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Comparism Between Analytical and Numerical Result of Stability Analysis of a Dynamical System

Innocent C. Eli and Godspower C. Abanum*
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Abstract *The system of ODE's involving Sickle Cell, HIV and T-Cells are formulated with the aid of biological mathematical model. The eigen values are obtained to test for the trivial steady state solution or points using characteristic equation which is analytical. Finally, numerical simulations were carried out to test the level of reliability of the result*

Keywords: *Biological mathematical model, Ordinary differential equations, sickle cells, HIV, T-cell, eigen values, ODE 45*

Innocent C. Eli
Department of Mathematics and Statistics,
Ignatius Ajuru University of Education,
Port Harcourt, Nigeria
Email: innocenteli35@gmail.com
Orcid id:

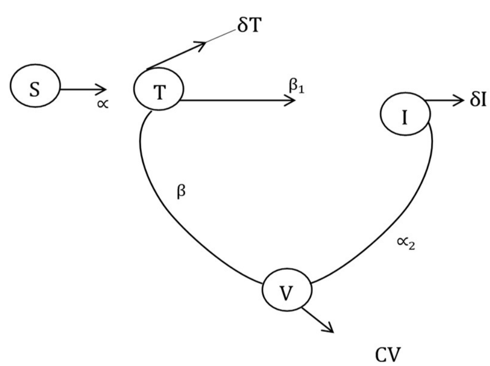
Godspower C. Abanum
Ignatius Ajuru University of Education,
Port Harcourt, Nigeria
Email: godspower.abanum@iaue.edu.ng
Orcid id:0000-0001-8173-226X

1.0 Introduction

Most times when real life (Biological) situation is model and analyzed, it usually leads to ordinary differential equation (Burden, and Faires, 2011). The spread of HIV is also attributed strongly with the spatial distribution of high risk groups. The distribution of AIDS cases do not just varies by cities and states but also by geographical regions (Lange Snyder *et al*., 1988). For the fact that there is no cure of HIV medically for now has given scientist great concern which lead to the system of non-linear first order ordinary differential equations that defines the number of infected cells of HIV infection of $CD4^{+}$T-Cells as one of the time-dependent variables is a popular system of three model equations (Song and Chen, 2005; Ekaka *et al*., 2014). . Other related mathematical models (Grossman *et al*., 1999; Herz *et al*., 1996; Tam, 1999) which explains the dynamics of HIV/AIDS without randomize implications and without weekly predicted assumptions only gives adequate theoretical bases with rare policy simulations. Notwithstanding, the stress associated with the stage of HIV infection (Antoni, 2003), it is shown that the parameters which were limited to explain the dynamics of HIV infection are beyond deterministic characteristic. Thus, random noise in the limited parameters and HIV infection data also aid this work (Elliot *et al*., 2002). In this modern age of medicine besides their importance in evolution, their relation to disease and environment is increasingly imperative (Waziralih *et al*., 2005). In this paper, we consider the comparism between analytical and numerical result of stability analysis of a dynamical system which result is quite different from [4].

2.0 Mathematical formulation

In this section, we would formulate a system of ODE's to study the dynamical interaction between Sickle cells and HIV infected cells with the constant supply of the active T cells from the thymus. However, before we delve into the formulation of the mathematical models, lets us consider the compartment diagram showing their relationship



Compartment Diagram

We formulate nonlinear ordinary differential equations with the following assumptions.

There is a constant supply of T cells from the thymus.

There is a death rate for T-cells, virus and infection.

- T = T-cells population
- V = Recovered human immune cell (T-Cell)

- S = Constant supply of T-cells
- δT = Death rate of T-cells
- β = Death due to interaction of T-cells with HIV virus
- $\propto$ = Is called the maximum proliferation rate of target cells
- $\propto_2$ = Progression rate of the infected T-Cell to the recovered cell group.
- δI = Death rate of infection
- CV = Death rate of virus.
- β_1 = Exponential equation. $\beta_1 = Be^{-m\tau}$ Whereas the term $e^{-m\tau}$ represents cells that are infected, τ is time unit later.
- K = Carrying capacity

2.1 Method of solution

$$\frac{dT}{dt} = S - \delta T + \beta V - \frac{\beta V^2}{K} - \beta_1 T \qquad 1$$

$$\frac{dI}{dt} = \beta_1 TV - \delta T - \propto_1 I \qquad 2$$

$$\frac{dV}{dt} = \propto_2 I - CV - \beta V \qquad 3$$

At steady state, $\frac{dT}{dt}, \frac{dI}{dt}$ and $\frac{dV}{dt} = 0$

$T = T_0 V = I_0 V = V_0,$

i.e.

$$S - \delta T_0 + \beta_0 V - \frac{\beta V_0^2}{K} - \beta_1 T_0 = 0 \qquad 4$$

$$\beta_1 T_0 V_0 - \delta I_0 - \propto_2 I_0 = 0 \qquad 5$$

$$\propto_2 I_0 - CV_0 - \beta V_0 = 0 \qquad 6$$

From equation 6,

$$I_0 = \frac{CV_0 + \beta V_0}{\propto_2} \qquad 7$$

Substituting 7 into 5 we get

$$\beta_1 T_0 V_0 - \delta\left(\frac{CV_0 + \beta V_0}{\propto_2}\right) - \propto_2\left(\frac{CV_0 + \beta V_0}{\propto_2}\right) = 0$$

$$\beta_1 T_0 V_0 - \frac{\delta CV_0 - \delta\beta V_0}{\propto_2} - CV_0 - \beta V_0 = 0$$

$$V_0\left(\beta_1 T_0 - \frac{\delta C - \delta\beta}{\propto_2} - C - \beta\right) = 0$$

$\Rightarrow V_0 = 0$ and

$$\beta_1 T_0 - \frac{\delta C - \delta\beta}{\propto_2} - C - \beta = 0$$

If we substitute $I_1 = 0 = V_0$, then eqn. 2.4 reduces to

$$S - \delta T_0 - \beta_1 T_0 = 0$$

$$S - T_0(\delta - \beta_1) = 0$$

$$S = T_0(\delta - \beta_1)$$

$$\therefore T_0 = \frac{S}{\delta - \beta_1} \qquad 8$$

Hence, the trivial steady state (T_0, V_0, I_0) is $\left(\frac{S}{\delta - \beta_1}, 0, 0\right)$

Using linearization, we get

$$\frac{dT}{dt} = S - \delta T + \beta V - \frac{\beta V^2}{K} - \beta_1 T = F_1(T, I, V) \qquad 9$$

$$\frac{dI}{dt} = \beta_1 TV - \delta I - \propto_2 I = F_2(T, I, V) \qquad 10$$

$$\frac{dV}{dt} = \propto_1 I - CV - \beta V = F_3(T, I, V) \qquad 11$$

$F_1T = \delta - \beta_1$

$F_1I = 0$

$F_1V = \beta - \frac{2\beta V}{K}$

$F_2T = \beta_1 V$

$F_2I = -\delta - \propto_2$

$F_2V = \beta_1 T$

$F_3T = 0$

$F_3I = \propto_2$

$F_3V = -C - \beta$

In order to get the eigen values using dacobian matrix $|J - \lambda I| = 0$ where I is the identity matrix.

Here $J = \begin{bmatrix} F_1T & F_1I & F_1V \\ F_2T & F_2I & F_2V \\ F_3T & F_3I & F_3V \end{bmatrix}$ and

Identify matrix $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that

$$|J - \lambda I| = 0 = \begin{bmatrix} -\delta - \beta_1 & 0 & \beta - \frac{2\beta V}{K} \\ \beta_1 V & -\delta - \propto_2 & \beta_1 T \\ 0 & \propto_2 & -C - \beta \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

But $V = V_0 = 0$, so that

$$= \begin{bmatrix} -\delta - \beta_1 - \lambda & 0 & \beta \\ 0 & -\delta - \propto_2 - \lambda & \beta_1 T \\ 0 & \propto_2 & -C - \beta - \lambda \end{bmatrix} = 0$$

$$[-\delta - \beta_1 - \lambda][(-\delta - \propto_2 - \lambda)(-C - \beta - \lambda) - \propto_2 \beta_1 T] = 0$$

Which shows that λ cannot be strictly negative hence not stable.

3. 0 Results and Discussion

The result of the numerical simulation with respect to (i) Quantifying the effect of decreasing and increasing the model parameter K = 1,300 and the type of stability of a trivial steady state solution under basic assumption using ODE 45 and (ii) Quantifying the effect of decreasing and increasing the model parameter $\alpha = 6.8$ and the type of stability of a trivial steady state solution using ODE 45 are presented in Tables 1 and 2 respectively.

Table 1: Quantifying the effect of decreasing and increasing the model parameter K = 1,300 and the type of stability of a trivial steady state solution under basic assumption using ODE 45

Example	$K\times 10^3$	$T \times 10^3$	I	V	λ_1	λ_2	λ_3	TOS
1	1.3000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
2	130.0000	16.6667	0	0	0.3000	-6.3000	-10.2000	Unstable
3	260.0000	16.6667	0	0	0.3000	-6.3000	-10.2000	Unstable
4	390.0000	16.6667	0	0	0.3000	-6.3000	-10.2000	Unstable
5	520.0000	16.6667	0	0	0.3000	-6.3000	-10.2000	Unstable
6	650.0000	16.6667	0	0	0.3000	-6.3000	-10.2000	Unstable
7	780.0000	16.6667	0	0	0.3000	-6.3000	-10.2000	Unstable
8	910.0000	16.6667	0	0	0.3000	-6.3000	-10.2000	Unstable
9	1.04000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
10	1.17000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
11	1.23500	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
12	1.43000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
13	1.56000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
14	1.69000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
15	1.82000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
16	1.95000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
17	2.08000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
18	2.21000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
19	2.34000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
20	2.47000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable

Table 2: Quantifying the effect of decreasing and increasing the model parameter $\alpha = 6.8$ and the type of stability of a trivial steady state solution using ODE 45

Example	α	$T \times 10^3$	I	V	λ_1	λ_2	λ_3	TOS
1	6.8000	0.0167	0	0	0.0003	-0.0063	-0.0102	Unstable
2	0.6800	16.6667	0	0	0.0003	-0.1800	-10.2000	Unstable
3	1.3600	16.6667	0	0	0.0003	-0.8600	-10.2000	Unstable
4	2.0400	16.6667	0	0	0.0003	-1.5400	-10.2000	Unstable
5	2.7200	16.6667	0	0	0.0003	-2.2200	-10.2000	Unstable
6	3.4000	16.6667	0	0	0.0003	-2.9000	-10.2000	Unstable
7	4.0800	16.6667	0	0	0.0003	-3.5800	-10.2000	Unstable
8	4.7600	16.6667	0	0	0.0003	-4.2600	-10.2000	Unstable
9	5.4400	16.6667	0	0	0.0003	-4.9400	-10.2000	Unstable
10	6.1200	16.6667	0	0	0.0003	-5.6200	-10.2000	Unstable
11	6.4600	16.6667	0	0	0.0003	-5.9600	-10.2000	Unstable
12	6.8680	16.6667	0	0	0.0003	-6.3680	-10.2000	Unstable
13	7.4800	16.6667	0	0	0.0003	-6.9800	-10.2000	Unstable
14	8.1600	16.6667	0	0	0.0003	-7.6600	-10.2000	Unstable
15	8.8400	16.6667	0	0	0.0003	-8.3400	-10.2000	Unstable
16	9.5200	16.6667	0	0	0.0003	-9.0200	-10.2000	Unstable
17	10.200	16.6667	0	0	0.0003	-9.7000	-10.2000	Unstable
18	10.88	16.6667	0	0	0.0003	-10.3800	-10.2000	Unstable
19	11.560	16.6667	0	0	0.0003	-11.0600	-10.2000	Unstable
20	12.240	16.6667	0	0	0.0003	-11.7400	-10.2000	Unstable

The result of the numerical simulation as seen in Tables 1 and 3 and analytical method of solving of the dynamical system (method of solution) shows that the dynamical system under consideration is asymptotically unstable. Looking at Table 1, when the model parameter value $K =$

1300 is decreased and increased deterministically from 10%, 20% to 190%, we have observed that the trivial steady state solution is dominantly unstable. In the same context , when the $\boldsymbol{\alpha = 6.8}$ parameter value is decreased and also increased from 10% followed by 20% and in step length of 10% up to 180%, we have also found that the type of stability remain the same for the trivial steady state solution (Table 2).

4.0 Conclusion

We have found that irrespective of increasing the decreased of the carrying capacity and the maximum proliferation rate of target cell, the trivial steady state solution does not change its type of stability. Other possible extension of this present study will be explore in our next investigation.

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