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Thinning-Based Models: A Comprehensive Literature Review

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Abstract: *Count time series are non-negative values that are collected in chronological order over time. This type of series posses characteristics such as under-dispersion, equi-dispersion, over-dispersion, and zero-inflation (deflation) and arise naturally from several areas of human endeavour. The quest for proper modelling of such series, so as to avoid biased estimation and misleading predictions, gave rise to thinning base models. In this paper, a comprehensive literature review on thinning based models is made. Different thinning based models such as integer-valued autoregressive models - with details on integer-valued autoregressive model of first order, its extensions-, integer-valued moving average and some other models that are related to these models were reviewed. The review, in addition to containing robust information on the thinning operators employ when constructing thinning based models, paid particular attention to integer-valued autoregressive models for the various dispersion structure (under-dispersion, equi-dispersion and over-dispersion) that may be present in a given count time series and for zero-inflated (deflated) count time series. Test of hypothesis required for the application of integer-valued autoregressive models, methods of estimation use and forecast functions for the models were discussed. From the review some possible areas for future research on thinning based models were suggested.*

Keywords: *Integer values, autoregressive models, thinning operators, thinning-based models, and literature review.*

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1.0 Introduction

The daily number of patients admitted in a hospital, the daily number of confirmed cases of a particular influenza reported during an outbreak, the number of transactions of a given stock observed every minute, the annual number of recorded claims in actuarial applications, the monthly number of auto accidents in a region, and many more of such data observed in several fields of human endeavour are referred to as integer-valued time series. According to Liu *et al.* (2023), integer-valued time series are clearly produced from counts of events that are recoded sequentially over time and can be divided into four different types: N-valued counts, whose values range as N = {0,1,2,…}, Z-valued counts, which assume values in the range Z = {…,−1,0,1,…}and can be obtained, for instance, from the difference between the number of goals scored in different football matches, bounded N-valued counts with range{0,1,2,…,n} for a given $n \in N$, such as the observed data set of estuary water quality, and multivariate counts, which are counts recorded on multiple variables simultaneously over time.

The aforementioned types of time series were first analyzed using well-known classical time series analysis techniques, such as Gaussian models dominated by autoregressive moving-average (ARMA) schemes, generalized linear models, generalized autoregressive conditional heteroscedasticity (GARCH), and integer-valued generalized autoregressive conditional heteroscedasticity (INGARCH) models. These models can handle continuous or discrete

observations, specify the conditional distribution for each observation, and extend linear regression for Gaussian data to exponential family distributed data (Li, 1994). Using the first and second moments, the classical time series models effectively describe a continuous-valued time series. Nevertheless, these models often describe integer-valued time series poorly. This is because the Gaussian models rely on the normality assumption and may not accurately characterize count time series, which are frequently skewed, heavy-tailed, and over-dispersed. Although INGARCH-type models have been successfully created for integer-valued time series with N-valued counts, its application to Z-valued counts, bounded N-valued counts, and multivariate counts has proven challenging. According to Liu *et al.* (2023), analyzing Z-valued counts using an INGARCH-type model necessitates the implementation of more intricate and subtle dynamic structures, and the accompanying theoretical proofs are frequently challenging.

As some of the aforementioned difficulties became apparent, efforts were directed toward developing models that accurately depict the dynamics and structure of time series with integer values. The most notable attempt was the introduction of the integer valued autoregressive model of first order (INAR(1)) model, which was independently developed by Al-Osh & Alzaid (1987) and McKenzie (1985). To maintain the discreteness of the count time series, the INAR(1) model was created by substituting the scalar multiplication in traditional time series models with a thinning operation and taking into consideration the adoption of a discrete distribution as the innovation distribution of the model. Many researchers in the field of count time series modeling were greatly intrigued by the model introduced by McKenzie (1985) and Al-Osh & Alzaid (1987) and thus directed their focus toward the creation of what are now known as models based on thinning operators. There are numerous works on thinning based models in literature today.

The beauty of thinning based models is in their construction, which uses thinning operators and appropriate innovations to preserve the discreteness, correlation, and dispersion features of count time series (McKenzie, 1985; Al-Osh & Alzaid, 1987). For thinning based models, various thinning operators have been created, as given in Section 3, to handle various correlations (positive or negative correlations) that are may be seen in integer values. Once more, a number of other authors have examined the choice of innovation to ensure that appropriate innovation is used to capture the various dispersions that are inherent in integer values: under-dispersion (variance less than mean), equi-dispersion (variance equal to mean), and over-dispersion (variance greater than mean) (Jazi *et al.* 2012a; Weiß, 2013). In integer-valued time series, over-dispersion is a prevalent feature that frequently arises from positive associations between observed occurrences (Weiß, 2009). Many statistical models have been developed to address over-dispersion. Nevertheless, these models can exaggerate the impact of some factors and underestimate standard errors. INAR models have gained popularity as a solution to the difficulties in modeling over-dispersion and zero-inflation in count time series through the use of some over-dispersed distributions such as geometric, negative binomial etc as the innovation distribution of the INAR model (Robinson *et al.*, 2010; Auer & Doerge, 2011: Pounds *et al.*, 2012). Interest has also been given to estimation methods and their associated estimators (Freeland, 1998: Bourguignon *et al.*, 2015), bootstrapping in estimation (Luisa and Margherita, 2016;Weiß & Jentsch ,2019), forecasting methods (Freeland and McCabe 2004; McCabe and Martin 2005; Jung and Tremayne 2006; Silva *et al.* 2009; McCabe *et al.* 2011; Bisaglia and



Canale 2016; Awale *et al.*, 2017), replication in INAR model (Silva *et al*., 2005), seasonality in count time series (Bourguignon *et al.*, 2015; Tian *et al.*, 2018; Okereke *et al.*, 2019), zero-inflation in the observations (Lambert, 1992; Dietz & Bohning, 2000; Ridout *et al.*, 2001; Jazi *et al.*, 2012b; Zhu, 2012; Maiti *et al.*, 2015; Bourguignon, 2018; Bakouch *et al.*, 2018), the use of exploratory variable (Brännäs ,1995; Monteiro *et al.,*2008), random thinning coefficients (Joe, 1996; Zheng *et al.*, 2007), non-stationarity (Kim & Park, 2008; Freeland, 2010; Wagner and Bourguignon, 2013) extension of model order (Alzaid and Al-Osh, 1990; Du & Li, 1991), inclusion of moving average component (McKenzie, 1986 ; Al-Osh & Alzaid, 1988), intervention INAR models (Moriña , *et al.*, 2020), treating missing data (Anderson & Karlis, 2008), controlling jumps in time series (Weiß, 2008a), long memory in INAR model (Quoreshi, 2014) and the development of bivariate and multivariate INAR models (Heinen and Rengifo, 2003; Karlis & Meligkotsidou, 2005; Bermúdez *et al*., 2013; Nikoloulopoulos, 2013; Karlis, 2015; Karlis, 2018; Bermúdez and Karlis, 2021).

All these achievements, no doubt, have been partially reviewed in one article or another depending on the focus of the articles. The most detailed reviews on thinning-based models in the literature to the best of our knowledge are found in Scotto *et al.* (2015) who focused on the evolution of thinning-based models in relation to the different thinning operators and in Serrao *et al.* (2026) who recently presented an overview of the methodological developments in integer-valued autoregressive (INAR) models alone, with particular emphasis on thinning operators, estimation methods, and model extensions. The aim of this paper is to carry out a more comprehensive literature review on thinning based models so as to expose to a greater extent the developments and achievements already made towards modeling count time series via these models as well as identifying the direction for future development. The paper is structured as follows; Section 2 dealt the various types of thinning based models and some of their properties. The review of thinning operators used in the construction of thinning based models and their applications on INAR models was made in Section 3. Section 4 contains the review of INAR models based on different dispersion structures. The reviews on INAR models for zero inflated or deflated count time series and that of test of hypothesis for INAR models were given in Sections 5 and 6 respectively. The review on the estimation methods for INAR model and that of the forecasting methods for INAR models are made in Sections 7 and 8 respectively. Possible areas for future development on thinning based models were discussed in Section 9. Lastly, the paper was concluded in Section 10.

2.0 Different Thinning Based Models

In this section, we briefly reviewed the most common thinning models and some of their important structural properties

2.1 The INAR Models

The first INAR model is the Poisson INAR(1) model independently introduced by McKenzie (1985) and Al-Osh & Alzaid (1987) using binomial thinning operator. An INAR(1) model is of the form

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t \quad , \quad t \in Z \qquad (1)$$

Basically, Eq. (1) is a convolution of two independent components. The first component $\alpha \circ X_{t-1}$, is the binomial thinning operator $\circ$ component that thins down the previous value of the process X_t with the thinning coefficient $\alpha \in (0,1)$ and as well guarantees the discreteness of X_t. The second component is the innovation ε_t, a sequence of uncorrelated non-negative integer valued random variables with mean μ_ε and finite variance σ_ε^2.

The basic properties of an INAR(1) process with binomial thinning are generally given as(…);

i. $\mu_X = E(X_t) = \frac{\mu_\varepsilon}{1-\alpha}$, the mean of X_t

ii. $\sigma_X^2 = Var(X_t) = \frac{\alpha\mu_\varepsilon + \sigma_\varepsilon^2}{1-\alpha^2}$, the variance of X_t

iii. $\gamma_X(k) = Cov(X_t, X_{t-k}) = \alpha^{|k|}\sigma_X^2$, $k \in Z$, the autocovariance function at lag *k*.

iv. $\rho_X^k = \frac{\gamma_X(k)}{\gamma_X(0)} = \alpha^{|k|}$, the autocorrelation function at lag *k*.

When the innovation ε_t follows Poisson distribution with parameter, say θ, as it is the case of the Poisson INAR(1) model of McKenzie (1985) and Al-Osh and Alzaid (1987), properties (i) and (ii) by virtue of equidispersion become

$$\sigma_X^2 = Var(X_t) = \frac{\theta}{1-\alpha} = \mu_X$$

Eq. (1) is second order stationary if $0 \le \alpha < 1$, where $\alpha = 0$ implies independence and $\alpha = 1$ non-stationarity (Al-Osh and Alzaid, 1987;Du and Li, 1991). Furthermore, the INAR(1) model in Eq. (1) shares many properties with the conventional AR(1) model which includes the same form of autocorrelation function (ACF) even though that the range of variation of the ACF in the INAR(1) is restricted to [0;1]. Another important property of the INAR(1) model is that its probability generating function (p.g.f.) satisfies $\phi_X(s) = \phi_X(1-\alpha+\alpha s)\phi_\mu(s)$

(2)

Eq. (2) implies that any discrete self-decomposable (DSD) distribution can serve as a marginal distribution for the model (Jazi *et al.* 2012a). DSD implies infinite divisibility and unimodality (Steutel & Van Harn, 2004). Obviously many common discrete distributions such as the Poisson distribution, the negative binomial distribution, the generalized Poisson distribution, the geometric distribution, etc belong to the class of DSD distributions (Jazi *et al.*, 2012a; Weiß, 2013).

Poisson INAR(1) model has been studied extensively and applied to different areas**,** see Freeland (1998), Freeland and McCabe (2004), Bu *et al.* (2008), Bourguignon (2014) for more account on the areas of application of model.

Relevant adjustment, extensions and generalizations have been made on the stationary INAR(1) model, some of which are presented in Table 1.

Table 1 Some Extensions of Stationary INAR(1) Model

Author(s)	Model	Remark
Brännäs (1995), Monteiro *et al.* (2008)	$X_t = \alpha \circ X_{t-1} + \varepsilon_t$, $t \in Z$, where the well-known specification $\alpha_t = 1/(1+e^{y_t\delta})$, with the fixed explanatory variable vector $y_t = [y_{t,1}, y_{t,2}, ..., y_{t,i}]$ and the corresponding vector of unknown parameters $\delta = [\delta_1, \delta_2, ..., \delta_l]^l$	Incorporated explanatory variables in INAR(1) model
McKenzie (1985)	$X_t = \alpha \circ X_{t-1} + \beta \circ (n - X_{t-1})$, $t \in Z$	For analysis of N-bound count time series. The term ε_t was

	where $\alpha = \beta + p, \beta = \pi(1-p)$ for $\pi \in (0,1)$ and $p \in (\max\{-1/(1-\pi), -(1-\pi)/\pi\}; 1)$	replaced by $\beta \circ (n - X_{t-1})$ and all thinning operators at time t independent of $(X_k)_{k<t}$
Brännäs and Hellström (2001)	$X_t = \alpha \circ X_{t-1} + \varepsilon_t \quad , \quad t \in Z$	Extended INAR(1) model that addressed the dependence structure of count time series.
Silva *et al.* (2005)	$X_{t,k} = \alpha \circ X_{t-1,k} + \varepsilon_{t,k} \quad , \quad t \in Z$	Replicated INAR(1) model, for kth time series $X_{t,k}, (k = 1,2,...,r)$,for more flexibility in modeling of real count data
Joe (1996), Zheng *et al.* (2007)	$X_t = \alpha \circ X_{t-1} + \varepsilon_t \quad , \quad t \in Z$	INAR(1) model with random coefficient thinning operator (RCINAR(1)) to overcome the restriction of the usual deterministic thinning coefficient
Bourguignon *et al.* (2015), Tian *et al.* (2018), Okereke *et al.*, (2019)	$X_t = \alpha \circ X_{t-s} + \varepsilon_t \quad , \quad t \in Z$	Captured seasonality in count time series. Marcelo *et al.* (2015) used Poisson innovation while Tian *et al.* (2018) and Okereke *et al.*, (2019) used geometric innovation, however, with different thinning processes.

Extension of INAR model to *p*th order was given by Alzaid and Al-Osh (1990) and, Du and Li (1991). The INAR(*p*) model was given as

$$X_t = \alpha_1 \circ X_{t-1} + \alpha_2 \circ X_{t-2} + ... + \alpha_p \circ X_{t-p} + \varepsilon_t \quad , \quad t \in Z, \tag{3}$$

where $0 < \alpha_1, \alpha_2, ..., \alpha_p < 0$ are the parameters of the model which are fixed and ε_t a sequence of independent and identically distributed (iid) Poisson random variable with parameter θ called the innovation.

Eq. (3) can be written in compact for as

$$X_t = \sum_{i=1}^{p} \alpha_i \circ X_{t-i} + \varepsilon_t \tag{4}$$

At $p = 1$, Eq. (4) corresponds to an INAR(1) process. Additionally, Alzaid and Al-Osh (1990) and, Du and Li (1991) paid particular attention to $p = 2$ which is an extension of INAR(1) model and gave the model of the form

$$X_t = \alpha_1 \circ X_{t-1} + \alpha_2 \circ X_{t-2} + \varepsilon_t \quad , \quad t \in Z. \tag{5}$$

However, the approaches they employed differ. While Alzaid and Al-Osh (1990) employed Eq. (5) as a direct extension of INAR(1) process such that the population, at any time period, is thinned down with the survivors following the two thinning operator coefficients α_1 and α_2 under the specification

$$\left((\alpha_1 \circ X_{t-1}, \alpha_2 \circ X_{t-1})^T \middle| X_{t-1}\right) \sim \text{Trinomial}(\alpha_1, \alpha_2, X_{t-1}),$$

Du & Li (1991) constructed Eq. (4) as a model analogous to Gaussian linear autoregressive model of higher order in which the two thinning operators $\alpha_1 \circ X_{t-1}$ and $\alpha_2 \circ X_{t-2}$ are independent of the innovation ε_t and thins each count twice at a given time period t, firstly by the coefficient α_1 at the time t+1 and secondly by the coefficient α_2 at the time t+2. In practice the INAR(p) model as given by Du and Li (1991) enables the application of the well known Yule-Walker autocorrelation function (ACF) equations structure of an autoregressive model of order p (AR(p)) and thus is commonly used in research (Weiß *et al.* 2019). $\sum_{i=1}^{p} \alpha_i < 1$ is assumed for the existence and stationarity of the INAR(p) process by Du and Li (1991). $\sum_{i=1}^{p} \alpha_i = 1$ implies non-stationarity while $\sum_{i=1}^{p} \alpha_i > 1$ implies an explosive model.

The major limitation common between the stationary INAR(p) models introduced by Alzaid and Al-Osh (1990) and, Du and Li (1991) is that the stationary marginal distributions of the models differ for $p = 1$ and $p \geq 2$. To surmount this problem a combined INAR(p) (CINAR(p)) model was developed by Zhu and Joe (2006) and Weiß (2008b) through the use of a probability mixing mechanism.The CINAR(p) model is of the form

$$X_t = D_{t,1}.(\alpha \circ X_{t-1}) + D_{t,2}.(\alpha \circ X_{t-2}) + \ldots + D_{t,p}.(\alpha \circ X_{t-p}) + \varepsilon_t, \; t \in Z \quad (6)$$

where (D_t) is an i.i.d random variable with $D_t = (D_{t,1}, D_{t,2}, \ldots, D_{t,p}) \sim \text{Mult}(1; \varphi_1, \varphi_2, \ldots, \varphi_p)$ independent of ε_t.

The main advantage of the CINAR(p) model when compared with the INAR(p) model is that if (X_t) is a stationary CINAR(p) model then its p.g.f. satisfies Eq. (2)(Zhu and Joe, 2006). Hence, the possible marginal distributions of a stationary CINAR(p) process include the DSD family.

Barczy *et al.* (2011) demonstrated that for the non-stationary model (i.e., when the characteristic polynomial has a unit root), the INAR (p) process converges weakly towards a squared Bessel process (also known as the Cox-Ingersoll-Ross process) under certain natural assumptions, such as the INAR (p) process being primitive (i.e., the greatest common divisor of the set being equal to one). The unstable INAR(p) process's asymptotic behavior is very different from that of its traditional unstable AR(p) counterpart: the characteristic polynomial's unit root has multiplicity one, whereas the unstable AR(p) model may have real or complex unit roots with different multiplicities. Additionally, while the unstable INAR(p) model converges to a well-defined limit, the unstable AR(p) model generally lacks such a limit. Barczy *et al.* (2011, 2014) provide more results for the unstable condition. The explosive INAR(p) process, in contrast to the other two occurrences, has not been discussed in the literature and is still a subject for further research.

Models for multivariate time series of count data have been developed in response to the current rise in count data availability. Multivariate time series of count data (counts collected several variables over time) are inherently in several areas of human endeavour (Bermúdez & Karlis, 2021). Heinen & Rengifo (2003), Karlis (2015), Bermúdez *et al*. (2013), Karlis (2018), and Bermúdez& Karlis (2021) are just a few of the numerous works that have examined modeling bivariate or multivariate count data under the assumption of an underlying Poisson distribution. Multivariate time series models based on the Poisson and double Poisson distributions were first presented by Heinen & Rengifo (2003) and was

later developed in Karlis and Meligkotsidou (2005) & Nikoloulopoulos (2013). The multivariate Poisson distribution of Karlis and Meligkotsidou (2005) is hampered by the requirement of assessing several sums, which can be laborious and prone to mistakes. However, this approach is limited to positive correlations between variables. Nikoloulopoulos (2013) provided a different approach that does not require evaluation of multiple sums, based on copula functions. Again, this approach has a disadvantage stemming from the fact that the joint probability mass function (pmf) cannot be expressed simply and often requires the evaluation of several terms which creates numerical problems.

In order to handle the cross-correlation in multivariate count time series, Bermúdez & Karlis (2021) use multivariate discrete distributions defined using the Sarmanov family, while the MINAR(1) model is used to explain the time correlation. Bermúdez & Karlis (2021) claimed that these distributions can offer intriguing benefits over alternative methods for the multivariate discrete situation.

According to Pedeli & Karli (2013) Multivariate integer-valued autoregressive model of order 1, MINAR(1) is of the form

$$\mathbf{X_t} = \mathbf{A} \circ \mathbf{X_{t-1}} + \boldsymbol{\eta_t}, \qquad t \in Z \tag{7}$$

where $\{\boldsymbol{\eta_t}\}_{t\in Z}$ is a sequence of non-negative integer-valued random vectors with mean $\boldsymbol{\mu}_{\boldsymbol{\eta}}$ and variance $\boldsymbol{\Sigma}_{\boldsymbol{\eta}}$, independent of $\mathbf{A} \circ \mathbf{X_{t-1}}$. $\mathbf{A}$ is a $m \times m$ matrix with entries α_{ij} with $0 \le \alpha_{ij} \le 1$ for i,j = 1,...,m, and $\{\mathbf{X_t}\}_{t\in Z}$ is a random vector with values in N^m. $\mathbf{A} \circ \mathbf{X_{t-1}}$ is an m-dimensional random vector with *i-th* component as

$$[\mathbf{A} \circ \mathbf{X_{t-1}}]i = \sum_{j=1}^{m} \alpha_{ij} \circ X_{j,t-1}, \quad i = 1,2,...,m \tag{8}$$

where the discrete series in all $\alpha_{ij} \circ X_{j,t-1}$, $i, j = 1,2,...,m$ are assumed to be independent. Useful properties of this matrices' operator can be found in Franke and Rao (1993) and Latour (1997).

Equivalently Eq. (7) can be written in matrix form as

$$\begin{bmatrix} X_{1t} \\ X_{2t} \\ . \\ . \\ . \\ X_{mt} \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 & . & . & . & 0 \\ 0 & \alpha_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \alpha_m \end{bmatrix} \circ \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ . \\ . \\ . \\ X_{m,t-1} \end{bmatrix} + \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \\ . \\ . \\ . \\ \eta_{mt} \end{bmatrix}, \quad t \in Z \tag{9}$$

Thus, the *i-th* element of MINAR(1) in Eq. (7) is

$$X_{it} = \sum_{j=1}^{m} \alpha_{ij} \circ X_{j,t-i} + \eta_{it}, \quad i = 1,2,...,m; t \in Z \tag{10}$$

where $\alpha_{ij} \circ X_{j,t-1}$ are assumed to be mutually independent binomial thinning operations.

Note that $\{\mathbf{X_t}\}_{t\in Z}$ is strictly the stationary solution of Eq. (7) if the largest eigenvalue of $\mathbf{A}$ is less than unity and $E\|\boldsymbol{\eta_t}\| < \infty$.

Using the basic properties of thinning operation, Pedeli & Karlis (2013) showed that the mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\gamma}(h)$ at lag h of the $\{\mathbf{X_t}\}_{t\in Z}$ are respectively given by



$$\boldsymbol{\mu} = E(\mathbf{X_t}) = (1-\mathbf{A})^{-1}\boldsymbol{\mu}_{\boldsymbol{\eta}}$$

and

$$\boldsymbol{\gamma}(h) = E\left[(\mathbf{X}_{t+h}-\boldsymbol{\mu})(\mathbf{X}_t-\boldsymbol{\mu})^T\right] = \begin{cases} \mathbf{A}\boldsymbol{\gamma}(0)\mathbf{A}^T + diag(\mathbf{B}\boldsymbol{\mu}) + \boldsymbol{\Sigma}_\eta, & h=0 \\ \mathbf{A}\boldsymbol{\gamma}(0), & h \geq 1 \end{cases}.$$

where $[\mathbf{B}]_{ij} = \alpha_{ij}(1-\alpha_{ij})$ for $i, j = 1,...,m$ and $\boldsymbol{\gamma}(0)$ is the stationary variable of the variance matrix of the process.

For $m=2$, Eq. (7) becomes the bivariate integer-valued autoregressive model of order 1 (BINAR(1)) which can be expressed in matrix form as

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \circ \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix}, \quad t \in Z \tag{11}$$

with the stationarity condition $\max(\alpha_1, \alpha_2) < 1$ and $E\|\boldsymbol{\eta}_t\| < \infty$, and the *i-th* element given as

$$X_{it} = \sum_{j=1}^{2} \alpha_{ij} \circ X_{j,t-i} + \eta_{it}, \quad i = 1,2; t \in Z. \tag{12}$$

Obviously, varieties of models of this kind can be created by assuming particular structure for the matrix **A** and the use of certain bivariate distributions for the innovations. Under the assumption of independence between and within thinning operations and the assumption that the innovation $\{\boldsymbol{\eta}_t\}_{t\in Z}$ for the BINAR(1) model is iid sequence with mean θ_j and variance $\delta_j^2 = v_j\theta_j$, for $v_j > 0$ and $j = 1,2$, the following properties hold;

i. $E(X_{jt}) = \mu_{Xj} = \dfrac{\theta_j}{1-\alpha_j}, \quad j = 1,2$

ii. $Var(X_{jt}) = \sigma_{Xj}^2 = \dfrac{(\alpha_j + v_j)\theta_j}{1-\alpha_j^2}, \quad j = 1,2$

iii. $Cov(X_{jt}, X_{j,t+h}) = \gamma_{Xj}(h) = \alpha_j^h \sigma_{Xj}^2, \quad h = 1,2,...$

iv. $Corr(X_{jt}, X_{j,t+h}) = \rho_{Xj}(h) = \alpha_j^h, \quad h = 1,2,...$

Since θ_j, σ_j^2 *and* α_j are all positive, it follows that the mean, variance, and autocovariance functions can only have positive values. Depending on whether $v_j = 1, v_j \in (0,1)$, or $v_j > 1$, the variance is equal to the mean (equidispersion), smaller than the mean (underdispersion), or bigger than the mean (overdispersion)

Allowing for dependency between η_{1t} and η_{2t} while keeping all the previous assumptions constant leads to dependence between the two series that make up the BINAR(1) process. It can be demonstrated that the covariance between the innovations of the two series at time *t* completely determines the covariance between the current value of one process and the innovations of the other process at the same point in time *t*, regardless of the underlying joint distribution of (η_{1t}, η_{2t}).

2.2 The INMA Models

Another form of thinning base models is the integer-valued moving average (INMA) models. McKenzie (1986) and Al-Osh and Alzaid (1988) also independently made a significant advancement on INAR-type models by introducing a moving average (MA)

component in the model. They gave an INMA(1) model as

$$X_t = \mu_t + \beta \circ \mu_{t-1}, \quad t \in Z \tag{13}$$

where $\beta \circ \mu_{t-1}$ is thinning operator, operating on the previous values of the innovation μ_t and $\beta \in (0,1)$ a binomial thinning operator coefficient. Since its introduction, the INMA model has drawn considerable attention in the field of count time series analysis. The INMA model was extended into a vector INMA (VINMA) model by Quoreshi (2008). The spillover effects of trades from one stock to another were examined using the VINMA. For example, Brännäs and Brännäs (2004) and Rydberg & Shephard (1999) examine additional extensions to conventional count data regression models. Sunecher *et al.* (2018) introduced a first-order bivariate integer-valued moving average process (BINMA(1)) with corresponding innovation series that follow COM-Poisson distribution, with the aim of capturing varying degrees and kinds of dispersion within the INMA (1) framework. A novel bivariate integer valued moving average of the first order (BINMA(1)) process with independent negative binomial (NB) innovations under non-stationary moment circumstances was presented by Ristic *et al.* (2018). More flexible conditional mean and heteroskedasticity specifications for the INMA model are discussed and introduced by Brännäs & Quoreshi (2010), who then applied the model to tick-by-tick financial market data. High frequency data is produced by these tick-by-tick data.

Lack of synchronization in the sense that there are seldom several transactions at a time is another characteristic of such financial data. Later on, Quoreshi (2006, 2008, and 2014) created the integer-valued autoregressive fractional integrated moving average (INARFIMA) model to capture the extended memory phenomenon and the BINMA and VINMA models to address the synchronization issue.

McKenzie (1986) and Al-Osh & Alzaid (1988) also extended INMA(1) model to order *q* and gave an INMA(*q*) model as

$$X_t = \mu_t + \beta_1 \circ \mu_{t-1} + \beta_2 \circ \mu_{t-2} + \ldots + \beta_q \circ \mu_{t-q}, \quad t \in Z \tag{14}$$

which can also be written in compact form as

$$X_t = \mu_t + \sum_{j=1}^{q} \beta_j \circ \mu_{t-j}, \tag{15}$$

where μ_t is an i.i.d. sequence of non-negative integer-valued r.v's with finite mean and variance, and where $\beta_0, \beta_1, \ldots, \beta_q \in [0,1]$ with $\beta_q \neq 0$ and, usually, $\beta_0 = 1$. In analogy to the INAR(*p*) case, each μ_t is involved in *q*+1 thinning operators, at *q*+1 different times *t*, *t*+1,...,*t*+*q*. Hence, for the same set of thinning parameters $(\beta_0, \beta_1, \ldots, \beta_q)$, a wide variety of models exhibiting very different autocovariance functions and joint distributions can be constructed by only changing the dependence structure of the thinning operators. Such types of INMA(*q*) models were proposed by McKenzie(1988) , Al-Osh & Alzaid(1988) and Brännäs & Hall (2001)

2.3 The INARMA Models

Following the INMA(*q*) model, McKenzie (1986) and Al-Osh & Alzaid (1988) further defined the compact form of the integer valued autoregressive model of order *p,q* (INARMA(*p*,*q*)) as

$$X_t = \sum_{i=1}^{p} \alpha_i \circ X_{t-i} + \sum_{j=1}^{q} \beta_j \circ \mu_{t-j} + \mu_t, \tag{16}$$

where $\alpha_i, \beta_j \in [0,1] \forall i, j$, all thinning operators act independently and the stationarity condition is $\sum_{i=1}^{p} \alpha_i < 1$, $\sum_{j=1}^{q} \beta_j < 1$.

The motivation for INARMA models is the modification of the Guassian ARMA recursion through replacing the scalar multiplications by binomial thinning operations. Observing the properties of binomial thinning operator given by Steutel and van Harn (1979) the INARMA (*p,q*) was written recursively by Dion *et al.* (1995) as

$$X_t - \alpha_1 \circ X_{t-1} - \alpha_2 \circ X_{t-2} - \ldots - \alpha_p \circ X_{t-p} = \mu_t + \beta_1 \circ \mu_{t-1} + \beta_2 \circ \mu_{t-2} + \ldots + \beta_q \circ \mu_{t-q} \quad (17)$$

where μ_t is the innovation, a sequence of iid random variable with finite mean λ and variance $\phi\lambda$, ϕ is the dispersion index (the ratio of the variance to mean) and, $\alpha_i, \beta_j \in [0,1]$ for $i = 1,2,\ldots, p-1$, $j = 1,2,\ldots, q-1$ and $\alpha_p, \beta_q \in (0,1]$. Setting all $\alpha_i = 0$, Eq. (16) becomes the INMA(q) model while setting all $\beta_j = 0$, it becomes the INAR(p) model. The thinning operators are assumed to be performed independently of each other and of the innovation. It is important to exercise caution when dealing with Eq. (16). The joint distribution between the thinnings and the joint distribution of the thinnings to the other random variables in Eq. (16) must be carefully considered because the binomial thinning operations are random operations. It is the usual practice to create multiple independence assumptions in order to acquire realistic stochastic qualities. This is demonstrated in the sequel by going over both the INARMA(1,1) model and the purely autoregressive INAR(p) models. Although practitioners are familiar with the stochastic features of INAR models, the INARMA(1,1) model lacks such properties.

The INARMA(1,1) model is defined as

$$X_t - \alpha_1 \circ X_{t-1} = \mu_t + \beta_1 \circ \mu_{t-1}, \quad (18)$$

where the parameters $\alpha_1, \beta_1 \in [0,1]$ are used within the binomial thinning operations and these are assumed to be executed independently of each other. Hence,

$$Cov(\alpha_1 \circ W, \beta_1 \circ Z) = \alpha_1 \beta_1 Cov(W, Z) \quad (19)$$

The INARMA(1,1) model is a special case of generalized INARMA(p,q) model by Dion *et al.* (1995). For $\beta_1 = 0$, the model in Eq. (18) reduces to the INAR(1) model by McKenzie (1985), Al-Osh and Alzaid (1987), and for $\alpha_1 = 0$, it reduces to the INMA(1) model recently surveyed by Aleksandrov and Weiß (2019). But the model in Eq. (18) differs from the INARMA models considered by Dungey *et al.* (2019), McKenzie (1988) and Bracher(2019). The INARMA (1,1)-like model by McKenzie (1988) includes the AR-component only with lag 2 (i.e., with X_{t-2} instead of X_{t-1}), and the one by Dungey *et al.* (2019) applies the β_1-thinning to μ_t instead of μ_{t-1}. In Bracher (2019), the MA-component is constructed in a different way.

The moment properties of INAMA (1,1) as given by Weiß *et al* (2019) are

i. $E(X_t) = \dfrac{(1+\beta_1)\mu}{1-\alpha_1}$, the mean of X_t.

ii. $Var(X_t) = \dfrac{[\alpha_1(1+\beta_1) + \beta_1(1-\beta_1)]\mu + [\beta_1(\beta_1 + 2\alpha_1) + 1]v\mu}{1-\alpha_1^2}$, the variance of X_t.

iii. $Cov(X_t, X_{t+k}) = \alpha_1^k Var(X_t) + \alpha_1^{k-1} v\mu$, the autocovaraince function at lag $k \geq 0$.

iv. $$Cov(X_t, \mu_{t+k}) = \begin{cases} Var(\mu_t), & k = 0 \\ (\alpha_1 + \beta_1) Var(\mu_t), & k = -1 \\ 0, & otherwise \end{cases}$$

v. $Cov(\alpha_1 \circ X_{t-1}, \beta_1 \circ \mu_{t-1}) = \alpha_1 \beta_1 Cov(X_{t-1}, \mu_{t-1}) = \alpha_1 \beta_1 Var(\mu_t)$

2.4 Other Thinning Based Models

The concept of Long memory in time series was first conceived and demonstrated by Hurst (1951,1956) using the long term storage requirement of the Nile River in which the cumulated water flow in a year was found to persistently impact on the water flow for the subsequent years. In other to address long memory in time series, Granger (1980) developed an autoregressive fractional integrated moving average model of order *p,d,q* (ARFIMA(*p,d,q*)). Even though Bhardwaj and Swason (2006) discovered strong evidence in favour of ARFIMA model in absolute squared and log-squared stock index returns, the drawback of the model in modeling count time series is that the model does not produce integer valued result. Long memory in count time series implies long range dependence in the data (Quoreshia *et al.*, 2020). Thus to adequately model integer valued time series in the presence of long memory, Granger & Joyeux (1980) and Hosking (1981) constructed an integer valued autoregressive fractional integrated moving average model of order *p,d,q* (INARFIMA (*p,d,q*)) as an analogy to the ARFIMA model of Granger (1980). Doukhan *et al.* (2006) extended the class thinning based model and gave a nonnegative integer valued bilinear process, denoted by INBL(*p, q, m, n*), similar to the real-valued bilinear model and as well demonstrated the existence of strictly stationary process and its condition. Again, Moriña, *et al.* (2020) noted that intervention analysis is employed when interest lies in the evaluation of the impact of certain events (interventions) on a time series and thus proposed an intervention INAR(*p*) model analogous to the intervention model of the classical time series models, for modeling the effect of interventions in integer-valued time series. Popovic′(2021) proposed a mixed nonlinear INAR(1) model (MINLINAR(1)) on the bases of negative binomial thinning operator and geometric marginal distribution (innovation), in which the autoregressive component can be a one point linear or nonlinear at some other point.

All the models so far discussed assume that the observed count time series is stationary, that is, the statistical properties such as the mean and the variance of the series remain constant over time. Nonetheless, real-world situations frequently involve non-stationary processes. Asymptotic characteristics of non-stationary INAR models for non-stationary time series have been studied. Hellstrom (2001) offered small sample distributions for the Dickey-Fuller test statistic and concentrated on testing the unit root in INAR(1) models. A practically non-stationary INAR(1) process was examined by Ispéany *et al.* (2003). It was demonstrated that the conditional least squares estimator for the coefficient has a normal limiting distribution. A non-stationary homogeneous INAR(1) process with a slowly convergent autoregressive type coefficient was proposed by Gyorfi *et al.* (2007). In order to manage a non-stationary integer-valued time series with a significant dispersion, Kim and Park (2008) examined a procedure known as the integer valued autoregressive process with signed binomial thinning. Barczy *et al.* (2011) demonstrated that the sequence of suitably

scaled random step functions formed from an unstable INAR(p) process that converges weakly towards a squared Bessel process. Drost *et al.* (2009) investigated the asymptotic properties of this "near unit root" scenario and discovered that the limit experiment is Poissonian. Similar to studies of non-stationary processes for continuous time series models, Kaizhi *et al.* (2013) examined a new non-stationary INAR(1) process in which the innovation follows a qth-order moving average process (NSINARMA(1,q)).

The form of the above mentioned thinning based models are summarized in Table 2.

3.0 Thinning Operators for Thinning Based Models

One of the choices to make when constructing a thinning based model is that of the thinning operator to employ. Thinning operators are probabilistic operations performed on random counts which randomly thin down the members of a population (Jazi & Alamatsaz, 2012). The basis for choosing a particular thinning operator has been reported in several articles (Jazi & Alamatsaz, 2012; Resti c′ *et al*., 2009; Nasti c′ *et al*., 2012; Nasti c′ *et al*., 2017).

Table2. Other Forms of Thinning Based Models

Authors	Name of Model	Model	Remark
Granger & Joyeux (1980), Hosking (1981)	INARFIMA (p,d,q)	$\alpha(L^{\circ})X_t = \beta(L^{\circ})(1-L^{\circ})^{-d}\mu_t$ where the lag polynomials of order p and q are $\alpha(L^{\circ}) = 1-\alpha_1 \circ L - \alpha_2 L^2 - \ldots - \alpha_p \circ L^p$ and $\beta(L^{\circ}) = 1+\beta_1 \circ L + \beta_2 \circ L^2 + \ldots + \beta_q \circ L^q$, and the requires $\alpha_i, \beta_j \in [0,1]$ for $i>0$; $j \geq 0$	
Morina, *et al.* (2020)	Intervention INAR(p)	$X_t = \alpha_1 \circ X_{t-1} + \alpha_2 \circ X_{t-1} + \ldots + \alpha_p \circ X_{t-p} + W_t(\lambda')$ where $W_t(\lambda')$ is the innovations which reflect the new counts produced at time t, $\lambda' = \lambda + \delta . I_t$ and I_t is a dummy variable that assumes 1 if t is within an intervention period and 0 otherwise.	The model also helps in the evaluation of seasonal variation in the time series
Popovic′ (2021)	MINLINAR(1)	$X_t = \begin{cases} \alpha * X_{t-1} + \varepsilon_t, & \text{w.p. } p \\ \alpha * (X_{t-1}\varepsilon_t) + \varepsilon_t, & \text{w.p.} 1-p \end{cases}$ $\{X_t\}$ evolves as a linear model with probability p and as a non-linear model with probability $1-p$.	This model has a random thinning coefficient that determines whether the autoregressive component is linear or not.
Doukhan *et al.* (2006)	INBL(p, q, m, n)	$X_t = \sum_{i=1}^{p} a_i \circ X_{t-i} + \sum_{j=1}^{q} c_j \circ \varepsilon_{t-j} + \sum_{k=1}^{m}\sum_{\ell=1}^{n} b_{\ell k} \circ (\varepsilon_{t-\ell} X_{t-k}) + \varepsilon_t$,	

		where the operators $a_i \circ (i = 1,..., p)$, $c_j \circ (j = 1,..., q)$ and $b_{\ell k} \circ (\ell = 1,..., n, k = 1,..., m)$ are mutually independent.	
Kaizhi *et al.* (2013)	NSINARMA(1, q)	$X_t = X_{t-2} + \mu_t$ $\mu_t = \varepsilon_t + \alpha_1 \circ \varepsilon_{t=1} + \alpha_2 \circ \varepsilon_{t-2} + ... + \alpha_q \circ \varepsilon_{t-q}$ $\{\varepsilon_t\}$ is a sequence of i.i.d. nonnegative integer-valued random variables with finite mean $\mu_\varepsilon < \infty$, variance $\sigma_\varepsilon^2 < \infty$, and probability mass function f_ε.	All counting series $\alpha_r \circ \varepsilon_{t-r}$ are mutually independent, and $\alpha_r \in [0,1], r = 1,2,..,q$

The choice depends on the underlying process that generated the count time series. The binomial thinning operator "$\circ$", operational form $\alpha \circ X = \sum_{i=1}^{X} B_i; \alpha \in [0,1)$ where B_i is the i.i.d Bernoulli random variable (r.v), introduced by Steutel & van Harn (1979) was the first thinning operator to be employed in the construction of INAR models. This operator was used by McKenzie (1985) and Al-Osh & Alzaid (1987) for the construction of their novel INAR (1) model. The thinning operator is suitable when the thinning process of the population units assume Bernoulli distribution with range $[0,1]$. That is, when the units of the population enter and exit the system from one time interval to another (Nastić *et al,* 2012). An example of data that may be generated from a system where binomial thinning operation is appropriate may include hospital admission count, detention count etc. In the bid to further improve modeling of count time series using INAR models, Restić *et al*. (2009) introduced negative binomial thinning operator "$*$" with the form $\alpha * X = \sum_{i=1}^{X} G_i; \alpha \in [0,1)$ where G_i is the i.i.d geometric $(\alpha / 1 + \alpha)$ r.v . This operator is designed for a thinning process that follows geometric distribution with range $[1, \infty)$ and thus is suitable for systems involving much more activity, such as the count of sex offenses (Nastić *et al,* 2017). Several other thinning operators have been developed and used in the construction of INAR models as noted in Serrao *et al.* (2026). Table 3 gives an overview of some other thinning operators for INAR models, author and suitable area of application.

4. 0 INAR Models for Various Dispersions in Count Time Series

Three distinct dispersion structures- over-dispersion, equi-dispersion, and under-dispersion- represent the dispersion of data points around their mean. If the variance-to-mean ratio of a set of count time series is more than one, the set of data is considered over-dispersed; if the ratio is less than one, the data are considered under-dispersed; when the ratio is equal to one, the data are considered equi-dispersed.

The two primary methods suggested for dealing with over-dispersion include modifying INAR (1) models by utilizing alternate marginal distributions and thinning operators, and employing the binomial thinning operator to accommodate various innovation distributions (Serrao, 2026).

Table3. Some Thinning Operators for Thinning Based Models

Name	Author (Date)	Form of Thinning Operator	Area of Data Application
Signed Binomial thinning operator	Kim and Park (2008)	$\alpha \Theta X = \text{sgn}(\alpha)\text{sgn}(X)\sum_{i=1}^{\lvert X\rvert} W_i; \alpha \geq 0$ where W_i is the i.i.d Bernoulli r.v, $\text{sgn}(x) = 1$, if $x \geq 0$ or $\text{sgn}(x) = -1$, if $x < 0$	-Monthly counts of robbery. - Monthly numbers of new patients diagnosed with AID.
Signed generalized power series thinning operator	Zhang *et al.* (2010)	$\alpha \otimes X = \text{sgn}(\alpha)\text{sgn}(X)\sum_{i=1}^{\lvert X\rvert} W_i; \alpha \geq 0$ where W_i is the i.i.d generalized power series distribution.	Counts of drugs reported in the police car beat.
Generalized Binomial thinning operator	Ristic *et al.*, (2013)	$\alpha \circ_\theta X = \sum_{i=1}^{X} U_i; \alpha \in [0,1)$ where $U_i = (1 - V_i)W_i + V_i Z$ and V_i, W_i, Z are sequences of i.i.d Bernoulli r.v with parameters respectively as θ, α, α.	Monthly count of vagrancy offenses.
Generalized Poisson thinning operator	Kai and Yang *et al.*, (2019)	$(\lambda, k) \oplus X = \sum_{i=1}^{X} C_i$ where $\lambda > 0, \max(-1, -\lambda/m) < k < 1$ and C_i is the i.i.d generalized Poisson r.v	-Monthly claims count. -Monthly family violence count.
Gómez-Déniz-Sarabia-Calderín-Ojeda(GSC) thinning operator	Kang *et al.* (2020)	$\alpha \langle\rangle X = \sum_{i=1}^{X} W_i; \alpha < 1, \alpha \neq 0$ where W_i is the i.i.d GSC $(\alpha, \exp\{-\lvert\alpha\rvert\})$	-Monthly count of criminal mischief -Number of different IP addresses
Generalized Negative Binomial thinning operator	Shamma *et al.* (2020)	$\alpha \circ_\theta X = \sum_{i=1}^{X} U_i;\ 0 \leq \alpha \leq \theta \leq 1$ where $U_i = V_i Z$, V_i is the sequence of i.i.d geometric r.v with parameter $(\theta/(1+\theta))$ and Z is a Bernoulli r.v with parameter (α/θ).	Weekly count of Hantavirus.
Extended Binomial thinning operator	Liu and Zhu (2021)	$(m, \alpha) \Theta X = \sum_{i=1}^{X} U_i(m, \alpha),\ 0 \leq \alpha \leq 1, m \geq 2$ where $U_i(m, \alpha)$ is the sequence of i.i.d r.v with common distribution $EB(m, \alpha)$.	-Monthly tallies of crime data. -Stock data in New York Stock Exchange. -Crime of family violence.
Generalized Neyman Type A (GNTA)	Amiri *et al.* (2024)	$\alpha \circ_\theta X = \sum_{i=1}^{X} U_i; \alpha \in [0,1), \alpha \leq \theta e^{\theta}$	-Monthly counts of drugs calls. -Daily counts of COVID-19.

thinning operator	where U_i is the dependent zero-inflated Bell distribution (ZIBe) with parameter $\left(1-\alpha/\theta e^{\theta}, \theta\right)$.	-Weekly counts of Tulavemia disease.

Over-dispersion in count time series has been addressed by several INAR (1) models. The New geometric INAR (1) (NG-INAR (1)) model was first proposed by Resti c′ *et al.* (2012), which extends the Poisson INAR (1) (PINAR (1)) model, using a negative binomial thinning operator $(*)$. To model over-dispersion in count time series, Jazi *et al.* (2012a) presented the INAR (1) model with geometric innovations. Resti c′ *et al.* (2013) developed the first-order dependent counting geometric INAR model for over-dispersion by introducing a generalized binomial thinning operator $(\circ_{\theta})$.Weiß (2014) presented a compound Poisson INAR(1) (CP-INAR (1)) model with a binomial thinning operator $(\circ)$ as an additional development. Bourguignon (2013) introduced the INAR (1) model with a Poisson-Geometric marginal distribution, which extends the capabilities of the Poisson INAR(1) (PINAR (1)) and geometric INAR(1) (GINAR (1)) models using a two-step CLS estimation process. Building on this, a stationary INAR (1) model with Poisson-Lindley innovations was presented by Mohammadpour *et al.* (2018).

Small sample sizes can lead to under-dispersion (Sellers & Morris, 2017). Under-dispersed count data can be analyzed using a number of well-known parametric models. These models include the Conway-Maxwell Poisson (CMP) distribution created by Sellers *et al.* (2012), the Double Poisson distribution suggested by Efron (1986), and the Generalized Poisson distribution introduced by Cousul (1989). Two parameters define each of these models. Nevertheless, these models are not very good at explaining under-dispersion. Weiß (2013) investigated the performance of a stationary INAR (1) process with two-parameter Good and Power Law distribution innovations in order to address this problem. This process showed improved characteristics for simulating under-dispersed counts. But adding higher-order autoregressive structures to these models-like Jin-Guan & Yuan (1991) INAR(*p*)-introduces more complexity. Interestingly, the resulting INAR(*p*) process does not always maintain this quality in the observed data, even if the innovations are under-dispersed. This restriction points to a crucial area for further investigation. Under-dispersion emphasizes the need for specialist models to adequately address this phenomenon, although it receives little attention in practical data analysis.

It is also possible to model over-dispersion and under-dispersion simultaneously within INAR framework. In order to model over-dispersion and under-dispersion in count data simultaneously, a number of models have been developed. Bourguignon & Vasconcellos (2015) published an INAR (1) model with power series advances. MacDonald and Bhamani (2020) adapted the Poisson-hidden Markov model to address under-dispersion within a hidden Markov (HM) framework by substituting the CMP distribution for the Poisson distribution, creating the CMP-HM model. $(\diamond)$. Kang *et al.* (2020) presented a new thinning operator based on the GSC distribution $(\diamond)$. To handle large-tailed count time series, Qian and Zhu (2020) created an INAR (1) model that integrates Generalized Poisson inverse Gaussian innovations into the

INAR(1) framework. This method can handle both equi-dispersion and over-dispersion and performs well with count data that has a high frequency of zero values. In a study similar to Kang *et al.* (2020), Liu & Zhu (2021) developed an extended binomial INAR(1) (EBINAR(1)) that simultaneously models over-dispersion and under-dispersion and introduced an extended binomial thinning operator ($\circledcirc$) with two parameters, providing more modeling flexibility. Later, Qian and Zhu (2023) used the generalized CMP distribution to create a flexible INGARCH model that adds a parameter to control heavy tail behavior. A zero-distorted generalize geometric INAR(1) (ZDGG-INAR (1)) model was recently presented by Kang *et al.* (2024). This model effectively represents all the features, including zero-inflation/deflation, zero truncation, and dispersion, while addressing the shortcomings of the generalized Poisson-inverse Gaussian INAR(1) (GPIG-INAR (1)) model.

When analyzing different kinds of count data, models that take into account all three types of dispersion give a great deal of flexibility and broad application. Both zero-inflation and zero-deflation in count data can be efficiently handled by the CMP model of Francis (2012). Subsequently, Burguignon & Weiß (2017) created an INAR (1) framework that offers more flexibility for count data modeling by integrating Bernoulli-Geometric marginal distributions and using a generalized thinning operator $(\circ_\theta)$. In a similar vein, Kim & Lee (2017) used Katz family innovations to suggest an INAR (1) model that combines binomial, Poisson, and negative binomial distributions to reflect a range of dispersion patterns. Bourguignon *et al.* (2018a) extended the INAR (1) process by creating the double Poisson INAR(1) (DP-INAR (1)) and generalized Poisson INAR(1) (GP-INAR (1)) models, which were inspired by the Double Poisson and Generalized Poisson distributions' capacity to capture a variety of dispersion patterns and their comparatively simple probability mass functions. This model captures various dispersion patterns using binomial thinning $(\circ)$ and Poisson innovations. Yang *et al.* (2019) investigated a related extension that included a generalized Poisson thinning operator ($\circledast$). Furthermore, a shifted INAR (1) model with Borel innovations and binomial thinning $(\circ)$ was proposed by Cunha *et al.* (2021). More recently, Mohammadi *et al.* (2024) introduced the weighted negative binomial Lindley (WNBL) distribution-based INAR(1) process, which combines binomial thinning operator $(*)$ with WNBL innovation. The innovation distribution is built as a weighted combination of the size-based and traditional negative binomial distributions, providing flexibility through adjustable parameters.

5.0 INAR Models for Zero Inflated or Deflated Count Time Series

Zero-inflation, a term coined by Neyman (1939) and Feller (1943), is the term for count data that often exhibit an excess of zeros. To solve the problem of zero-inflation in count data, zero-inflated models were created, which combine a count distribution like Poisson or negative binomial with a degenerate distribution at mass zero. Lambert (1992) introduced a generalized linear model utilizing the Zero-Inflated Poisson (ZIP) distribution, which was specifically created for a manufacturing process where additional zeros resulted from transitions between flawless and defective production states. This model significantly addressed the problem of zero-inflation. For datasets exhibiting equi-dispersion, the ZIP model of (Lambert, 1992) is suitable. Other distributions, such as the zero-inflated negative binomial (ZINB) distribution (Ridout *et al.*, 2001) and the zero-inflated CMP distribution (Sellers & Raim, 2016), have been created as alternatives for datasets with a large prevalence of zero counts. The ZINB model takes into account population

heterogeneity, where the likelihood of zero counts vary among people, in address both excess zeros and over-dispersion. The ZINB model is better suited for datasets with both excess zeros and over-dispersion than the ZIP model, which is primarily intended for equi-dispersed data (Lambert, 1992). The construction of several INAR processes to simulate zero-inflated and zero-deflated count time series is one recent breakthrough in modelling count data using INAR models. Jazi *et al.* (2012b) presented the zero-inflated Poisson INAR(1) (ZIP-INAR (1)) model, which combines ZIP innovations with the INAR (1) process to handle excess zeros. Zhu (2012) presented related expansions within the INGARCH models, including ZIP and negative binomial INGARCH models that were fitted to real-world datasets using the EM technique. Yang *et al.* (2015) presented a flexible state-space model that makes use of the ZINB distribution within the state-space framework. The study used a Monte Carlo-based Expectation-Maximization algorithm, which incorporates particle filtering and smoothing for parameter estimation, due to the drawbacks of conventional estimation methods like the Kalman filter and Kalman smoother. To facilitate the use of this model, the authors also created the R package Zero Inflated Models(ZIM). Maiti *et al.* (2015) examined a new ZIP-INAR (1) model that integrated an independent Bernoulli process with the PINAR (1) process. Barreto-Souaza (2015) proposed the zero modified geometric marginal INAR(1) (ZMG-INAR (1)) model and presented a test for identifying zero-inflation/deflation based on the model in order to overcome the flaws of the ZIP-INAR (1) model.

Barreto-Souza (2015) introduced the ZMG-INAR (1) model, which has a complex innovation structure because of its parameter restrictions. Bourguignon (2018) used binomial thinning $(\circ)$ in an INAR (1) model with ZMG to overcome these restrictions. Equi-dispersion, under-dispersion, and over-dispersion are also supported by this model, and its adaptability is increased by including the GINAR (1) model as a special case. Furthermore, Moller *et al.* (2018) extended the binomial autoregressive model to accommodate a variety of zero patterns by introducing four variants: BAR (1) Process with zeros at random, BAR (1) Process with innovational zeros, Zero inflated binomial thinning for AR (1) model and BAR (1) Process with zero threshold. These models use maximum likelihood estimation to handle different zero structures in count data. Furthermore, Bourguignon *et al.* (2018b) suggested a new thinning operator based on zero-modified geometric counting series and the generalized zero-modified geometric distribution. Bakouch *et al.* (2018) investigated an INAR-based model that included a zero-inflated geometric distribution with random coefficients and a randomized binomial thinning operator $(\circ)$. Recently the concept of zero inflation (deflation) was addressed in seasonal count time series by Almuhayfith *et al.* (2023). The authors developed a seasonal integer-valued autoregressive model for count time series and discussed the probabilistic properties of the general seasonal INAR(p) while applying the first order of the model to influenza data. Marinho *et al.* (2024) employed zero-modified distribution in the family of Power Series in developing models and determined the suitability of the models for modelling count time series that exhibit both zero-inflation and zero-deflation using a full Bayesian method through the Hamiltonian Monte Carlo (HMC) technique.

6.0 Relevant Tests of Hypothesis and Procedures for INAR Models

When employing thinning-based models several tests of hypotheses may be required. The first important test is the test for INAR effect, that is a test to determine if the given

count data exhibits autoregressive behavior ($H_0 : \alpha = 0$ (serial dependence) against $H_1 : \alpha > 0$ (INAR process)). The score test has been used to determine the existence or otherwise of serial dependence in a set of count data. Freeland (1998) obtained the asymptotic distribution of a score test for Poisson INAR(1) model. In comparing the score tests and other nonparametric tests for INAR models, Jung and Tremayne (2003) discovered that modified score tests compete well with its nonparametric counterparts. Explicit formulas for the score test statistic for various types of innovation distributions were derived by Sun and McCabe (2013). In order to investigate size and power, they also conducted simulation studies, just like Jung and Tremayne (2003) have already done, pertaining to the score test and some non-parametric tests. In terms of size, a modified version of the score test was shown to be quite competitive with the non-parametric tests. Furthermore, it was perceived as having more strength under an INAR(1) option. Additionally, they looked at the power under INAR(2) framework of Alzaid and Al-Osh (1990). In this instance, they discovered a reduction in power when the ACF exhibits oscillating behavior. Sten (2019) developed approximate Likelihood Ratio (LR) tests based on score tests and proposed asymptotically similar modified statistic. Awale *et al.* (2019) applied score tests to assess the constancy of the thinning parameter in a random coefficient INAR model. Rolf (2020) carried out a score test for serial dependence for INAR(1) model with Poisson innovation and proposed different explicit approximations of the likelihood ratio (LR) statistic and derived the limiting distributions of the likelihood ratio (LR) statistic under the null hypothesis. In a simulation study, also Rolf (2020) also compares the size and power of our tests with the score test proposed by Sun & McCabe (2013).

Whether a particular count data follows Poisson distribution or not is of interest in the use of an INAR model. This can be determined through the dispersion index, which is equal to one when the data is Poisson, thus equi-dispersed, or greater than one, meaning over-dispersion, or less than one for under-dispersion (Jazi *et al.*, 2012a). However a number of authors have developed techniques to check for over-dispersion in INAR models. These techniques mainly involve examining the dispersion index (I_d). Schweer & Weiß (2014) introduced a test for over-dispersion in INAR models. They noted that the most accepted index is Fisher's index of dispersion of Fisher (1950), which measures the ratio of variance and mean, i.e., $f_d\left(\mu,\sigma^2\right)=\sigma^2/\mu \rightarrow I_d=\sigma^2/\mu$, with sample estimate given by $\hat{I}_d = S^2/\overline{X}$ with range $[0,\infty)$. If $I_d = 1$ then the count data is equi-dispersed and the null hypothesis of Poisson distribution holds. $I_d > 1$ implies over-dispersion while $I_d < 1$ indicates under-dispersion and in both cases the null hypothesis Poisson distribution does not hold. For details on the application of Fisher's index of dispersion in testing over(under)-dispersion of count data see ((Fisher, 1950; Rao & Chakravarti, 1956; Böhning,1994; Schweer & Weiß, 2014 Weiß & Schweer, 2015).

The use of the proportion of zero $\hat{\rho}_0 = \frac{1}{T}\sum_{t=1}^{T} 1_{\{X_t=0\}}$ in a given count data, in the detection of departure from Poisson (over(uder)-dispersion) can be seen Sáez-Castillo & Conde-Sánchez (2015). In this case $\rho_0 = e^{-\mu}$ implies a Poisson distribution; otherwise (i.e for any value of ρ_0 greater of less than $e^{-\mu}$) there is departure from Poisson which may be over-dispersion or under-dispersion. Sáez-Castillo & Conde-Sánchez (2015) further gave a general Poisson zero index of the form $f_Z\left(\hat{\rho}_0, \overline{X}\right)$, where

$f_Z : \Re^2 \to \Re$ is a real valued function such that $f_Z\left(e^{-\mu}, \mu\right), \forall \mu$ and also implies that $f_Z(\rho_0, \mu) \neq 0$ if $\rho_0 \neq e^{-\mu}$. Puig and Valero (2006) gave a particular case of the general Poisson zero index and designate it as *PV* zero index (I_{pv}). The I_{pv} is expressed as $f_Z(\rho_0, \mu) = 1 + \text{In}\rho_0/\mu \to I_{pv} = 1 + \text{In}p_0/\mu$ with sample estimate as $\hat{I}_{pv} = 1 + \text{In}\hat{p}_0/\overline{X}$ whose range is $(-\infty, 1]$. A set of count data is zero inflated or deflated and hence does not follow Poisson distribution when the value of I_{pv} exceed 1 while the data is distributed as Poisson if $I_{pv} = 0$. The numerator of the score test given by van den Broek (1995) gave another form of Poisson zero index as $f_Z(\rho_0, \mu) = \rho_0 e^{\mu} - 1 \to I_{vdB} = \rho_0 e^{\mu} - 1$ which is estimated by $\hat{I}_{vdB} = \rho_0 e^{\overline{X}} - 1$ with range $[-1, \infty)$ such that if $\hat{I}_{vdB} = 0$,then the underlying distribution of a count data is Poisson, otherwise, the data set is over(under)-dispersed. Weiß *et al.* (2016) argued that the above mention indexes on individual bases may not suffice in the detection of departure from Poisson distribution. They cited a case where, for a count data, the $I_{\text{pv}} = I_{\text{vdB}} = 0$ and $I_{\text{d}} = 1$ and yet the data was also found not to be distributed as Poisson and therefore suggested the use of the approximately bias-corrected form of the indexes simultaneously. See Weiß *et al.* (2016) for the asymptotic distribution of the joint model of the indexes and their approximately bias-corrected form.

Another important test when using thinning based models in modelling count time series is the test for structural change. For instance, the parameter (thinning coefficient) of INAR(1) model is assume to be constant. This condition is practically restrictive since it has been noted that some real life situations may demand that the thinning coefficient be random (Joe, 1996; Zheng *et al.*, 2007). Szabo′ (2011) introduced a detection procedure for change in parameter of integer valued autoregressive processes and carried a simulation study to investigate the power of the test in order to determine how effectively the test procedure can detect changes in the main parameters of the process. According Szabo′ (2011) the test procedure involve splitting the parameter vector of an INAR(p) model , $\theta_k := [\alpha_{1,k}, \alpha_{2,k}, ..., \alpha_{p,k}, \mu_{\varepsilon,k}]^T$ into parameter vectors ϕ_k and nuisance vectors η such that $\theta_k = (\phi_k, \eta_k)$. The null hypothesis to be tested is;

$$H_0 : \theta_1 = \theta_2 = ... = \theta_n \text{ and } \varepsilon_1, \varepsilon_2, ..., \varepsilon_k \text{ are i.i.d}$$

against the alternative

$$H_1 : \exists K^* \in [1, ..., n-1] : \phi_1 = \phi_2 = ... = \phi_{k^*} \neq \phi_{k^*+1} = ... = \phi_n,$$

where $\eta_1, \eta_2 ..., \eta_k$, $\varepsilon_1, \varepsilon_2 ..., \varepsilon_k$ and $\varepsilon_{k^*+1}, .., \varepsilon_n$ are i.i.d.

The test statistic according to Szabo′ (2011), rely on the sequence

$$M_t = X_t - E(X_t | \delta_{t-1}) = X_t - \alpha_1 X_{t-1} - ... - \alpha_p X_{t-p} - \mu_\varepsilon,$$

from which the CLS estimate can be obtained $\hat{\theta}^{(t)} := [\hat{\alpha}_1^{(t)}, ..., \hat{\alpha}_p^{(t)}, \hat{\mu}_\varepsilon^{(t)}]$ and hence the corresponding stochastic process given by

$$\hat{M}_t^{(n)} = \hat{I}_t\left(\hat{\theta}^{(t)}\right)^{-1/2}\sum_{k=1}^{[nt]}\begin{bmatrix}\hat{M}_k^{(t)}X_{k-1}\\ \vdots \\ \hat{M}_k^{(t)}X_{k-p}\\ \hat{M}_k^{(t)}\end{bmatrix},\ t\in Z, n\in[0,1]$$

where $X \in Z$ is a non-negative integer real number, $\hat{I}\left(\hat{\theta}^{(t)}\right)$is a strongly consistent estimator analogous of the Fisher information matrix.

The test procedure can simultaneously test change in several parameters. However, if *d* different test are performed together with significant level $1-\alpha^*$ each, then for each test H_0 is accepted with the probability$1-\alpha^*$, if all the independent *d* tests are not significant. The *d* tests can be carried out simultaneously with a probability of accepting H_0 as $\left(1-\alpha^*\right)^d$ thus leading to a significance level of $\alpha^* = 1-\left(1-\alpha\right)^{1/d}$. Based on this level of significance, Szabo′ (2011) gave the decision rules for three different constructs of a test of a change in parameters as follows;

i. One tailed test:

If $\underset{0\le n\le l}{\text{Sup}}\,\hat{M}_t^{(i)}(n) > Q_1\left(1-\alpha^*\right)$, then H_0 is rejected and one can conclude that there is a downward change in the ith parameter. However, If $\underset{0\le n\le l}{\inf}\,\hat{M}_t^{(i)}(n) < -Q_1\left(1-\alpha^*\right)$, then H_0 is rejected and one can conclude that there is an upward change in the ith parameter.

ii. Two tailed test:

If $\underset{0\le n\le l}{\text{Sup}}\left|\hat{M}_t^{(i)}(n)\right| > Q_2\left(1-\alpha^*\right)$, then H_0 is rejected which implies that there is a change in the ith parameter.

iii. Temporary Change:

If $\underset{0\le n\le l}{\text{Sup}}\,\hat{M}_t^{(i)}(n) - \underset{0\le n\le l}{\inf}\,\hat{M}_t^{(i)}(n) > Q_3\left(1-\alpha^*\right)$, then H_0 is rejected meaning that there is a temporary change in the ith parameter.

Szabo′ (2011) discovered that the test may identify changes in any of the parameters mentioned. However, for modifications that take place relatively near the start and finish of the trajectory $X_1, X_2, ..., X_t$, the power of the test diminishes. A shift in mean of the stationary distribution significantly increases the power of the test and increases the likelihood of detecting larger changes. The power of the test is influenced by the absolute, not the relative, magnitude of the change of innovation. Tests that are one-sided have greater power than those that are two-sided. Unfortunately, there are also a number of issues with the test. If one of the other parameters changes significantly, the individual test will mistakenly identify change. The CLS fitted process has a high variance and the test fails nearly all the time if the coefficient is near to 1 or the change in the parameters is significant enough. Consequently, the test works best when there is a slight change in the parameters and the coefficient is not too near to 1.

7.0 Estimation Methods for INAR Models

The conventionally used estimation methods for the estimation of the parameters of INAR models are Yule-Walker (YW), conditional least squares (CLS), maximum likelihood (ML) and conditional maximum likelihood (CML) estimations.

7.1 Yule-Walker (YW) estimation

Let $\hat{\gamma}(k) = \frac{1}{T}\sum_{t=2}^{T}\left(X_t - \overline{X}\right)\left(X_{t+k} - \overline{X}\right)$ be the sample autocovariance function at lag *k* of an INAR process, where $\overline{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$ is the sample mean. Let $\hat{\rho}(k) = \hat{\gamma}(k)/\hat{\gamma}(0)$be the

corresponding autocorrelation. Then given the INAR(1) model in Eq.(1) the YW estimator for the parameter α is given by

$$\hat{\alpha}_{YW} = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} = \frac{\sum_{t=2}^{T}\left(X_t - \overline{X}\right)\left(X_{t-1} - \overline{X}\right)}{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)^2} \tag{20}$$

This result is common for INAR(1) models. However the YW estimator of the parameter of the innovation distribution depends on the discrete distribution adopted. For instance, the YW estimator of the innovation distribution parameter λ of the PoINAR(1) of McKenzie (1985) and Al-Osh & Alzaid (1987) is given by

$$\hat{\lambda}_{YW} = \left(1 - \hat{\alpha}_{YW}\right)\overline{X} \tag{21}$$

Du and Li (1991) state that since $\hat{\gamma}_{YW}$ and $\rho(k)$ are strong consistent, therefore $\hat{\alpha}_{YW}$ and $\hat{\lambda}_{YW}$ are also consistent estimators. With respect to T, the estimators $\hat{\alpha}_{YW}$ and $\hat{\lambda}_{YW}$ are asymptotically unbiased normally distributed, with variances provided by (Silva *et al*, 2005).

7.2 Conditional least squares (CLS) estimation

Klimko & Nelson (1978) introduced the conditional least squares (CLS) approach, which has been widely applied in the context of count time series analysis, especially for estimating the parameters of INAR processes (Du & Li, 1991).

The CLS estimators of the parameters of an INAR model are obtained by minimizing the criterion

$$Q(\psi) = \sum_{t=2}^{T}\left[X_t - g(\psi)\right]^2, \tag{22}$$

Where $g(\psi) = E\left(X_t \middle| X_{t-i}\right)$ is the conditional mean of X_t given X_{t-1} and can be expressed for PoINAR(1) as $g(\psi) = E\left(X_t \middle| X_{t-i}\right) = \alpha X_{t-1} + \lambda$.

Minimizing Eq. (22) results in obtaining the estimator of α for any INAR(1) model as

$$\hat{\alpha}_{CLS} = \frac{(T-1)\sum_{t=2}^{T} X_t X_{t-1} - \sum_{t=2}^{T} X_t \sum_{t=2}^{T} X_{t-1}}{(T-1)\sum_{t-1}^{T} X_t^2 - \left(\sum_{t=2}^{T} X_{t-1}\right)^2}, \tag{23}$$

and assuming Poisson innovation distribution, the estimator of the parameter λ is given by

$$\hat{\lambda}_{CLS} = \frac{\sum_{t=2}^{T} X_t - \hat{\alpha}_{CLS}\sum_{t=1}^{T} X_{t-1}}{T-1}. \tag{24}$$

7.3 Maximum likelihood and Conditional Maximum Likelihood estimations

The log-likelihood function of an INAR(1) model for a fixed value X_1 is given by

$$\ell(\psi) = \sum_{t=2}^{T} \log \Pr\left(X_t \middle| X_{t-1}\right), \tag{25}$$

where $\Pr(X_t|X_{t-1})$ is the conditional density of X_t given X_{t-1}. For PoINAR(1) model, the conditional density can be expressed as

$$\Pr(X_t|X_{t-1}) = \exp(-\lambda)\sum_{k=0}^{\min(X_{t-1},X_t)} \frac{\lambda^{X_t-k}}{(X_t-k)}\binom{X_t}{k}\alpha^k(1-\alpha)^{X_t-k}. \quad (26)$$

The steady state density corresponding to Eq. (26) is

$$\Pr(X_1) = \left[\frac{\lambda}{1-\alpha}\right]^{X_1}\frac{1}{X_1!}\exp\left[-\frac{\lambda}{1-\alpha}\right].$$

The exact ML estimators of the parameter of an INAR(1) model minimize the value of the function

$$\ell(\psi) = \ell_1(\psi) + \Pr(X_1),$$

while the CML estimators minimize Eq.(25) directly.

The ML and the CML estimators of INAR models most often lack closed-form expressions and are often obtained using numerical process. However the ML is effective for small sample sizes where the effect of the initial data point counts while the CML performs better in large sample sized (Wang & Chen, 2019).

Several authors have employed these methods in the estimation of the parameters of INAR models. The YW, the CLS, and the MLE are important estimation techniques employed by several aurthors that use INAR model. Jazi *et al.* (2012a) introduced INAR (1) model with geometric innovations and used CML and approximate full maximum likelihood (AFML) methods to estimate its parameters. The CLS and the MLE techniques were used by Restic *et al.* (2012) to estimate INAR(1) parameters. Francis *et al.* (2012) highlighted the flexibility of the CMP model by employing MLE for estimate. Jazi *et al.* (2012b) used CML and AFML estimation techniques to estimate the parameters of the INAR (1) process with ZIP innovations. Maiti *et al.* (2015) used YW, CLS, and MLE estimation techniques in conjunction with an independent Bernoulli process and the PoINAR (1) process. Vasconcellos (2015) used YW, CLS, and MLE to estimate the parameters of the INAR (1) model with power series innovations. Kim & Lee (2017) used MLE to estimate the parameters of the model. The intricacy of the marginal distribution, which frequently lacks a closed-form representation, makes the practical application of both CML and MLE approaches difficult. In their stationary INAR (1) model with Poisson-Lindley innovations, Mohammadpour *et al.* (2018) used CLS, YW, and MLE for estimation. The DP-INAR (1) model was proposed by Bourguignon *et al.* (2018a), and techniques including CML, CLS, and YW were used to estimate the parameters. Bourguignon (2018) used CML, YW, and CLS to estimate the parameters of the INAR (1) method using ZMG innovations. CLS and MLE approaches are used by Bakouch *et al.* (2018) to analyze count time series with complicated patterns with more flexibility. Yang *et al.* (2019) used CLS, modified quasi-likelihood (MQL), and CML approaches to estimate model parameters for INAR(1). Qian and Zhu (2020) employed CML approach for parameter estimation while modeling count data with a large frequency of zero values using the INAR(1) model. Kang *et al.* (2020) used CLS, weighted CLS, and MQL techniques to estimate the model's parameters while creating an INAR (1) model. In addition to discussing the CML estimation approach, Zhu (2021) introduced two parameter EBINAR(1) and used two-step CLS estimation to obtain the parameter estimates. The ZDGG-INAR (1) model of Kang *et al.* (2024) is a flexible tool for managing different count data features inside the INAR framework since its parameters were evaluated utilizing techniques

including YW, CLS, CML, and Bayesian approaches. The CML approach was used to estimate the parameters of the INGARCH model provided by Qian and Zhu (2023). YW, CLS, and CML techniques were used in the Cunha *et al.* (2021) to estimate the parameters INAR(1) model. In order to demonstrate the practical applicability and versatility of the INAR(1) process, model parameters were estimated using CML, YW, and CLS approaches and applied to real-world datasets from criminology and epidemiology (Mohammadi *et al.,* 2024).

Some novelty estimation approaches for the parameters of INAR models have been proposed and comparison to the conventional approaches. The stochastic approximation algorithm (SAA), which iteratively updates the parameters of an INAR model using the gradient of the log-likelihood function, was presented by Wang and Chen (2019) as a novel technique for estimating the parameters of integer-valued autoregressive (INAR) models. They argued that the maximum likelihood estimation (MLE) approach may not be effective for large datasets because it requires addressing a non-linear optimization problem and can be computationally demanding. The authors use simulation experiments and real-world data to show how effective their proposed approach is. They demonstrate that, particularly when the sample size is large, their approach can converge to the genuine parameter values faster than conventional MLE methods. Li & Li (2020) suggest a Bayesian method for estimating the parameters of an INAR model with missing values using the Markov Chain Monte Carlo (MCMC) technique. The fully conditional specification on which the MCMC algorithm is based enables effective INAR model estimation and prediction. The authors show that their Bayesian estimation method beats conventional MLE methods in terms of estimation accuracy, and they can accurately estimate the parameters of the INAR model even when some data points are missing. Additionally, they show that even in cases when the data contains missing values, their approach can produce precise forecasts for the future values of a count time series. The effectiveness of several techniques for estimating the parameters of integer-valued autoregressive (INAR) models is compared by Chen and Wang (2020). MLE, Bayesian inference using MCMC, Bayesian inference using Laplace approximation, neural network-based estimation, and a hybrid approach that combines MLE and MCMC are the five methods for estimating INAR models that the authors compare in terms of performance. They assess the accuracy of each method, computing efficiency, and robustness using a simulation analysis. Additionally, they use real-world data to compare the performance of each method. Their findings demonstrate that although the Bayesian inference using MCMC approach is computationally demanding, it performs well in terms of accuracy and robustness. Although the neural network-based estimation approach is computationally efficient, its accuracy may be inferior to that of the Bayesian methods. Again the hybrid approach incorporates the advantages of MCMC and MLE; its robustness may not match those of the Bayesian approaches. The authors also discover that the particular features of the data, such as the sample size, the number of parameters, and the degree of noise in the data, influence the technique selection (Chen and Wang, 2020). A hybrid Bayesian-neural network method for estimating the parameters of integer-valued autoregressive (INAR) models is proposed by Li & Li (2020). To estimate the parameters of INAR models, the authors suggest a hybrid method that combines the advantages of neural networks and Bayesian inference. The neural network component is employed to increase the accuracy and efficiency of estimation, while the Bayesian component offers a flexible and



reliable framework for modeling the uncertainty in the parameter estimates. The authors create a hybrid algorithm that consists of two primary steps: first, an MCMC algorithm is used to train a Bayesian neural network to estimate the parameters of an INAR model; second, the conditional mean and variance of the model are estimated using the posterior distribution of the parameters. They use real-world data and simulated studies to show how effective their approach is. They demonstrate how their approach can outperform conventional Bayesian methods in terms of computational efficiency and accuracy, as well as accurately estimate the parameters of INAR models with complicated topologies. By contrasting their approach with other Bayesian and neural network-based approaches, they also show how flexible their method is. They demonstrate that even in cases when the data is noisy or contains missing values, their approach can yield more reliable and precise estimations of the parameters of INAR models. Recently, Bamdadi & Mohammadpour (2024) employed saddle point maximum likelihood (SPML) method in the estimation of the parameters of their integer-valued bilinear model and found, through simulation study, that the method is an efficient approach to estimation of parameters of integer- valued processes.

8. 0 Forecast Methods for INAR Models

One of the key applications of the fitted INAR model is to forecast future outcomes of the process. Specifically, for an observed INAR(1) process $X_1, X_2, ..., X_T$, one may wish to obtain the future value of the process X_{T+h} for some forecast horizon $h \geq 1$. This is achieved this, the conditional mean, which optimal in the sense of the mean squared error (MSE) (Gneiting 2011), is often employed.

The h-step-ahead conditional mean for the INAR(1) model in Eq.(1) is given by

$$\begin{aligned} E\left(X_{T+h}|\delta_T\right) &= E\left(\alpha \circ X_{T+h-1} + \varepsilon_{T+h}|\delta_T\right) \\ &= E\left(\alpha \circ \left(\alpha \circ X_{T+h-2} + \varepsilon_{T+h-1}\right) + \varepsilon_{T+h}|\delta_T\right) \\ &\overset{d}{=} .... \\ &\overset{d}{=} E\left(\alpha^h \circ X_T + \sum_{j=0}^{h-1} \alpha^j \circ \varepsilon_{T+h-j}|\delta_T\right) \\ &\overset{d}{=} E\left(\alpha^h \circ X_T|\delta_T\right) + E\left(\sum_{j=0}^{h-1} \alpha^j \circ \varepsilon_{T+h-j}|\delta_T\right) \\ &\overset{d}{=} \alpha^h X_T + \mu_\varepsilon \frac{1-\alpha^h}{1-\alpha} \end{aligned} \tag{27}$$

Freeland and McCabe (2004), McCabe & Martin (2005), Jung & Tremayne (2006), Silva *et al.* (2009), McCabe *et al.* (2011) and Bisaglia and Canale (2016) observed that the major drawback in the use of conditional mean in the forecast of the future values of an INAR(1) process is that it returns non-discrete values while surely will be a count value and thus suggested coherent forecasting techniques that produce forecast value in N = {0,1,2,…}. To achieve coherent forecast requires the computation the conditional distribution of X_{T+h} given the past values of and the derivation of an integer value and the forecast. The most common coherent central point forecast is the conditional median, which minimizes the mean absolute error (MAE) and thus satisfies optimality property (Gneiting 2011).The median of the k-step-ahead conditional distribution is used by Freeland & McCabe (2003) to achieve the goal of

maintaining the integer structure of the data in forecasts. A broad framework for generating coherent predictions of low count data is developed by McCabe & Martin (2005). They permit the arrivals to follow any other distribution in the integer class, in contrast to the typical applications of the model INAR(1), where the arrival process is typically Poisson. An estimate of the k-step–ahead predictive probability mass function serves as the foundation for the forecasts. Bayesian techniques are applied to remove undesired values from the conditioning set of the forecast function. By suggesting a computer-intensive technique for producing coherent, integer out-of-sample predictions, specifically obtaining the h-step-ahead predictor for the INAR(2), Jung & Tremayne (2006) expand on some of the concepts employed by Freeland and McCabe (2004) in higher-order dependent structure. They suggest a Monte Carlo method that respects the integer structure of the data by estimating the sample distributions as a mean of producing one-step and multi-step-ahead forecasts using bootstrap techniques.

The conditional mode may also be important in certain situations, irrespective of the fact that the conditional median aims for a consistent central forecast. A non-central point forecast may be more desirable in some applications. For instance, when evaluating a loss distribution in risk analysis, especially large outcomes of a process X_t are seen as an undesirable event (such as a high number of defects, complaints, etc.) and are therefore considered a risk (Göb 2011). Homburg *et al.*(2019) noted that one of the most popular risk measures for a fixed risk level $\rho \in (0,1)$ is the Value at Risk (VaR_p), which is defined as the ρth quantile of the (loss) distribution, $VaR_p = \min\{x \in \mathrm{N}_0 | P(X/ < x) \geq \rho\}$ is regarded as a threshold that is only exceeded in the worst $(1-\rho)100\%$ circumstances. Given the previous values $X_T,...,X_1$, the VaR_p derived from the conditional distribution of X_{T+h} is a non-central (and coherent) point forecast. Just like the central median forecast, the conditional VaR_p can be estimated, in an approximate manner, by discretizing the conditional VaR_p obtained from a real-valued process model or can be calculated based on the actual count process model, resulting in a coherent forecast. While there are methods for computing coherent point forecasts for count processes, they are frequently disregarded in practice in favor of real-valued time series methods (with discretization). It is therefore reasonable to inquire about the performance of these approximations in comparison to coherent forecasting methods. This question has been addressed only in a small number of works in literature. Central point projections based on specific datasets of traffic incidents in Great Britain were examined by Quddus (2008). It turned out that a discrete forecasting model exhibits the best predictive performance, particularly for disaggregated data sets. Bisaglia & Gerolimetto (2015) found that the Gaussian AR(1) approximation should be avoided after conducting a simulation study to compare the median forecast for a so-called PoINAR(1) process with the rounded-mean forecast. Alwan & Weiß (2017) examined quantile predictions for INAR(1) processes in the context of demand prediction and demonstrated that, when compared to coherent model-based forecasts, approximate demand forecasts typically result in much higher costs. A more mixed picture can be seen in the simulation findings provided by Maiti *et al.* (2016), who once more compared Gaussian AR approximations to AR-like count processes. Although the approximation fared similarly well with regard to (non-coherent) mean forecasts, it performed badly with regard to quantiles. It would be premature to

categorically advise against the approximation technique, even though these studies suggest that such discretized approximate point forecasts would perform very poorly.

9. 0 Possible Areas for Future Development of Thinning-Based Models

Despite the tremendous works that have been done for over four decades in the development of thinning based models for the analyses of count time series, there also exist a whole lot of areas that are either scarcely developed or completely unattended to. The possible areas to explore in future researches may include the following;

i. An important area that is scarcely developed in the INAR framework is the non-stationary INAR models. Most existing INAR models were constructed on the assumption of stationarity. However few authors have looked at cases non-stationarity (see Section 2.4) but more needs to be done in this area.

ii. Incorporating regime-switching in INAR models, building time-varying parameter INAR models and nonlinear dynamics INAR models with threshold effects and structural breaks are other areas that are not common in literature.

iii. It might be of interest in further research to embark on the development of high-dimensional and multivariate INAR models that addressed parameter explosion, difficulty in capturing cross-correlations and lack of scalable estimation methods. Due to the large volume of data associated with multivariate models it might be worthwhile to combine thinning base modelling with deep machine learning (DML), that is, developing hybrid statistical–DML thinning base models for count processes.

10.0 Conclusion

This review has given comprehensive information on thinning-based models which are specifically constructed for modeling count time series. The review covered different thinning based models such as INAR model, where detailed information on INAR(1) model and the associated extensions and adjustments were given, the INMA models, the INARMA models and models related to either INAR or INMA models (eg. INARFIMA, INBL, MINLINAR, INSINARMA). A robust review was also made on various thinning operators used in the construction of the thinning based models. Particular attention was also given to INAR models for various dispersion structures (under-dispersion, equi-dispersion and over-dispersion) inherent in count time series and INAR models for zero-inflated (deflated) count time series. Also reviewed are test of hypothesis associated to INAR models, various methods of estimation employ in estimating the parameters of INAR models, and forecast functions for INAR(1) model. Lastly, possible areas for future research on thinning based models were suggested.

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