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On the Reformulation of Cramer-Like Methods

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Abstract: *Recently, in 2021 and 2022, Okoli reviewed the result of the work Gabriel Cramer did in 1750 that yield the well-known formula for solving an arbitrary number of unknowns in a square linear system of equations. However, it is observed that the reformulation recorded by (Okoli et al., 2021, 2022) was particularly for Cramer's method and not for related modified Cramer's methods. It is our purpose to extend this reformulation to some of these modified Cramer's methods in this work, as could be seen in the work of (Babarinsa & Kamarulhaili, 2017, 2019), using detailed examples to demonstrate the validity of the result obtained in this research work.*

Keywords *Cramer's rule; Cramer-like methods; determinants; linear systems; linear algebra.*

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1.0 Introduction

The Swiss mathematician Gabriel Cramer (1704–1752) is credited with formulating the rule for solving an n×n system of linear equations that subsequently became known as Cramer's rule (Debnath, 2013). Cramer's rule provides explicit expressions for the unknown variables of a square linear system in terms of determinants and has consequently played an important role in the development of determinant-based methods for solving linear systems. Although related determinant-based procedures had been used before Cramer's publication (MacLaurin, 1748; Boyer, 1966; Hedman, 1999), Cramer is widely credited with giving the method a systematic and recognizable formulation.. The solution of systems of linear equations is fundamental to mathematics and has wide applications in science, engineering, economics, statistics, and other quantitative disciplines Historically, Cramer's rule was often regarded as computationally impractical for large systems because of the computational cost associated with determinant evaluation (Moler, 1974)., which gave room to some 'favoured' alternative methods, such method includes; However, subsequent studies have challenged the view that Cramer's rule is inherently computationally impractical, particularly when efficient procedures for evaluating determinants are employed (Habgood, 2010; Dunham, 1980). These developments suggest that the computational limitations traditionally associated with Cramer's rule may depend substantially on the determinant-evaluation technique employed. Consequently, computational methods such as Gaussian elimination, Gauss–Jordan elimination, and LU decomposition have generally been preferred for the numerical solution of large linear systems.

It is important to note that much of the criticism concerning the computational impracticality of Cramer's rule is associated with the use of Laplace expansion for determinant evaluation,

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which becomes increasingly expensive as the order of the matrix increases (Habgood, 2010). The development and rediscovery of alternative determinant-evaluation techniques, including Sylvester's identity, Chio's condensation, and Jacobi's identity as related to Dodgson's condensation, have provided alternative approaches to determinant computation (Chio, 1853; Dodgson, 1866; Muir, 1960; Akritas *et al*., 1996; Francine, 2008, 2014). These approaches have attracted attention because appropriate modifications of the condensation process can reduce the arithmetic operations and computational time required for determinant evaluation Furthermore, some modifications in the condensation process by some authors have yielded the same arithmetic computational complexity and operational time when compared to the Gaussian elimination method and its likes (Habgood & Arel, 2012). Such developments are particularly relevant to Cramer-based methods because the computational burden of Cramer's rule is closely related to the evaluation of the determinants appearing in its formula.

Despite these developments, existing reformulations have largely focused on the classical Cramer's rule. Less attention has been given to determining whether the reformulation developed for Cramer's rule can be systematically extended to modified Cramer's methods or other Cramer-like formulations. Thus, there remains a methodological gap concerning the applicability of the reformulation framework to these related methods. Addressing this gap is important for determining whether the advantages associated with the reformulation of Cramer's rule can also be realized in its modified forms.

Okoli *et al*. (2021, 2022) previously examined the reformulation of Cramer's rule and demonstrated its applicability to the classical formulation. However, the extent to which this reformulation applies to modified Cramer's methods remains insufficiently investigated. Therefore, this study aims to extend the reformulation of Cramer's rule to selected modified Cramer-like methods and to establish the validity of the resulting formulations through theoretical derivations and illustrative numerical examples

Based on the aforementioned computational improvement in evaluating the determinant of a matrix, it is significant to develop the Cramer's approach unlike when it was believed that it is computational impracticable, which some scholars have already done as we shall discuss some of these developments in the sequel.

The remainder of this paper is organized as follows. The next section presents Cramer's rule and selected modified Cramer-like methods. The main reformulation results are then established, followed by illustrative examples that demonstrate their validity. The paper concludes with a summary of the findings and recommendations for further research. The study is significant because it broadens the existing reformulation framework, provides a unified perspective on related Cramer-like methods, and contributes to the development of alternative determinant-based procedures for solving systems of linear equations

1.1 Cramer's Method and some of it Variants

Let R denote the set of all real numbers such that $a_{ij} \in R = \{x :\ \infty < x < \infty\}$ for all $i, j \ \in [n] = \{1,2,3,\cdots,n\}$, then a system of n linear equations in n variables is given by

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{matrix}\right\} \tag{1}$$

Its equivalent matrix equation is given by

$$AX = B \tag{2}$$

Where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \vdots & \vdots & \vdots \ \ddots \ \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{pmatrix} ; \ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ; \ B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Where a_{ij} is the (i, j)-th component (entry) of the matrix A. The Transpose of the matrix A is denoted by A^T. Let

$$A_{r'|B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,r-1} & b_1 & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,r-1} & b_2 & a_{2,r+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,r-1} & b_3 & a_{3,r+1} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4,r-1} & b_4 & a_{4,r+1} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,r-1} & b_n & a_{n,r+1} & \cdots & a_{nn} \end{pmatrix} ; A_{r\pm B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,r-1} & a_{1,r} \pm b_1 & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,r-1} & a_{2,r} \pm b_2 & a_{2,r+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,r-1} & a_{3,r} \pm b_3 & a_{3,r+1} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4,r-1} & a_{4,r} \pm b_4 & a_{4,r+1} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,r-1} & a_{n,r} \pm b_n & a_{n,r+1} & \cdots & a_{nn} \end{pmatrix}$$

Let the coefficient matrix of the system whose r^{th} column is replaced by (restricted to) the column vector B and the matrix obtained by adding the column vector B to the r^{th} column vector of the coefficient matrix of the system with their corresponding determinant denoted as $D_{r'|B}$ and $D_{r\pm B}$.

Also let D_0 denote the determinant of the coefficient matrix of the system A given by

$$D_0 \coloneqq |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,r-1} & a_{1,r} & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,r-1} & a_{2,r} & a_{2,r+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,r-1} & a_{3,r} & a_{3,r+1} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4,r-1} & a_{4,r} & a_{4,r+1} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,r-1} & a_{n,r} & a_{n,r+1} & \cdots & a_{nn} \end{vmatrix} = \sum_{j_1 j_2 j_3 \cdots j_n \in S_n} \mu_{j_1 j_2 j_3 \cdots j_n} a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n}$$

Where the summing is taken over the permutation set (symmetric permutation group) S_n and

$$\mu_{j_1 j_2 j_3 \cdots j_n} = \begin{cases} +1; \ if \ j_1 j_2 j_3 \cdots j_n \ is \ an \ even \ permutation \\ -1; \ if \ j_1 j_2 j_3 \cdots j_n \ is \ an \ odd \ permutation \end{cases}$$

Cramer (1750), propose the following theorem for solving square system of linear equations

Lemma 2.1 (Cramer's Theorem) If the matrix of coefficients A of system (1.1) is non-singular, then the unique solution $(x_1, x_2, x_3, \cdots, x_n)$ to the system is given by

$$x_r = \frac{D_{r'|B}}{D_0} \ \ \forall \ r = 1,2,3, \cdots, n \tag{3}$$

In an attempt to modify the work of Cramer's, Babarinsa and Kamarulhaili (2017) used the fact that

$$D_{r\pm B} = D_0 \pm D_{r'|B} \tag{4}$$

to obtain the following modification on dividing through by D_0;

$$\frac{D_{r\pm B}}{D_0} = 1 \pm \frac{D_{j'|B}}{D_0}; \Longrightarrow \ \frac{D_{j'|B}}{D_0} = \pm\left(\frac{D_{r\pm B}}{D_0} - 1\right)$$

So that by (1.3) we have

$$x_r = \begin{cases} 1 - \dfrac{D_{r-B}}{D_0} \\ \dfrac{D_{r+B}}{D_0} - 1 \end{cases} \tag{5}$$

Kitaro (2018), used the transpose property of a matrix $|A| = |A^T|$ to obtain the following modification

$$x_r = \frac{D_{r'|B}}{D_0} = \frac{D_{r'|B}}{|A|} = \frac{D_{r'|B}}{|A^T|} \tag{6}$$

Which is quite obvious and trivia.

Proposition 2.2 Let $(y_1, y_2, y_3, \cdots, y_n)^T \in R^n$ be such that

$$\left.\begin{array}{l} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \cdots + a_{1n}y_n = \sum_{j=1}^{n} a_{1j} \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \cdots + a_{2n}y_n = \sum_{j=1}^{n} a_{2j} \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + \cdots + a_{3n}y_n = \sum_{j=1}^{n} a_{3j} \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ a_{n1}y_1 + a_{n2}y_2 + a_{n3}y_3 + \cdots + a_{nn}y_n = \sum_{j=1}^{n} a_{nj} \end{array}\right\} \tag{.7}$$

Then

$$D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T} = D_0 \quad \forall \ r \in [n] \tag{8}$$

Proof.

Observe that the constant b_i $(i = 1,2,3,\cdots,n)$ has been replaced by the summation of the row entries $\sum_{j=1}^{n} a_{ij}$ $(i = 1,2,3,\cdots,n)$ of the coefficient matrix of the linear system (1.1). Thus the only solution set $\{y_j : j = 1,2,3,\cdots,n\}$ that satisfies the system of equations (1.7) is $\{y_{\square} = 1 \ \forall \, r = 1,2,3,\cdots,n\}$ so that it follows from Cramer's method that

$$y_r = \frac{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T}}{D_0} = 1 \ \forall \ r = 1,2,3,\cdots,n; \Longrightarrow$$

$$D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T} = D_0 \quad \forall \ r \in [n]$$

This completes the proof. Furthermore, more recently, (Babarinsa and Kamarulhaili, 2019) used equation (1.8) to obtain the modification that follows; observe that by equation (1.4) it follows that

$$D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T \pm B} = D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T} \pm D_{r'|B} \, ; \Longrightarrow$$

$$\frac{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T \pm B}}{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T}} = 1 \pm \frac{D_{r'|B}}{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T}} \, ; \Longrightarrow$$

$$\frac{D_{r'|B}}{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T}} = \pm\left(\frac{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T \pm B}}{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T}} - 1\right) ; \Longrightarrow$$

So that by (18) we have

$$x_r = \pm\left(\frac{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T \pm B}}{D_{r'|\left(\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T}} - 1\right) ; \Longrightarrow$$

$$x_r = \begin{cases} 1 - \dfrac{D_{r'|\left(\sum_{j=1}^{n} a_{1j},\cdots,\sum_{j=1}^{n} a_{nj}\right)^T - B}}{D_{r'|\left(\sum_{j=1}^{n} a_{1j},\cdots,\sum_{j=1}^{n} a_{nj}\right)^T}} \\ \dfrac{D_{r'|\left(\sum_{j=1}^{n} a_{1j},\cdots,\sum_{j=1}^{n} a_{nj}\right)^T + B}}{D_{r'|\left(\sum_{j=1}^{n} a_{1j},\cdots,\sum_{j=1}^{n} a_{nj}\right)^T}} - 1 \end{cases} \tag{9}$$

It is interesting to note that Cramer's method and its modifications, as presented in Equations [insert equation numbers], all require [insert number] determinants to determine the solution of the system completely. Since a single evaluation of a determinant using the cofactor method requires [insert number] terms for an n×n system, the computational burden becomes increasingly cumbersome as n becomes arbitrarily large. Furthermore, the proposed modifications introduce additional arithmetic operations, contrary to the expectation that such modifications would reduce the computational complexity. Consequently, these modifications do not necessarily provide a better computational time than the classical Cramer's method. This limitation was duly acknowledged by Babarinsa and Kamarulhaili (2017, 2019).

Therefore, this study aims to develop a reformulation of the modified Cramer-like methods that minimizes the computational time involved in obtaining the solution of the system. Specifically, the proposed formulation seeks to reduce the number of determinant evaluations required to determine the complete solution, compared with the classical Cramer's method and its existing modifications. The proposed approach is subsequently demonstrated and validated using appropriate mathematical derivations and illustrative examples.

3.0 Main Results

At this point we are now ready to give some important theorems that justify the purpose of this research work as stated earlier. We shall begin with some important definitions and lemmas which we shall use in the sequel.

Lemma 3.1 (Okoli *et al*., 2021): let the entries of the column matrix $\left(D_{1'|B}, D_{2'|B}, \cdots, D_{n'|B}\right)^T$ represent the determinant defined above, then the determinant D_0 ($|A|$) of the coefficient matrix A is given by

$$D_0 = \frac{1}{b_i}\left(\sum_{j=1}^{n} a_{ij} D_{j'|B}\right) : \forall \ i \in [n] \tag{10}$$

Lemma 3.2 let the entries of the column matrix $(D_{1\pm B}, D_{2\pm B}, \cdots, D_{n\pm B})^T$ represent the determinant defined above then for every $i \in [n]$ the determinant D_0 ($|A|$) of the coefficient matrix A is given by

$$D_0 = \begin{cases} \dfrac{1}{\sum_{j=1}^{n} a_{ij} + b_i}\left(\sum_{j=1}^{n} a_{ij}\, D_{j+B}\right) \\ \dfrac{1}{\sum_{j=1}^{n} a_{ij} - b_i}\left(\sum_{j=1}^{n} a_{ij} D_{j-B}\right) \end{cases} \tag{11}$$

Proof.

Observe that system (1.1) above can be written as

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \ ; \quad i = 1,2,3, \ldots \ , n$$

By using the result of (Babarinsa and Kamarulhaili, 2017), $x_r = \pm\left(\frac{D_{r\pm B}}{D_0} - 1\right)$, it suffices to prove this for one part of the value say $x_r = \frac{D_{r\pm B}}{D_0} - 1$ then the other follows similarly. Now

$$b_i = \sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} a_{ij}\left(\frac{D_{j+B}}{D_0} - 1\right) = \frac{1}{D_0}\sum_{j=1}^{n} a_{ij}D_{j+B} - \sum_{j=1}^{n} a_{ij} \;;\Rightarrow$$

$$D_0 = \frac{1}{b_i + \sum_{j=1}^{n} a_{ij}}\left(\sum_{j=1}^{n} a_{ij}\, D_{j+B}\right)$$

This completes the proof of the lemma.

Lemma 3.3 let the entries of the column matrix $\left(D_{r'|E\pm B}\,, D_{r'|E\pm B}\,, \cdots, D_{r'|E\pm B}\right)^T$ represent the determinant defined above then for every $i \in [n]$ the determinant D_0 ($|A|$) of the coefficient matrix A is given by

$$D_0 = \begin{cases} \dfrac{1}{\sum_{j=1}^{n} a_{ij} + b_i}\left(\displaystyle\sum_{j=1}^{n} a_{ij}\, D_{r'|E+B}\right) \\ \dfrac{1}{\sum_{j=1}^{n} a_{ij} - b_i}\left(\displaystyle\sum_{j=1}^{n} a_{ij} D_{r'|E-B}\right) \end{cases} \qquad (12)$$

where $E = \left(\sum_{j=1}^{n} a_{1j}\,, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T$.

Proof.

Similarly, we follow the same argument as in lemma 3.2. This completes the proof of the lemma.

The importance of these lemmas is that it eliminate the computation of D_0 from the matrix of coefficients A of system (1.1), thus reducing the amount of time required to complete the computation and guarantees the fact that for $j = 1,2,3, \dots ,n$ the collection $\{D_{j'|B}\}, \{D_{j\pm B}\}, \{D_{j'|E\pm B}\}$ completely determines the solution set $\{x_j\}$ of the linear system (1.1) respectively, unlike in the case of (1.3), (1.5), (1.6) and (1.9) above, which requires the direct computation of D_0 from the matrix of coefficients A of the system. The next theorem actually furnishes us with this computing formula.

Theorem 3.4 If the matrix of coefficients A of system (1.1) is non-singular, then $\forall\, i \in [n]$ the unique solution $(x_1, x_2, x_3, \cdots, x_n)$ to the system is given by

$$x_r = \begin{cases} \dfrac{b_i D_{r'|B}}{\sum_{j=1}^{n} a_{ij} D_{j'|B}} \\ \dfrac{\left(\sum_{j=1}^{n} a_{ij} + b_i\right) D_{r+B}}{\sum_{j=1}^{n} a_{ij}\, D_{j+B}} - 1 \\ 1 - \dfrac{\left(\sum_{j=1}^{n} a_{ij} - b_i\right) D_{r-B}}{\sum_{j=1}^{n} a_{ij}\, D_{j-B}} \\ \dfrac{\left(\sum_{j=1}^{n} a_{ij} + b_i\right) D_{r'|E+B}}{\sum_{j=1}^{n} a_{ij} D_{j'|E+B}} - 1 \\ 1 - \dfrac{\left(\sum_{j=1}^{n} a_{ij} - b_i\right) D_{r'|E-B}}{\sum_{j=1}^{n} a_{ij} D_{j'|E-B}} \end{cases} \qquad (13)$$

where $E = \left(\sum_{j=1}^{n} a_{1j}\,, \sum_{j=1}^{n} a_{2j}\,, \cdots, \sum_{j=1}^{n} a_{nj}\right)^T$.

Proof.

The proof of this theorem easily follows from the results of Lemma 2.1, equation (1.4), Lemma 3.1, Lemma 3.2 and Lemma 3.3.

To further validate Theorem 3.4 we consider a numerical example to demonstrate the theorem in the section that follows.

In this section, we will use the example in (Babarinsa and Kamarulhaili 2017) below to illustrate the validity of our theorems as concern to fhe (Cramer's, 1750) and the (Babarinsa and Kamarulhaili 2017) modified method and the method proposed in this work.

Application

Example 4.1 Solve for x_j for $j \in \{1,2,3,4\}$ respectively in the system of linear equations

$$\begin{gathered}2x_1 + 5x_2 - 9x_3 + 3x_4 = 151\\ 5x_1 + 6x_2 - 4x_3 + 2x_4 = 103\\ 3x_1 - 4x_2 + 2x_3 + 7x_4 = 16\\ 11x_1 + 7x_2 + 4x_3 - 8x_4 = -32\end{gathered}$$

Solution

The (Cramer's, 1750) and the (Babarinsa and Kamarulhaili, 2017) modified formula $x_r = \frac{D_{r+B}}{D_0} - 1$, requires the computation of five (5) determinants (using any convenient method) of the set $\{D_0, D_B, D_B, D_B, D_B\}$ and $\{D_0, D_{1+B}, D_{2+B}, D_{3+B}, D_{4+B}\}$ given by

$$\left.\begin{array}{l}
D_0 = \begin{vmatrix} 2 & 5 & -9 & 3\\ 5 & 6 & -4 & 2\\ 3 & -4 & 2 & 7\\ 11 & 7 & 4 & -8\end{vmatrix} = -2373\\
D_{1|B} = \begin{vmatrix} 151 & 5 & -9 & 3\\ 103 & 6 & -4 & 2\\ 16 & -4 & 2 & 7\\ -32 & 7 & 4 & -8\end{vmatrix} = -7119\\
D_{2|B} = \begin{vmatrix} 2 & 151 & -9 & 3\\ 5 & 103 & -4 & 2\\ 3 & 16 & 2 & 7\\ 11 & -32 & 4 & -8\end{vmatrix} = -11865\\
D_{3|B} = \begin{vmatrix} 2 & 5 & 151 & 3\\ 5 & 6 & 103 & 2\\ 3 & -4 & 16 & 7\\ 11 & 7 & -32 & -8\end{vmatrix} = 26103\\
D_{4|B} = \begin{vmatrix} 2 & 5 & -9 & 151\\ 5 & 6 & -4 & 103\\ 3 & -4 & 2 & 16\\ 11 & 7 & 4 & -32\end{vmatrix} = -16611
\end{array}\right\};\Longrightarrow
\begin{array}{l}
x_1 = \frac{D_{1|B}}{D_0} = \frac{-7119}{-2373} = 3\\
x_2 = \frac{D_{2|B}}{D_0} = \frac{-11865}{-2373} = 5\\
x_3 = \frac{D_{3|B}}{D_0} = \frac{26103}{-2373} = -11\\
x_4 = \frac{D_{4|B}}{D_0} = \frac{-16611}{-2373} = 7
\end{array}$$

and

$$\left.\begin{aligned}
D_0 &= \begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix} = -2373 \\
D_{1+B} &= \begin{vmatrix} 2+151 & 5 & -9 & 3 \\ 5+103 & 6 & -4 & 2 \\ 3+16 & -4 & 2 & 7 \\ 11+(-32) & 7 & 4 & -8 \end{vmatrix} = -9492 \\
D_{2+B} &= \begin{vmatrix} 2 & 5+151 & -9 & 3 \\ 5 & 6+103 & -4 & 2 \\ 3 & -4+16 & 2 & 7 \\ 11 & 7+(-32) & 4 & -8 \end{vmatrix} = -14238 \\
D_{3+B} &= \begin{vmatrix} 2 & 5 & -9+151 & 3 \\ 5 & 6 & -4+103 & 2 \\ 3 & -4 & 2+16 & 7 \\ 11 & 7 & 4+(-32) & -8 \end{vmatrix} = 23730 \\
D_{4+B} &= \begin{vmatrix} 2 & 5 & -9 & 3+151 \\ 5 & 6 & -4 & 2+103 \\ 3 & -4 & 2 & 7+16 \\ 11 & 7 & 4 & -8+(-32) \end{vmatrix} = -18984
\end{aligned}\right\};\Longrightarrow
\begin{aligned}
x_1 &= \frac{D_{1+B}}{D_0} - 1 = \frac{-9492}{-2373} - 1 = 3 \\
x_2 &= \frac{D_{2+B}}{D_0} - 1 = \frac{-14238}{-2373} - 1 = 5 \\
x_3 &= \frac{D_{3+B}}{D_0} - 1 = \frac{-23730}{-2373} - 1 = -11 \\
x_4 &= \frac{D_{4+B}}{D_0} - 1 = \frac{-18984}{-2373} - 1 = 7
\end{aligned}$$

In our formula $x_r = \frac{\left(\sum_{j=1}^{n} a_{ij} + b_i\right) D_{r+B}}{\sum_{j=1}^{n} a_{ij}\, D_{j+B}} - 1; r = 1, 2, 3, 4,$ we take four (4) determinants as contained in the set $\{D_{1+B}, D_{2+B}, D_{3+B}, D_{4+B}\} = \{-9492, -14238, -23730, -18984\}$ to determine the solution. So that for $i = 1$

$$\left.\begin{aligned}
D_{1+B} &= \begin{vmatrix} 2+151 & 5 & -9 & 3 \\ 5+103 & 6 & -4 & 2 \\ 3+16 & -4 & 2 & 7 \\ 11+(-32) & 7 & 4 & -8 \end{vmatrix} = -9492 \\
D_{2+B} &= \begin{vmatrix} 2 & 5+151 & -9 & 3 \\ 5 & 6+103 & -4 & 2 \\ 3 & -4+16 & 2 & 7 \\ 11 & 7+(-32) & 4 & -8 \end{vmatrix} = -14238 \\
D_{3+B} &= \begin{vmatrix} 2 & 5 & -9+151 & 3 \\ 5 & 6 & -4+103 & 2 \\ 3 & -4 & 2+16 & 7 \\ 11 & 7 & 4+(-32) & -8 \end{vmatrix} = 23730 \\
D_{4+B} &= \begin{vmatrix} 2 & 5 & -9 & 3+151 \\ 5 & 6 & -4 & 2+103 \\ 3 & -4 & 2 & 7+16 \\ 11 & 7 & 4 & -8+(-32) \end{vmatrix} = -18984
\end{aligned}\right\}
\begin{aligned}
x_1 &= \frac{(1+151)(-9492)}{2(-9492)+5(-14238)-9(23730)+3(-18984)} - 1 = 3 \\
x_2 &= \frac{(1+151)(-14238)}{2(-9492)+5(-14238)-9(23730)+3(-18984)} - 1 = 5 \\
x_3 &= \frac{(1+151)(23730)}{2(-9492)+5(-14238)-9(23730)+3(-18984)} - 1 = -11 \\
x_4 &= \frac{(1+151)(-18984)}{2(-9492)+5(-14238)-9(23730)+3(-18984)} - 1 = 7
\end{aligned}$$

It is important to note that (you may wish to verify) if we take $i = 2,3,4$ we will still have the same result. This fact gives the method proposed some degree of flexibility or choices when determining the solution to the linear system. Similarly, using the other part of (Babarinsa and Kamarulhaili 2017) modified formula $x_r = 1 - \frac{D_{r-B}}{D_0}$, one still need to compute (another) five (5) determinants (using any convenient method) of the entries in the set $\{D_0, D_{1'|B}, D_{2'|B}, D_{3'|B}, D_{4'|B}\}$ which is given by

$$\left.\begin{aligned}
D_0 &= \begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix} = -2373 \\
D_{1-B} &= \begin{vmatrix} 2-151 & 5 & -9 & 3 \\ 5-103 & 6 & -4 & 2 \\ 3-16 & -4 & 2 & 7 \\ 11-(-32) & 7 & 4 & -8 \end{vmatrix} = 4746 \\
D_{2-B} &= \begin{vmatrix} 2 & 5-151 & -9 & 3 \\ 5 & 6-103 & -4 & 2 \\ 3 & -4-16 & 2 & 7 \\ 11 & 7-(-32) & 4 & -8 \end{vmatrix} = 9492 \\
D_{3-B} &= \begin{vmatrix} 2 & 5 & -9-151 & 3 \\ 5 & 6 & -4-103 & 2 \\ 3 & -4 & 2-16 & 7 \\ 11 & 7 & 4-(-32) & -8 \end{vmatrix} = -28476 \\
D_{4-B} &= \begin{vmatrix} 2 & 5 & -9 & 3-151 \\ 5 & 6 & -4 & 2-103 \\ 3 & -4 & 2 & 7-16 \\ 11 & 7 & 4 & -8-(-32) \end{vmatrix} = 14238
\end{aligned}\right\};\Longrightarrow
\begin{aligned}
x_1 &= 1-\frac{D_{1-B}}{D_0} = 1-\frac{4746}{-2373} = 3 \\
x_2 &= 1-\frac{D_{2-B}}{D_0} = 1-\frac{9492}{-2373} = 5 \\
x_3 &= 1-\frac{D_{3-B}}{D_0} = 1-\frac{-28476}{-2373} = -11 \\
x_4 &= 1-\frac{D_{4-B}}{D_0} = 1-\frac{14238}{-2373} = 7
\end{aligned}$$

In our formula $x_r = 1-\frac{\left(\sum_{j=1}^{n} a_{ij}-b_i\right)D_{r-B}}{\sum_{j=1}^{n} a_{ij}\, D_{j-B}}; r = 1,2,3,4,$ we take four (4) determinants as contained in the set $\{D_{1-B}, D_{2-B}, D_{3-B}, D_{4-B}\} = \{4746, 9492, -28476, -14238\}$ to determine the solution. So that for $i = 1$

$$\begin{aligned}
D_{1-B} &= \begin{vmatrix} 2-151 & 5 & -9 & 3 \\ 5-103 & 6 & -4 & 2 \\ 3-16 & -4 & 2 & 7 \\ 11-(-32) & 7 & 4 & -8 \end{vmatrix} = 4746 \\
D_{2-B} &= \begin{vmatrix} 2 & 5-151 & -9 & 3 \\ 5 & 6-103 & -4 & 2 \\ 3 & -4-16 & 2 & 7 \\ 11 & 7-(-32) & 4 & -8 \end{vmatrix} = 9492 \\
D_{3-B} &= \begin{vmatrix} 2 & 5 & -9-151 & 3 \\ 5 & 6 & -4-103 & 2 \\ 3 & -4 & 2-16 & 7 \\ 11 & 7 & 4-(-32) & -8 \end{vmatrix} = -28476 \\
D_{4-B} &= \begin{vmatrix} 2 & 5 & -9 & 3-151 \\ 5 & 6 & -4 & 2-103 \\ 3 & -4 & 2 & 7-16 \\ 11 & 7 & 4 & -8-(-32) \end{vmatrix} = 14238
\end{aligned}
;\Longrightarrow
\begin{aligned}
x_1 &= 1-\frac{(1-151)(4746)}{2(4746)+5(9492)-9(-28476)+3(14238)} = 3 \\
x_2 &= 1-\frac{(1-151)(9492)}{2(4746)+5(9492)-9(-28476)+3(14238)} = 5 \\
x_3 &= 1-\frac{(1-151)(-28476)}{2(4746)+5(9492)-9(-28476)+3(14238)} = -11 \\
x_4 &= 1-\frac{(1-151)(14238)}{2(4746)+5(9492)-9(-28476)+3(14238)} = 7
\end{aligned}$$

Similarly, (you may wish to verify) if we take $i = 2,3,4$ we will still have the same result.

4.0 Conclusion

In this study, a reformulated formula-based method has been developed to reduce the computational time associated with the Cramer-like methods proposed by Babarinsa and Kamarulhaili (2017, 2019), particularly as the order of the system increases. The validity of the proposed formulation was demonstrated through the illustrative example presented in this study. Similar to the reformulation reported by Okoli *et al*. (2020), the proposed method requires only determinant evaluation(s) to determine the complete solution of the system, compared with the determinant evaluations required by the classical Cramer's method and its variants.

The proposed formulation therefore provides a more computationally efficient alternative to the existing Cramer-type and modified Cramer-like methods considered in this study.

Accordingly, the proposed methods are recommended for solving systems of linear equations where reduction in determinant evaluations, arithmetic operations, and computational time is desirable.

Furthermore, the study demonstrates that the proposed formulation provides a degree of flexibility in solving the system, allowing the problem to be approached in different ways while maintaining the same minimal computational time and arithmetic complexity. This flexibility is an additional advantage over the classical Cramer's method and its variants. The proposed reformulation therefore extends the applicability of Cramer-type methods and provides a basis for further investigation of computationally efficient determinant-based techniques for solving systems of linear equations.

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Odico Chukwuemeka Okoli conceptualized the study, developed the reformulated Cramer-like methods, and supervised the mathematical analysis. Mark Laisin contributed to the literature review, mathematical derivations, verification of examples, and interpretation of results. Amarachi C. Eze contributed to data validation, computational checking, manuscript preparation, and critical revision. All authors reviewed and approved the final manuscript and agreed to be accountable for its content.