (Arumugam & Rajathi, 2021). Furthermore, Block Designs and Mutually Orthogonal Latin Squares (MOLS) are widely applied in statistical experiments to control variability, precision in estimation , proper data analysiswithin an experimental design, used to achieve research objectives clearly and efficiently within treatment comparisons (Sewenet, 2019).Furthermore, BIBDs minimize experimental bias and enhances statistical precision by grouping homogeneous experimental units and implementing randomization strategies within experimental settings (Huang, 2017). BIBDs effectively manage heterogeneity by accommodating fewer treatments per block, "thereby increasing experimental precision (Mahanta, 2018).However, many incomplete designs are available for experimental design construction, a major challenge lies in their existence for all parameter combinations and so, the aspect of their construction becomes very important, since no universal construction methodology currently exists (Pachamuthu & Ramya, 2019). Despite the extensive applications in various fields the construction of new BIBDs is said to have some construction challenges because of the allocations of treatments in different blocks, such that each pair of treatments is replicated a constant number of times is not straight forward (Mahanta, 2018), and so remains a significant challenge. "Despite these advantages, BIBDs remain underutilized in applied research due to their mathematical complexity and the difficulty of construction, particularly when satisfying conditions of replication, orthogonality, and block resolution and orthogonality, which allows for incomplete pairing of treatments, however maintaining balance and efficiency.

This research aims to develop and evaluate an efficient method for the construction of a Symmetric Balanced Incomplete Block Design (SBIBD) by superimposing Latin Squares, constructed using Galois Field GF(4) and combinatorial techniques, Mutually Orthogonal Latin Squares (MOLS) to verify the symmetric balance of the design by constructing a pair of Mutually Orthogonal Latin Squares of order four, treatment combinations suitable for BIBDs by superimposing Mutually Orthogonal Latin Squares (MOLS) and Correction: "validating that the constructed BIBD satisfies balance and replication conditions"

.The significance of the study is important in experimental and statistical research"

, enabling the investigation of challenges that limit the development of statistically efficient and practically feasible designs, especially in real-world applications, where ideal conditions for complete randomization or replication are often difficult to achieve due to time, cost, or logistical limitations, thereby providing critical insights into the relationship between theoretical advancement and practical implementation in building a balanced design that is robust and virtually flexible. However, existing methods for constructing SBIBDs using MOLS remain computationally complex and lack a systematic GF(4)-based framework capable of ensuring symmetry and balance simultaneously.

This study integrates the construction and application of Symmetric Balanced Incomplete Block Designs (SBIBDs) via the implementation of Latin Squares, Mutually Orthogonal Latin Squares (MOLS), with emphasis on theoretical formulation and practical experimental applications, while demonstrating how SBIBDs can improve experimental balance and replication within the context of new design.

2.0 Materials and Methods

2.1 Conceptual Framework

Kelechi (2012) stated that a design is symmetric when the treatment and block effects can be interchanged without altering the error sum of square, "In symmetric BIBDs with $\lambda = 1$, every pair of treatments appears together in exactly one block. Additionally, highlighted that symmetric BIBDs which satisfy $b = v$, $k = r$, $\lambda=1$ are commonly used due to their simplicity and ease of analysis, although necessary and sufficient conditions for their existence remain an open problem, also enabling the derivation of new BIBDs through methods such as block omission and residual designs (Benedict *et al.*, 2023; Hasan *et al.,* 2016). Jaisankar and Pachamuthu (2019) outlined a process where MOLS are generated by systematically applying multipliers to initial Latin squares, ensuring orthogonality conditions are met. Similarly, researchers emphasized that the use of finite fields ,*where elements of Galois Fields (GF) are used in constructing Latin squares*

facilitates MOLS construction, while backtracking algorithms are used for smaller orders (Saka & Oyadare (2018); Saka (2020)). The integration of SBIBD and MOLS offers a powerful approach, addressing design challenges in experimental setups, improving treatment balance and reducing experimental error. , whereas a multi-part Balanced Incomplete Block design combines many orthogonal designs in the same block (Bailey and Cameron, 2019), improving the efficiency and precision of the estimates (Isah, *et al.* (2023;Kelechi (2012)). Academics proposed that the integration enhances the practical implementation of combinatorial designs in various fields, whereby various concepts can lead to a powerful and more efficient experimental designs, that allows for the construction of designs that can handle complex scenarios (Akong, 2023'; guyen & Zhang2, 018).

2.2 Theoretical Framework

Gupta et al. (2016) expanded on Fisher's approach to controlling nuisance variables, contributing to the advancement of BIBD methodologies, particularly evident when experimental error significantly influences treatment effects. In addition,Balanced Incomplete Block Designs (BIBDs) are crucial for maintaining balance and ensuring a valid design through various systematic substitution as well as other construction methods discovered and evaluated by several researchers (Ekpo *et al. (*2021); Abeynayake and Jaggi (2016); Pachamuthu and Ramya (2018); Shekar *et al.* (2018)). Prior to this study, Mutually Orthogonal Latin Squares (MOLS) were constructed of Latin Squares on "n" that, when superimposed, generate unique ordered pairs, which have been developed and examined by several researchers in the domain of "MOLS" (Pachamuthu & Ramya, 2019; Saka, 2018; Saka *et al.,* 2020).

2.3 Empirical Review

Recent studies, such as Dauran *et al.* (2020), have analyzed the construction of Balanced Incomplete Sudoku Square Designs (BISSD) by selectively eliminating certain cells from a complete Latin square. This technique, inspired by orthogonal Sudoku designs, has provided insights into relative efficiencies in experimental design. "Benedict et al. (2023) discussed symmetric BIBDs As prescribed by, Benedict *et al.* (2023) discussed symmetric BIBDs with $\lambda = 1$, a common class used in incomplete experimental block designs due to their simplicity in setup and analysis. While research has established necessary conditions for the existence of symmetric BIBDs, sufficient conditions remain unverified. Moreover, various construction methods have been proposed, but they have not been adequate in constructing all possible BIBDs. Koutra *et al,* (2020) examined the representation of block designs as integrating graph theory with Experimental design (MOLS) properties,Furthermore, scholars explored the application of Mutually Orthogonal Latin Squares (MOLS), demonstrating their connection to BIBDs, using the classical problem of Euler (1782), construction methods for maximal sets of MOLS, Galois Field theory and properties, Graeco-Latin squares etc. with advancement in computing power (Varghese, 2020; Pachamuthu and Ramya, 2019; Das and Giri, 1986; Dukes *et al.*,2014; Walsh, 2021, 2022). Incomplete Block Designs (IBDs) namely; A, B, C, D, E and F of size $t=9, b=9, k=3, r=3$ due to Nguyen, (1994) were extended by the construction of additional seventeen (17) new IBDs namely; G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V and W using all the initial blocks in Nguyen's using cyclic methods and Java programming (Ekpo & Oladugba, 2025).

2.4 Source of Data

This research builds on the foundational work of Pachamuthu and Ramya (2019), which was based on the Construction of Balanced Incomplete Block Design through Factorization and Coloring Graphs Using Mutually Orthogonal Latin square designs. The information for this research, such as generated Latin Squares, is derived from the existing study stated above. "The finite field GF(4) consists of the elements {0, 1, α, α+1}, where α satisfies $\alpha^2 = \alpha + 1$. Arithmetic in GF(4) is performed using polynomial operations modulo the irreducible polynomial $x^2 + x + 1$.. By substituting $\alpha = 2$ and reducing to Mod 4 technique, the first Latin square (L_1) was obtained,

Table 1 : First Latin square derived by Pachamuthu and Ramya (2019)

0	**1**	**2**	**3**
1	2	3	0
2	3	0	1
3	0	1	2

Likewise, the second and third Latin squares of the principal columns were obtained by multiplying the entries (primitive elements) in the principal column of the first summation table by α and second summation table by α + 1. Then, adding the corresponding entries of rows and columns we get other elements of second and third LSDs. The various GF (4) primitive element for the summation of L_1, L_2, L_3 are:dd

i. L_1 was obtained by substituting $\alpha = 2$
ii. L_2 was obtained by using $\alpha + 1$
iii. L_3 was obtained by using $\alpha + 2$

The Latin square is a table where the rows follow. It is a the rule of a tested layout used in combinatorial experiments that can be used to manipulate two sources of nuisance variability. However, different methods were used to construct 3 Latin Squares, which are derived from Pachamuthu and Ramya's first latin square layout, and structured similarly to L_1, but follow a different pattern of alteration of the GF (4) primitive elements. Correspondingly, for L_4, L_5, L_6, the next steps in the pattern of the mod 4 order will be:

i. $L_4 \rightarrow 2\alpha + 1 \; or \; \alpha + 3$
ii. $L_5 \rightarrow 2\alpha + 2 \; or \; \alpha + 4$
iii. $L_6 \rightarrow \alpha + 5$

The method used for L_4, L_5, L_6, follow the same structure, but with different substitution from the original GF (4) summation table. For the above analysis, it's been observed that the Latin Squares are generated using primitive elements summation:

Formula transformation on L_k GF (4) → $L_k(i,j) = (i+j)_{mod\ 4} + substitution\ elements$

$$L_k(i,j) = (i+j)_{mod\ 4} + \alpha_k,$$

Where α_k is a primitive element from GF (4):

$$\alpha_k\ follows\ the\ cycle\ [L_1 = 2,\ L_4 = 1, L_5 = 2, L_6 = 3]$$

Furthermore, laying emphasis to $\alpha_4 = 1, \alpha_5 = 2, \alpha_6 = 3$

Where the substitution component is determined by the corresponding GF (4) elements used as stated above, leading to each transformation preserving the Latin Square property, while satisfying the symmetric BIBD conditions and properties

Generating Latin Square (L_4, L_5, L_6), the elements of GF (4) used in transforming a set of data whereby the fourth Latin Square of the principal columns are systematically obtained by adding the entries (primitive elements) in the principal column of the first summation table by $\alpha + 3, \alpha + 4, \alpha + 5$. Therefore, adding the corresponding entries of rows and columns, we get the other elements for the fourth, fifth and sixth Latin Squares. This method is used to generate a new Latin Square from L_1 which involves relabeling the elements in the square and observing the fact that each element still appears once per row and once per column (a requirement for a Latin Square Design), thus enabling in the construction of Symmetric Balanced Incomplete Block Design. To derive the L_4, L_5, L_6, the following steps are observed:

i. Start with L_1 (First summation),
ii. Transform L_1 using GF (4) elements to get L_4, L_5, L_6,
iii. Proceed with the formula $L_k(i,j) = (i,j)_{mod\ 4} + \alpha_k$

iv. Confirm the α_k order follows GF (4) to maintain balance.
v. Construction of L_4 ($\alpha_4 = 1$, transformed from L_1)
vi. Construction of L_5 ($\alpha_4 = 1$, transformed from L_1)
vii. Construction of L_6 ($\alpha_6 = 3$, transformed from L_1)

Mutual Orthogonality is said to be the condition in which, a fundamental issue is to determine the maximum number of MOLS of order n, whereby this quantity is denoted $N(n)$. Subsequently, if any two Latin squares of order 1 are orthogonal, we say that $N\ (1) = \infty, For\ all\ n > 1$. Notwithstanding, ensuring the orthogonality n of Latin Squares and their utility in experimental designs that control for multiple factors, the key equation and principles underpin their derivation and application that aids MOLS conditions:

Step 1: Definition of a Latin Square

A Latin Square of order n is an $n\ x\ n$ array filled with n distinct symbols (i.e. $1, 2, \dots, n$), it satisfies:

i. Each of n symbols appear exactly once in each row
ii. Each of n symbols appear exactly once in each column.

This gives a combinatorial structure, where the entries $L\ (i, j)$ are drawn from the set of n symbols.

$\boldsymbol{Theorem\ 1:}$ Any finite group G has the following properties:

i. Its Cayley table forms a Latin square , in other words, a square matrix$\|L_{ik}\|$ in which each row and each column is a permutation of the elements of G.
ii. Provide a formal definition or citation because the criterion is introduced abruptly. which means that, for any indices i, j, k, l and i', j', k', l', it follows from the equations $L_{ik} = L_{i'\ k'}$, $L_{il} = L_{i'\ l'}$, $L_{jk} = L_{j'\ k'}$, and $L_{jl} = L_{j'\ l'}$. and so, any matrix satisfying properties (i) and (ii) can be bordered in such a way that it becomes the summation table of a group.

$\boldsymbol{Theorem\ 2:}$ If L is a Latin Square of order n which satisfies the quadrangle criterion and possess at least one transversal, then L has a decomposition into n disjoint transversal.

Proof of theorem 2: If L has a transversal formed by taking the symbol L_1 from the first row, L_2 from the second row, …, L_n from the n^{th} row, then it follows easily from the group axioms that another transversal can be obtained by taking $L_1 g$ from the first row, $L_2 g$ from the second row, …, $L_n g$ from the n^{th} row, where g is any fixed element of the group. As g varies through the n elements of the group. We shall however obtain n disjoint transversals. Furthermore, suppose that $L_i = g_i g_{i\ (i)}$ where the sequences $L_1, L_2, \dots, L_n$ and $g_1, g_2, \dots, g_n$ both represent orderings of the elements of G, the latter corresponding to the ordering of the rows and columns of L in the summation table of G. In other words, L_i is the element to be found in the cell which occurs in the i^{th} row and j^{th} column of L. Hence, since c_i form a transversal, the integer j is a function of i such that $j(i_1) \neq j(i_2)$ if $i_1 \neq i_2$. Then because G is a group,

$$L_i g = (g_i g_{j\ (i)})g = g_i(g_{j\ (i)} g) = g_i g_{k(i)}$$

where, g_j varies through the elements of G, so does g_k ,. Consequently, $L_{i1} g$ and $L_{i2} g$ are always in distinct columns and so the $L_i g$ form a transversal. Similarly, the transversals corresponding to two different choices of the elements g are disjoint.

Step 2: Orthogonality of two Latin squares is a function

For two Latin Squares $L_1 = \|a_{ij}\|$ and $L_2 = \|b_{ij}\|$ of order n symbols are said to be orthogonal if every ordered pair of symbols occurs exactly once in n^2 cells of the grid; when overlaid, every n ordered pair $(a_{ij,}\ b_{ij})$, $i, j = 1, 2, \dots, n$, which can also be. shown as, $(L_1\ (i, j), L_2\ (i, j))$. Thus, noting that there are

n^2 unique pairs distributed one per cell, which ensures the conditions of the total number of pairs as stated:

i. There are n^2 cells in the $n * n$ grid
ii. Each pair x and y must appear exactly once across all cells (where x is from L_1 and y is from L_2).

Proof of orthogonality:

For any two Latin Squares L_k and L_l (assembled as above):

$L_k\,(i,j) = (i + k\,.j)\,mod\,n.$

$L_l\,(i,j) = (i + l\,.j)\,mod\,n.$

Overlay these squares. The ordered pairs are:

$(L_k\,(i,j), L_l\,(i,j)) = ((i + k.j)\,mod\,n, (i + l.j)\,mod\,n).$

Suppose

$(L_k\,(i,j), L_l\,(i,j) = (L_k\,(i',j'), (L_l\,(i',j'))$

Then:

$(i + k\,.j)\,mod\,n = (i' + k\,.j')\,mod\,n$

$(i + l\,.j)\,mod\,n = (i' + l\,.j')\,mod\,n.$

Subtracting these equations:

$k\,.(j - j')\,mod\,n = 0,$

$l\,.(j - j')\,mod\,n = 0.$

Since k − l is nonzero in the finite field GF(n), it possesses a multiplicative inverse, the only solution is $j = j'$. Substituting back, we find $i = i'$. Hence, the ordered pairs are unique.

Step 3: Construction of Mutually Orthogonal Latin Squares (MOLS)

$\boldsymbol{Theorem\ 3}$ **:** A Latin square of order $n\ possess$ an orthogonal mate, if and only if it has n disjoint transversal.

$\boldsymbol{Theorem\ 4}$: Not more than $n-1$ mutually orthogonal Latin squares of order n exist.

Proof: Each of the squares may have its symbols given a set of new distinct values without affecting the orthogonality of the set. By such distinct values, we may arrange the symbols which occur in the first rows of all the squares as $1, 2, \ldots, n$ in natural order (Keedwell and Denes, 2015). The symbols in the first cells of the second rows of the squares must then all be different; for suppose two of them were the same, both containing the symbol i, say. Then the ordered pair (i, j) would occur in both the $(1, i)$-th position and the $(2, i)$-th position in the two squares and the squares could not be orthogonal. Therefore, none of the squares can have the symbol 1 as the entry in the first cell of the second row, otherwise the symbols would occur twice in the first column of the square. As a result, at most $n-1$ MOLS can exist corresponding to the $n-1$ different symbols distinct from 1 which can appear in the first cells of their respective second rows. Since no larger set is possible, a set of MOLS achieving the bound in $Theorem$ 3 is said to be complete.

Nonetheless, if a Latin Square is the multiplication table of a group, then we know more about its structure and hence we can strengthen $Theorem$ 1, where the existence of a single transversal is sufficient, by $Theorem$ 2 to guarantee existence of an orthogonal mate.

$\boldsymbol{Theorem\ 5}$**:** Let L be a Latin Square based on a finite group G, of which the following statements are equivalent:

i. G has a complete mapping,
ii. L has a transversal,
iii. L can be decomposed into disjoint transversals,
iv. There exist a Latin Square Orthogonal to L.

In the same way, for MOLS of order n, we extend this orthogonality condition:

$(L_x\,(i,j), L_y\,(i,j)) \neq (L_x\,(i',j'), L_y\,(i',j'))\ \forall\,(i,j) \neq (i',j'), \forall\, a \neq b$

Whereby, L_x and L_y are any two squares from the set of m Latin Squares, of which the maximum number of Mutually Orthogonal Latin Squares using finite fields, can be shown that the maximum number of MOLS of order n is $n - 1$, if n is a prime or a power of a prime, hence, the construction are as follows:

i. Take a finite field $\mathbb{F}_n$ of order n (exist if n is a prime power)
ii. Define $n - 1$ Latin Square using;

$L_k\,(i,j) = (i + k.j) mod\, n, \quad k \epsilon \{1, 2, \ldots, n-1\},$
Where i and j are the row and column indices, and so k identifies the square.

Step 4: Handling non-orthogonality in Latin square construction

To develop a robust experimental design, ensuring that the set of Latin Squares used in mutually orthogonality is critical in some instances where the constructed Latin Squares fail to satisfy mutual orthogonality, the methodology applies column and row permutations as corrective procedures to restore the integrity of the design.

2.5 Column and Row permutation:

In Latin Squares Design the properties stay well-preserved under permutation of Columns and Rows, which is particularly effective when failure of orthogonality is due to localized repetition in certain cells, the methodology involves:

i. Permuting columns or rows of one or more Latin Squares
ii. Observing the effect of these permutations on orthogonality
iii. Ensuring consistency across all affected squares if permutations are applied globally.

Proof of Orthogonality of two Latin squares:

Suppose L_1 and L_2 are not orthogonal, then, there exist two cells (i, j) and (i', j'), where:
$(L_1\,(i,j), L_2\,(i,j)) = (L_1\,(i',j'), L_2\,(i',j'))$.
Thereby, showing the same ordered pair (x, y) appears in at least two different cells, violating the condition of uniqueness, and so orthogonality holds if and only if all pairs are distinct.

2.6 Symmetric Balanced Incomplete Block Design (SBIBD)

A BIBD in which $b = v$ is called a symmetric Balanced Incomplete Block Design. As $bk = vr$, and $b = v$ for a symmetric BIBD, $r = k$. Moreover, since $\lambda\,(v-1) = r\,(k-1)$ and $r = k$, for a symmetric BIBD, the aforementioned follows that $\lambda\,(v-1) = k^2 - k$ (Pavarthy; Jen, *et al*. (2007); Moura (2017)).

For a Symmetric Balanced Incomplete Block Design that forms the cornerstone relationship governing the conditions of the equations, are fully characterized by three equations:

i. Symmetric Condition $(b = v)$
ii. Total incidence Equation $(vr = bk)$

As a result:

$$b = r\,\frac{\binom{v}{k}}{\binom{v-1}{k-1}} = \mathrm{r}\,\frac{v!}{k!(v-k)!} \div \frac{(v-1)!}{(k-1)!(v-k)!}$$
$$= r\,\frac{v!}{k!(v-k)!} \times \frac{(k-1)!(v-k)!}{(v-1)!}$$
$$= r\,\frac{v\,(v-1)!}{k(k-1)!(v-k)!} \times \frac{(k-1)!(v-k)!}{(v-1)!}$$
$$= \frac{rv}{k}$$
$$= vr = bk \Rightarrow r = k \quad (1)$$

(The replication number *r* is equal to the block ize *k*)

iii. Pairwise equation $(\lambda(v-1) = r\,(k-1))$

As a result:

$$b = \lambda\frac{\binom{v}{k}}{\binom{v-2}{k-2}} = \lambda\,\frac{v!}{k!(v-k)!} \div \frac{(v-2)!}{(k-2)!(v-k)!}$$

$$= \lambda \frac{v!}{k!(v-k)!} \times \frac{(k-2)!(v-k)!}{(v-2)!}$$
$$= \lambda \frac{v(v-1)(v-2)!}{k(k-1)(k-2)!(v-k)!} \times \frac{(k-2)!(v-k)!}{(v-2)!}$$
$$= \lambda \frac{v(v-1)}{k(k-1)} \quad (2)$$

But it is stated above that $b = \frac{rv}{k}$.

Hence, it can be shown that

$$= \frac{rv}{k} = \lambda \frac{v(v-1)}{k(k-1)}$$
$$= r = \lambda \frac{v(v-1)k}{k(k-1)v} = \lambda \frac{v-1}{k-1}$$
$$= \lambda(v-1) = r(k-1) \quad (3)$$

3.0 Results and Discussion

The outcomes of the research obtained through the constructed methodology are evaluated for balance and symmetry, confirming the effectiveness of using MOLS as a reliable framework in the construction of new design for experimental design structure.

3.1 Generating the First Set of Latin Squares

The Latin Square design was constructed using Pachamuthu and Ramya's (2019) First Latin Square design as $L1$, with three additional Latin Squares following the same approach and structure, with the substitution from the $GF(4)$ summation table using its primitive elements

Table 2: Latin Square (L_1)

0	**1**	**2**	**3**
1	2	3	0
2	3	0	1
3	0	1	2

Table 3: Latin Square (L_4)

1	**2**	**3**	**0**
2	3	0	1
3	0	1	2
0	1	2	3

Table 4: Latin Square (L_5)

2	**3**	**0**	**1**
3	0	1	2
0	1	2	3
1	2	3	0

Table 5: Latin Square (L_6)

3	**0**	**1**	**2**
0	1	2	3
1	2	3	0
2	3	0	1

3.2 Verification of Orthogonality

The three Latin Squares were verified for orthogonality by checking whether all possible ordered pairs appear exactly once when superimposed to form a Balanced Incomplete Block Design. The verification confirmed that the Latin Squares when superimposed together do not confirm that they are Mutually Orthogonal Latin Squares (MOLS). And so, three additional new Latin Squares were generated again in order to confirm mutual orthogonality. These new superimposed grids will form the foundation data for constructing the BIBD blocks, using the Column and Row Permutation method to get L_7, L_8, L_9.

Table 6: Latin Square (L_7)

2	**3**	**1**	**0**
3	0	2	1
0	1	3	2
1	2	0	3

Table 7: Latin Square (L_8)

3	**2**	**0**	**1**
0	3	1	2
1	0	2	3
2	1	3	0

Table 8: Latin Square (L_9)

1	**0**	**3**	**2**
2	1	0	3
3	2	1	0
0	3	2	1

3.3 Verification of Mutual Orthogonality

Three Mutually Orthogonal Latin Squares (MOLS) of order 4 was constructed and superimposed with the new sets of Latin Squares, to generate a grid containing triplets of values at each cell position, each possible ordered pair occurred exactly once, confirming mutual orthogonality, thus, ensuring the superimposed set of treatments forms the foundational data for constructing the SBIBD blocks. The generated superimposed MOLS is displayed in Table 9.

Table 9: Superimposed MOLS of order four

Row/column	0	1	2	3
0	2, 3, 1	3, 2, 0	1, 0, 3	0, 1, 2
1	3, 0, 2	0, 3, 1	2, 1, 0	1, 2, 3
2	0, 1, 3	1, 0, 2	3, 2, 1	2, 3, 0
3	1, 2, 0	2, 1, 3	0, 3, 2	3, 0, 1

3.4 Construction of the New Design from MOLS

Firstly, the aim is to construct a statistical design that satisfies the Symmetric BIBD, from the superimposed MOLS, within which 16 unique treatment combinations were derived. while maintaining balance and ensuring all treatment pairs occur equally across blocks, thereby, satisfying the Symmetric BIBD design ($v = 4$, $b = 4$, $k = 3$, $r = 3$, $\lambda = 2$) and condition
($v = b, b$

Table 10: New Balanced Incomplete Block Design

BLOCKS	TREATMENTS			BLOCKS	TREATMENTS		
1	2	3	1	**9**	0	1	3
2	3	2	0	**10**	1	0	2
3	1	0	3	**11**	3	2	1
4	0	1	2	**12**	2	3	0
5	3	0	2	**13**	1	2	0
6	0	3	1	**14**	2	1	3
7	2	1	0	**15**	0	3	2
8	1	2	3	**16**	3	0	1

$k = vr$, and each pair occurs in λ blocks).

Table 11: Blocks formed from each superimposed BIBD

	BIBD 1			BIBD 2			BIBD 3			BIBD 4		
Block 1	2	3	1	3	0	2	0	1	3	1	2	0
Block 2	3	2	0	0	3	1	1	0	2	2	1	3
Block 3	1	0	3	2	1	0	3	2	1	0	3	2
Block 4	0	1	2	1	2	3	2	3	0	3	0	1

3.5 Verification of SBIBD Properties for each BIBD

The new design constructed which forms a Symmetric Balanced Incomplete Block Design, treat each unique tuple in the superimposed table as a block. This resulted in each class independently analysed to verify that it satisfies the properties of a SBIBD with parameters (4,4,3,3,2). The occurrence frequency of each pair of treatments in each class provides a summary of the parameters verified in Tables 12.

Table 12: SBIBD parameters summary for each design

BIBD	v	b	k	r	λ	Status
BIBD 1	4	4	3	3	2	Verified
BIBD 2	4	4	3	3	2	Verified
BIBD 3	4	4	3	3	2	Verified
BIBD 4	4	4	3	3	2	Verified

4..0 Conclusion

The results demonstrated the successful construction of a Balanced Incomplete Block Design (BIBD) from a pair of Mutually Orthogonal Latin Squares (MOLS), highlighting an adaptable block structure based on Latin square design principles. , through systematic grouping suitable for block formation, using a corrective strategy based on column and row permutation techniques , without compromising balance, replication, or design flexibility fthereby making the design applicable to various experimental settings . An orthogonal design structure was achieved through the superimposition of multiple BIBDs. . The orthogonal BIBD was generated from an initial set of block designs. . Furthermore, the balanced distribution within MOLS provides an ideal structure for generating BIBDs, for representing the properties of the initial block designs across different squares ensuring uniform treatment allocation and consistent pairwise occurrences.

The resulting BIBD structure also demonstrated symmetry, observing equal numbers of treatments per block and replications. This method efficiently embraced the fundamental BIBD conditions, "confirming that MOLS can effectively satisfy experimental design requirements , about treatment replication and pairwise combinations. This further suggests that symmetric designs can be obtained with minor modifications. The study successfully demonstrated that Mutually Orthogonal Latin Squares (MOLS) of order 4 provide a reliable framework for constructing Symmetric Balanced Incomplete Block Designs (SBIBDs). for generating Balanced Incomplete Block Designs, particularly for smaller orders like four. The design maintained key properties such as $v = b$ (number of treatments equals blocks) and constant block size k , validating its mathematical exactness. Although the process used in constructing a BIBD via MOLS proved forthright, it reduced the complexity often related to constructing balanced designs manually or via random allocation. The SBIBD approach therefore provides a reliable and adaptable framework for constructing balanced experimental designs. This study establishes a practical and efficient foundation for using Mutually Orthogonal Latin Squares in the construction of Balanced Incomplete Block Designs. Future studies may investigate the effects of different combinatorial arrangements on BIBD efficiency. to further validate the robustness of the approach. Further analysis should be conducted to extend the method to higher orders and more complex parameter configurations. 4, to explore various balanced conditions.

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